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PRIME FILTERS WITH CIP

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Many definitions and theorems in general topology have common background in the theory of filters. It seems to me that the most convenient object for the study filter properties is paved space which is defined to be a set endowed with a finitely additive and finitely multiplicative cover (called paving). It appears then, e.g., that normality and extremal disconnectedness of a topological space have formally the same definition, and the description by means of filters is stimulating.

Here we consider the properties related to realcompact spaces, ultracompact spaces, almost realcompact spaces etc. In other words, we study the tightness of two valued σ-additive functions and their extensions. Thus the cardinal $\aleph_0$ enters all the definition, e.g. we consider filters with the countable intersection property and various kinds of completeness are introduced by properties of these filters. Almost all the results hold if the cardinal $\aleph_0$ is replaced by any larger cardinal, and in addition, the proofs work. The reader is invited to read this paper in this more general setting.

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For more results on extensions we refer to Frolík [5] where the extensions to hyper-extensions of pavings are studied - abstract "discrete" unions are taken as admissible operation. For more examples we refer to Frolík [6] where BB-complete spaces (= every Borel two-valued σ-neasure is a Dirac measure in Baire sets) are studied. Finally, in Frolík [7] a neat theory of multivalued maps into a paved space is developed.

The basic definitions and results were discussed in my 1970-71 seminars, and one of the main results was included in Herrlich s lecture on the 3rd Prague Symposium. The aim of the present paper is to furnish the proofs and the background material. The approach to the problems seems to be more important than the results.

1. Filters. In this paragraph let X be a set, and let $\mathcal{F}$ be a paving of X. The elements of $\mathcal{F}$ are called stones, and the elements of $\gamma\mathcal{F}$ are called co-stones. We denote by $t\mathcal{F}$ the smallest additive and completely multiplicative collection which contains $\mathcal{F}$. The elements of $t\mathcal{F}$ are called closed sets (more precisely, $\mathcal{F}$-closed sets), the elements of $\gamma t\mathcal{F}$ are called open sets.

1.1. Filters. An $\mathcal{F}$-filter is a finitely multiplicative subset Φ of $\mathcal{F}-(\emptyset)$ such that $F \supset F' \in \Phi$, $F \in \mathcal{F}$ implies $F \in \Phi$. A maximal $\mathcal{F}$-filter is a filter which is a proper subset of no $\mathcal{F}$-filter. Clearly an $\mathcal{F}$-filter Φ is maximal if and only if each stone which meets each element of Φ belongs to Φ. If $\mathcal{F} = \gamma\mathcal{F}$ then Φ is maximal if and only if for each F in $\mathcal{F}$ either $F \in \Phi$ or $X - F \in \Phi$.

1.2. <u>Definition</u>. An $\mathcal{F}$-filter Φ is called prime if the mutually equivalent conditions in the following proposition are fulfilled.

<u>Proposition.</u> The following conditions on an $\mathcal{F}$-filter Φ are mutually equivalent:

a) If $F_1 \cup F_2 \in \Phi$, $F_i \in \mathcal{F}$ then $F_1 \in \Phi$ or $F_2 \in \Phi$.

b) The collection $\widetilde{\Phi} = \{X - F \mid F \in \mathcal{F} - \Phi\} = \gamma(\mathcal{F} - \Phi)$ is a filter in $\gamma\mathcal{F}$.

c) $\Phi \cup \widetilde{\Phi}$ is a maximal centred collection in $\mathcal{F} \cup \gamma\mathcal{F}$.

d) There is a maximal centred collection Ψ in $\mathcal{F} \cup \gamma\mathcal{F}$ such that $\Phi = \Psi \cap \mathcal{F}$.

e) There is an ultrafilter Ψ' on X with $\Phi = \Psi' \cap \mathcal{F}$.

<u>Proof</u>. I. Conditions a) and b) are equivalent because if F is the union of a finite family $\{F_i\}$ in $\mathcal{F} - \Phi$ then

$$X - F = \bigcap\{X - F_i\},$$

and hence condition a) holds if and only if $\widetilde{\Phi}$ is finitely multiplicative.

II. We will check that conditions a) and b) imply condition c), and that condition c) implies condition d). Put $\Psi = \Phi \cup \widetilde{\Phi}$. If $Y \in \Psi$ then either $Y \in \Phi$ or $X - Y \in \Phi$, and hence Ψ is a maximal centred collection in $\mathcal{F} \cup \gamma\mathcal{F}$ if Ψ is centred. Assume conditions a) and b), and prove that Ψ is centred. It is enough to show that if $F \in \Phi$, $U \in \widetilde{\Phi}$, then $F \cap U$ is non-void. If $F \cap U = \emptyset$ then $F \subset X - U$, and hence $X - U \in \Phi$, and hence $U \notin \widetilde{\Phi}$, which contradicts our assumption $U \in \widetilde{\Phi}$. This proves that a),b) imply c). To prove that condition c) implies condition d) it suffices to show that

$$\Psi \cap \mathcal{F} \subset \Phi .$$

If $F \in \tilde{\Phi} \cap \mathcal{F} - \Phi$ then both F and $X - F$ belong to $\tilde{\Phi} \subset \Psi$ which contradicts our assumption that Ψ is centred.

III. Assume d), and let Ψ' be any ultrafilter on X such that $\Psi \subset \Psi'$. Clearly, $\Psi' \cap (\mathcal{F} \cup \gamma \mathcal{F}) = \Psi$ (because if Ψ is a maximal centred collection in $\mathcal{N}$ with $\gamma \mathcal{N} = \mathcal{N}$ then each ultrafilter $\Psi' \supset \Psi$ meets $\mathcal{N}$ in Ψ). Hence e) holds.

IV. Assume condition e) and prove condition a). If $F_1 \cup$ $\cup F_2 \in \Phi$ with $F_i \in \mathcal{F}$ then $F_1 \cup F_2 \in \Psi'$ and hence $F_1 \in \Psi'$ or $F_2 \in \Psi'$; since $\Psi' \cap \mathcal{F} = \Phi$, we conclude that either $F_1 \in \Phi$ or $F_2 \in \Phi$.

<u>Remark</u>. In what follows we shall freely use Proposition, and the following relation

$$\tilde{\tilde{\Phi}} = \Phi$$

which holds for every prime filter Φ .

<u>Corollary</u>. Every maximal filter is prime, and the converse holds if the stones and the co-stones coincide. Prime filters are just the traces of ultrafilters. An $\mathcal{F}$-filter Φ is prime if and only if $\tilde{\Phi}$ is a prime $\gamma\mathcal{F}$-filter. If Φ is a prime filter in $\mathcal{F}$, and if $\mathcal{F}' \subset \mathcal{F}$ then $\Phi \cap \mathcal{F}'$ is a prime filter in $\mathcal{F}'$.

A prime filter need not be maximal. For example, usually comaximal filters (see 1.3) are not maximal.

1.3. <u>Definition</u>. An $\mathcal{F}$-filter Φ is called comaximal if $\tilde{\Phi}$ is a maximal $\gamma\mathcal{F}$-filter.

By Corollary 1.2 every comaximal filter is prime.

<u>Proposition.</u> An $\mathcal{F}$-filter Φ is comaximal if and only if

$$\Phi = E\{F \mid F \in \mathcal{F},\ F \supset \mathcal{U} \in \tilde{\Phi} \text{ for some } \mathcal{U}\} .$$

Proof. Assume that $\tilde{\Phi}$ is maximal. If $F \in \Phi$ then $X - F \notin \tilde{\Phi}$, and since $\tilde{\Phi}$ is maximal, $\mathcal{U} \cap (X - F) = \emptyset$ for some $\mathcal{U}$ in $\tilde{\Phi}$; hence $\mathcal{U} \subset F$. This proves the inclusion $\subset$. The inverse inclusion is always true. Conversely assume that Φ is given by the formula. Then $\tilde{\Phi}$ is a filter (and hence both Φ and $\tilde{\Phi}$ are prime by 1). If $\tilde{\Phi}$ were not maximal then there would exist an element $\mathcal{U}$ in $\gamma\mathcal{F} - \tilde{\Phi}$ which would meet each element of $\tilde{\Phi}$; then $F = X - \mathcal{U} \in \mathcal{F}$ and $F \supset \mathcal{U}$ for no $\mathcal{U}$ in $\tilde{\Phi}$.

The result in Proposition is restated as follows. The meaning of the term basis is clear (a basis need not be a part of the filter).

Theorem. An $\mathcal{F}$-filter Φ is maximal if and only if Φ is a basis for $\tilde{\Phi}$, a filter Φ is comaximal if and only if $\tilde{\Phi}$ is a basis for Φ.

Remark. The following conditions on $\mathcal{F}$ are mutually equivalent:

a) Each prime $\mathcal{F}$-filter is maximal.

b) If Φ is prime, and if $F \in \mathcal{F} - \Phi$ then $F' \subset X - F$ for some F' in Φ.

c) If Ψ is an ultrafilter on X and $F \in \mathcal{F} - \Psi$ then $F' \subset X - F$ for some F' in $\Psi \cap \mathcal{F}$.

d) If Φ_1 and Φ_2 are two distinct prime filters in $\mathcal{F}$, then there exist $F_i \in \Phi_i$ such that $F_1 \cap F_2 = \emptyset$.

e) If Φ_1 and Φ_2 are prime, and if $\Phi_1 \subset \Phi_2$ then $\Phi_1 = \Phi_2$.

f) Each prime $\gamma\mathcal{F}$-filter is maximal.

Examples. 1. Let $\mathcal{F}$ be the collection of all closed sets in a topological space X. The following conditions are equivalent:

a) Each closed prime filter is maximal.

b) Each open prime filter is maximal.

c) The interior of each non-void closed set is non-void.

d) X is a topological sum of indiscrete spaces.

The proof is routine, and will be left to the reader.

2. Assume that Φ is a prime closed filter in a topological space X. Then Φ is maximal and comaximal only if for each F in Φ there exists an F' in Φ such that $F' \subset \mathit{int}\, F$ (use Theorem).

1.4. Separation axioms

Proposition A. The following three conditions on $\mathcal{F}$ are equivalent:

a) If $F_1 \in \mathcal{F}$, $F_2 \in \mathcal{F}$, $F_1 \cap F_2 = \emptyset$ then there exist $\mathcal{U}_i$ in $\gamma\mathcal{F}$ such that $F_i \subset \mathcal{U}_i$ and $\mathcal{U}_1 \cap \mathcal{U}_2 = \emptyset$.

b) If Φ is a prime $\mathcal{F}$-filter, and if Φ_1 and Φ_2 are maximal $\mathcal{F}$-filters with $\Phi \subset \Phi_1 \cap \Phi_2$ then $\Phi_1 = \Phi_2$.

c) If Φ is a comaximal $\mathcal{F}$-filter with $\Phi \subset \Phi_1 \cap \Phi_2$ where Φ_i are maximal then $\Phi_1 = \Phi_2$.

A paving $\mathcal{F}$ which satisfies these three conditions is called normal.

Proof. Assume a), and let Φ_1, Φ_2 be maximal $\mathcal{F}$-filters, and let $\Phi \subset \Phi_1 \cap \Phi_2$ be prime. If $\Phi_1 \neq \Phi_2$ then there are $F_i \in \Phi_i$ with $F_1 \cap F_2 = \emptyset$, and by Condition a) we can choose $\mathcal{U}_i \in \gamma\mathcal{F}$ with $F_i \subset \mathcal{U}_i$, $\mathcal{U}_1 \cap \mathcal{U}_2 = \emptyset$; put $F_i' = X - \mathcal{U}_i$. Clearly $F_i' \notin \Phi_i$, and hence $F_i' \notin \Phi$, $i = 1, 2$. On the other hand, since Φ is prime and $F_1' \cup F_2' = X \in \Phi$ we have either $F_1' \in \Phi$ or $F_2' \in \in \Phi$ and this is a contradiction. Clearly b) implies c).

Assume Condition c) and let $F_i \in \mathcal{F}$ with $F_1 \cap F_2 = \emptyset$. Let Ψ_i be the set of all $\mathcal{U} \in \gamma\mathcal{F}$ with $F_i \subset \mathcal{U}$. If $\mathcal{U}_1 \cap \mathcal{U}_2 \neq \emptyset$ for each $\mathcal{U}_i$ in Ψ_i then we can take a maximal $\gamma\mathcal{F}$-filter $\Psi \supset \Psi_1 \cup \Psi_2$, and the comaximal $\mathcal{F}$-filter $\widetilde{\Psi}$ has at least two distinct extensions. Indeed, F_i meets each element of $\widetilde{\Psi}$ and hence there exists a maximal $\mathcal{F}$-filter $\Phi_i \supset \widetilde{\Psi} \cup (F_i)$. Since $F_1 \cap F_2 = \emptyset$, necessarily $\Phi_1 \neq \Phi_2$

Examples. a) If $\mathcal{F}$ = closed X where X is a topological space then $\mathcal{F}$ is normal if and only if X is a normal topological space.

b) If $\mathcal{F}$ = open X where X is a topological space then $\mathcal{F}$ is normal if and only if X is extremally disconnected. (See 2.6.)

c) If $\mathcal{F} = zero\ X$ (= the collection of all zero sets in X) then $\mathcal{F}$ is normal.

d) If $\mathcal{F} = cozero\ X$ then $\mathcal{F}$ is normal if and only if X is an F-space.

Another characterization of normality is given in 2.6.

Proposition B. The following conditions on $\mathcal{F}$ are equivalent:

a) If Φ is a prime (comaximal) $\mathcal{F}$-filter then $\bigcap\Phi$ is empty or a singleton.

b) If $x_1 \in X$, $x_2 \in X$, $x_1 \neq x_2$ then $\mathcal{U}_1 \cap \mathcal{U}_2 = \emptyset$ for some $\mathcal{U}_i \in \gamma\mathcal{F}$ with $x_i \in \mathcal{U}_i$.

c) If $x_1 \in X$, $x_2 \in X$, $x_1 \neq x_2$ then $X = F_1 \cup F_2$ for some $F_i \in \mathcal{F}$, $x_i \notin F_i$.

A paving $\mathcal{F}$ which satisfies these three equivalent conditions is called separated.

Proof. Obviously the conditions b) and c) are equivalent, and c) implies a). Assume there are $x_1 \neq x_2$ such that if Ψ_i is the collection of all $\mathcal{U} \ni x_i$ then there exists a maximal $\gamma\mathcal{F}$-filter $\Psi \supset \Psi_1 \cup \Psi_2$. Clearly $x_i \in \bigcap \Phi$ where Φ is the comaximal $\mathcal{F}$-filter $\widetilde{\Psi}$. This concludes the proof.

If X is a topological space then X is Hausdorff if and only if closed X is separated. Similarly, X is regular if and only if closed X is regular in the sense of the following definition.

Proposition C. The following conditions on $\mathcal{F}$ are equivalent:

a) If $F \in \mathcal{F}$ and $x \in X - F$ then there exist $\mathcal{U} \in \gamma\mathcal{F}$ and $V \in \gamma\mathcal{F}$ such that $x \in \mathcal{U}$, $F \subset V$, $\mathcal{U} \cap V = \emptyset$.

b) If Φ is a prime $\mathcal{F}$-filter, and if Ψ is a comaximal filter contained in Φ , then $\bigcap \Phi = \bigcap \Psi$.

c) If Φ_1 and Φ_2 are prime $\mathcal{F}$-filters, and $\Phi_1 \subset \Phi_2$ then $\bigcap \Phi_1 = \bigcap \Phi_2$.

A paving $\mathcal{F}$ is called regular if it satisfies these conditions.

Proof. Since every prime filter contains a comaximal filter (which is prime), the two conditions c) and b) are equivalent.

Assume that Condition a) holds, and $\Phi_1 \subset \Phi_2$ are prime filters. If $x \in \bigcap \Phi_1 - \bigcap \Phi_2$ then there exists an F in $\Phi_2 - \Phi_1$ with $x \notin F$. If $\mathcal{U}$ and V are as in Condition a), then $(X - \mathcal{U}) \cup (X - V) = X$, and hence

$X-\mathcal{U} \in \Phi_1$,and hence $x \in X - \mathcal{U}$; this contradicts our assumption $x \subset \mathcal{U}$.

Now assume that condition a) does not hold, and choose $F \in \mathcal{F}$ and $x \in X - F$ such that x and F are not separated by sets in $\gamma\mathcal{F}$. Consider a maximal $\gamma\mathcal{F}$-filter Ψ which contains all $\mathcal{U} \cap V$ where $x \in \mathcal{U}$, $F \subset V$, $\mathcal{U} \in \gamma\mathcal{F}$, $V \in \gamma\mathcal{F}$. If $C \in \widetilde{\Psi}$ then $x \in C$, and hence $x \in \bigcap\widetilde{\Psi}$. Each element of $\widetilde{\Psi}$ meets F , and hence we can take a maximal $\mathcal{F}$-filter $\Phi \supset \widetilde{\Psi}$ which contains F. Clearly $x \notin \bigcap\Phi$.

It seems to me that condition c) explains why regularity enters so many theorems as an assumption.

1.5. <u>Compact and almost compact pavings</u>. A paving $\mathcal{F}$ is called compact (almost compact) if the intersection of each $\mathcal{F}$-filter (comaximal $\mathcal{F}$-filter) is non-void.

Each $\mathcal{F}$-filter is contained in a maximal $\mathcal{F}$-filter and hence, compact implies almost compact, and each of the following conditions is necessary and sufficient for $\mathcal{F}$ to be compact:

a) If Φ is prime $\mathcal{F}$-filter, then $\bigcap\Phi \neq \emptyset$.

b) If Φ is maximal $\mathcal{F}$-filter then $\bigcap\Phi \neq \emptyset$.

<u>Proposition.</u> If $\mathcal{F}$ is regular (see 1.4) and almost compact, then $\mathcal{F}$ is compact.

<u>Proof</u>. Let Φ be a prime $\mathcal{F}$-filter, and let Φ_1 be a comaximal filter contained in Φ (let $\Psi \supset \widetilde{\Phi}$ be a maximal $\gamma\mathcal{F}$-filter and put $\Phi_1 = \widetilde{\Psi}$). By regularity $\bigcap\Phi = \bigcap\Phi_1$.

<u>Theorem</u>. The following properties of a compact paving $\mathcal{F}$ are equivalent:

a) $\mathcal{F}$ is separated.

b) $\mathcal{F}$ is regular, and if $x_1 \neq x_2$ then $x_1 \in F$, $x_2 \notin F$ for some $F \in \mathcal{F}$.

c) $\mathcal{F}$ is separated and normal.

Remark. It is easy to see that $\mathcal{F}$ is compact if and only if $t\mathcal{F}$ is compact. For more results related to compactness and t we refer to Frolík [1], see also J. de Groot [1].

2. Filters with CIP

A collection $\mathfrak{M}$ of sets has the countable intersection property (abb. CIP) if the intersection of any countable subset of $\mathfrak{M}$ is non-void.

2.1. Proposition. The following conditions on a prime $\mathcal{F}$-filter Φ are equivalent:

(a) Φ has CIP.

(b) If $\mathcal{U} \subset \gamma\mathcal{F}$ is a countable cover of X, then $\tilde{\Phi} \cap \mathcal{U} \neq \emptyset$.

Proof. If $F_i \searrow \emptyset$, $F_i \in \Phi$ then $\mathcal{U} = \{X - F_i\}$ is a countable cover of X, and $\mathcal{U} \cap \tilde{\Phi} = \emptyset$. If $\mathcal{U} \subset \gamma\mathcal{F}$ is a countable cover of X with $\mathcal{U} \cap \tilde{\Phi} = \emptyset$ then $\{X - U \mid U \in \mathcal{U}\}$ is a countable subset of Φ with empty intersection

Definition. If $\mathfrak{M} \subset \exp X$ then an $\mathcal{F}$-filter Φ is called $\mathfrak{M}$-Cauchy if for each countable cover $\mathfrak{X} \subset \mathfrak{M}$ of X, some element of $\mathfrak{X}$ contains an element of $\mathcal{F}$.

Theorem. Let Φ be a prime $\mathcal{F}$-filter. Then Φ has CIP if and only if $\tilde{\Phi}$ is $\gamma\mathcal{F}$-Cauchy, Φ is $\mathcal{F}$-Cauchy iff $\tilde{\Phi}$ has CIP, and both Φ and $\tilde{\Phi}$ have CIP iff Φ is $\mathcal{F}$-Cauchy and $\tilde{\Phi}$ is $\gamma\mathcal{F}$-Cauchy.

Proof. From Proposition.

Corollary. If Φ is a maximal $\mathcal{F}$-filter, then the following conditions are equivalent:

(a) Φ has CIP;

(b) $\tilde{\Phi}$ is $\gamma\mathcal{F}$-Cauchy;

(c) Φ is $\gamma\mathcal{F}$-Cauchy.

Proof. Use 1.2.

Remark. Assume that for each sequence $\{F_n\}$ in $\mathcal{F}$ with $F_n \searrow \emptyset$ there exists a sequence $\{\mathcal{U}_n\}$ in $\gamma\mathcal{F}$ such that $\mathcal{U}_n \searrow \emptyset$ and $\mathcal{U}_n \supset F_n$ for each n. A paving $\mathcal{F}$ with this property is called $\aleph_0$-paracompact. Then obviously, for a prime Φ, if $\tilde{\Phi}$ has CIP, then Φ has CIP. Hence, we can add two more equivalent conditions to the list in Corollary:

(d) $\tilde{\Phi}$ has CIP;

* (e) Φ is $\mathcal{F}$-Cauchy.

Example. If $\mathcal{F}$ are the closed sets in a topological space X, then the spaces with the property in Remark are called countably paracompact.

2.2. Proposition. Assume that $\mathcal{F} = \delta\mathcal{F}$. The following two conditions on a prime $\mathcal{F}$-filter Φ are equivalent:

(a) $\delta\Phi = \Phi$,

(b) if $\mathcal{U} \subset \gamma\mathcal{F}$ is countable, and if $\cup\mathcal{U} \in \tilde{\Phi}$ then $\mathcal{U} \cap \tilde{\Phi} \neq \emptyset$.

Proof. Let $F_n \in \mathcal{F}$, $F = \bigcap\{F_n\}$, $\mathcal{U}_n = X - F_n$ and $\mathcal{U} = \{\mathcal{U}_n\}$. Clearly $F \notin \Phi$ iff $\cup\mathcal{U} \in \tilde{\Phi}$. Now if $F \notin \Phi$ then $\cup\mathcal{U} \in \tilde{\Phi}$ but $\mathcal{U} \cap \tilde{\Phi} = \emptyset$. If $\cup\mathcal{U} \in \tilde{\Phi}$ but $\mathcal{U} \cap \tilde{\Phi} = \emptyset$ then $F \notin \Phi$ for F_n, F defined as above.

This result generalizes as follows:

Theorem. The following conditions on a prime $\mathcal{F}$-filter Φ

are equivalent:

(a) if $\{F_n\}$ is a sequence in Φ, then any $F \in \mathcal{F}$ with $F \supset \bigcap \{F_n\}$ belongs to Φ (i.e. if $F = \bigcap\{F_n\}$, $F_n \in \Phi$, $F \in \mathcal{F}$ then $F \in \Phi$).

(b) If $U \in \widetilde{\Phi}$ and if $U \subset \bigcup\{U_n\}$ where $\{U_n\}$ is a sequence in $\gamma\mathcal{F}$ then $U \cap U_n \in \widetilde{\Phi}$ for some n (i.e. if $\bigcup\{U_i\} \in \widetilde{\Phi}$ then $U_i \in \widetilde{\Phi}$ for some i).

A paving $\mathcal{F}$ is called $\aleph_0$-normal if $\mathcal{F}$ is normal and $\aleph_0$-paracompact.

<u>Lemma</u>. If $\mathcal{F}$ is $\aleph_0$-normal and a prime $\mathcal{F}$-filter Φ has CIP then each $\mathcal{F}$-filter $\Phi' \supset \Phi$ has CIP.

<u>Proof</u>. If $\mathcal{F}$ is $\aleph_0$-normal and $F_n \searrow \emptyset$ where $F_n \in \mathcal{F}$, then there exist $F'_n \in \mathcal{F}$ and $U_n \in \gamma\mathcal{F}$ such that $F_n \subset U_n \subset F'_n$, and $F'_n \searrow \emptyset$. Now if Φ' has not CIP, and if $\{F_n\}$ is a sequence in Φ' with $F_n \searrow \emptyset$, and if $\{F'_n\}$ is a sequence as above, then $F'_n \in \Phi$ (since Φ is prime), and hence Φ has not CIP.

2.3. <u>Completeness.</u>

<u>Definition</u>. A paving $\mathcal{F}$ is called Lindelöf if $\bigcap \Phi \neq \emptyset$ for each $\mathcal{F}$-filter Φ with CIP. A paving $\mathcal{F}$ is called maximal-complete (or simply complete) or prime-complete or comaximal-complete if $\bigcap \Phi \neq \emptyset$ for each $\mathcal{F}$-filter Φ with CIP which is, accordingly, maximal or prime or comaximal. (Compare with 1.5.) A paving $\mathcal{F}$ is called countably compact if each $\mathcal{F}$-filter has CIP.

<u>Theorem</u>. Compact implies Lindelöf, Lindelöf implies prime-complete, prime-complete implies both comaximal-complete, and maximal-complete (this is obvious). For a regular $\mathcal{F}$,

prime-complete and comaximal-complete are equivalent (1.4), and hence comaximal-complete implies maximal-complete. For $\aleph_0$ -normal $\mathcal{F}$ (Lemma 2.2), maximal-complete implies prime-complete, and hence maximal-complete and prime-complete are equivalent. For countably compact $\mathcal{F}$, all properties mentioned except for comaximal-complete are equivalent, and comaximal-complete is equivalent to almost compact (here countable compactness may be weakened to almost countably compact (defined in an obvious way x)), and in addition, a paving $\mathcal{F}$ is almost compact if and only if it is comaximal-complete and almost countably compact .

2.4. Examples of $\aleph_0$ -normal pavings.

A paving $\mathcal{F}$ is called perfect if $\gamma\mathcal{F} \subset \sigma\mathcal{F}$ (equivalently, if $\mathcal{F} \subset \delta\gamma\mathcal{F}$).

Proposition. Every perfect paving is $\aleph_0$ -paracompact. Every normal perfect (= perfectly normal) paving is $\aleph_0$ -normal.

Proof. Assume $\mathcal{F}$ is perfect and $F_n \searrow \emptyset$. Write $F_n = \bigcap\{G_n^k \mid k \in N\}$, and put $G_n = \bigcap\{G_k^\ell \mid k \leqq n,\ \ell \leqq n\}$. Clearly $G_n \supset F_n$ and $G_n \searrow \emptyset$.

Example. If X is a topological space then $zero\, X$ (the paving which consists of all zero-sets in X) is $\aleph_0$ -normal and regular. Hence (2.3) the following conditions on $\mathcal{F} = zero\, X$ are equivalent:

a) $\mathcal{F}$ is maximal-complete;

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x) If $\{U_n\}$ is a centred sequence in $\gamma\mathcal{F}$, then $\bigcap\{F \mid F \in \mathcal{F},\ F \supset U_n$ for some $n\} \neq \emptyset$

b) $\mathcal{F}$ is prime-complete;

c) $\mathcal{F}$ is comaximal-complete.

<u>Definition</u>. A paving $\mathcal{F}$ is called normally perfect if for each element U of $\gamma\mathcal{F}$ there exist sequences $\{U_n\}$ in $\gamma\mathcal{F}$ and $\{F_n\}$ in $\mathcal{F}$ such that $\cup F_n = U$ and $F_n \subset U_{n+1} \subset F_{n+1}$. A paving $\mathcal{F}$ is called normally $\aleph_0$-paracompact, if for each $F_n \searrow \emptyset$, $F_n \in \mathcal{F}$ there exist $\{U_n\}$ in $\gamma\mathcal{F}$ and $\{F'_n\}$ in $\mathcal{F}$ such that $F_n \subset U_n \subset F'_n$ and $F'_n \searrow \emptyset$.

<u>Theorem A</u>. The category of normally perfect paved spaces is reflective in the category of all paved spaces. Every normally perfect space is normally $\aleph_0$-paracompact. Every $\aleph_0$-normal space is normally $\aleph_0$-paracompact.

The proof is left to the reader.

<u>Theorem B</u>. Let $\mathcal{F}$ be normally $\aleph_0$-paracompact. If Φ is a prime $\mathcal{F}$-filter with CIP then each $\mathcal{F}$-filter $\Phi_1 \supset \Phi$ has CIP.

<u>Proof</u>. If $F_n \in \Phi$ and $F_n \searrow \emptyset$ then there exist $U_n \in \gamma\mathcal{F}$ and $F'_n \in \mathcal{F}$ such that $F'_n \searrow \emptyset$ and $F_n \subset U_n \subset F'_n$. Since $F_n \subset U \subset F'_n$, necessarily $F'_n \in \Phi$.

<u>Corollary</u>. If $\mathcal{F}$ is normally $\aleph_0$-paracompact and regular then Conditions a), b) and c) in Example are mutually equivalent.

<u>Remark</u>. We have considered three kinds of completeness properties. There are three more of certain interest: $\cap\Phi \neq \emptyset$ if

a) $\tilde{\Phi}$ has CIP;

b) Φ is maximal and $\tilde{\Phi}$ has CIP;

c) Φ is comaximal and $\tilde{\Phi}$ has CIP;

The statements in a),b) and c) may be restated as follows:

a$'$) Φ is prime and $\mathcal{F}$-Cauchy;

b$'$) Φ is maximal and $\mathcal{F}$-Cauchy;

c$'$) Φ is comaximal and $\mathcal{F}$-Cauchy.

The first two properties have been studied in the case when $\mathcal{F}$ is closed X where X is a topological space, e.g. Frolík [1] and [2]. The third property is not interesting in the topological space if the cardinal of X is non-countable because:

if maximal open filter Ψ in X has CIP and $\bigcap \widetilde{\Psi} = \emptyset$ then the cardinal of X is measurable.

Indeed, consider a maximal disjoint family $\{U_a \mid a \in A\}$ in open $X - \Psi$. The union U of $\{U_a\}$ is dense in X, and hence $U \in \Psi$. Let $\mathcal{A}$ be the filter in $\exp A$ which consists of all $M \subset A$ such that the union of $\{U_a \mid a \in M\}$ is in Ψ. Clearly $\mathcal{A}$ is a maximal filter with CII in $\exp A$ and $\bigcap \mathcal{A} = \emptyset$. Thus A is of a measurable cardinal, and hence X is of a measurable cardinal.

The first two completeness properties will be studied in subsequent paper on the space of prime filters.

2.5. <u>Invariance of completeness under mappings</u>. Let $\mathcal{F}$ and $\mathcal{Z} \subset \mathcal{F}$ be pavings of X. Then $\mathcal{F}$ is called $\mathcal{Z}$-normal if each two disjoint $\mathcal{F}$-stones are separated by two disjoint $\mathcal{Z}$-costones. Similarly, $\mathcal{F}$ is $\mathcal{Z}-\aleph_0$-paracompact if for each $F_n \searrow \emptyset$, $F_n \in \mathcal{F}$, there exists $\{G_n\}$ in $\gamma\mathcal{Z}$ such that $G_n \supset Z_n$ and $G_n \searrow \emptyset$. Finally $\mathcal{F}$ is $\mathcal{Z}-\aleph_0$-normal if it is $\mathcal{Z}$-normal and $\mathcal{Z}-\aleph_0$-paracompact.

<u>Remark</u>. Two pavings $\mathcal{F}$ and $\mathcal{Z}$ are called topologically

equivalent if $t\mathcal{F} = t$. If $\mathcal{Z} \subset \mathcal{F}$ then $\mathcal{F}$ and $\mathcal{Z}$ are topologically equivalent if and only if

$$F = \bigcap \{ Z \mid Z \in \mathcal{Z} , Z \supset F \}$$

for each F in $\mathcal{F}$.

Theorem A. Let $\mathcal{F}$ and $\mathcal{Z} \subset \mathcal{F}$ be two topologically equivalent pavings of X. If $\mathcal{Z}$ is prime-complete then so is $\mathcal{F}$. If $\mathcal{F}$ is prime-complete and $\mathcal{Z}$ – $\aleph_0$ -normal then $\mathcal{Z}$ is prime-complete.

Proof. Assume that $\mathcal{Z}$ is prime-complete, and let Φ be a prime $\mathcal{F}$-filter with CIP. Since $\mathcal{Z} \subset \mathcal{F}$, $\Psi = \mathcal{Z} \cap \Phi$ is a prime $\mathcal{Z}$-filter which has the CIP. Hence $\bigcap \Psi \neq \emptyset$. Since $\mathcal{F}$ and $\mathcal{Z}$ are topologically equivalent and $\mathcal{Z} \subset \mathcal{F}$, $\bigcap \Phi = \bigcap \Psi$.

Now assume $\mathcal{F}$ is prime-complete and $\mathcal{Z}$ – $\aleph_0$ -normal. Let Ψ be a prime $\mathcal{Z}$-filter with CIP and let Φ be a prime $\mathcal{F}$-filter with $\Phi \cap \mathcal{Z} = \Psi$. As in the first part $\bigcap \Psi = \bigcap \Phi$, and it is easy to check that Φ has CIP. If $F_n \searrow \emptyset$, $F_n \in \Phi$, then there exists $Z_n \in \mathcal{Z}$ with $Z_n \searrow \emptyset$ and $Z_n \supset F_n$. Clearly $Z_n \in \Psi$. This contradiction concludes the proof.

The second theorem concerns the invariance of completeness properties under mappings. It gives the natural setting for the author's theorems on invariance of almost realcompact spaces, and in the light of the present theorem the old theorems do not seem pure from the conceptual point of view.

Theorem B. Let f be a perfect map of a paved space $\mathfrak{X} = \langle X, \mathcal{F} \rangle$ onto a paved space $\mathfrak{X}' = \langle X', \mathcal{F}' \rangle$ (that means, $f[\mathcal{F}] \subset \mathcal{F}'$, $f^{-1}[\mathcal{F}'] \subset \mathcal{F}$, and all sets $f^{-1}[(y)]$, $y \in X'$, are $\mathcal{F}$-compact).

Then $\mathcal{F}$ is prime-complete if and only if $\mathcal{F}'$ is, $\mathcal{F}$ is maximal-complete if and only if $\mathcal{F}'$ is (and of course, $\mathcal{F}$ is compact if and only if $\mathcal{F}'$ is).

Proof. a) Assume that Φ is an $\mathcal{F}$-filter and Φ' is an $\mathcal{F}'$-filter such that $f[\Phi] = \Phi'$. Then

$$f[\bigcap \Phi] = \bigcap \Phi' .$$

The inclusion $\subset$ is always true. Assume $y \in \cap \Phi'$, and consider the $\mathcal{F}$-compact set $K = f^{-1}[(y)]$. Clearly $K \cap \cap [\Phi]$ is centred (= has the finite intersection property), and hence $K \cap \bigcap \Phi \neq \emptyset$. Take some x in this intersection; clearly $fx = y$.

It follows that $\mathcal{F}$ is compact if and only if $\mathcal{F}'$ is compact. Notice that Φ has CIP if and only if Φ' has CIP.

b) Now let Φ be an $\mathcal{F}$-filter, and put $\Phi' = f[\Phi]$. Clearly $f^{-1}[\Phi'] \subset \Phi$, and hence Φ and Φ' satisfy the assumptions in part a). If Φ is prime or maximal then so is Φ'. Indeed, if $F_1 \cup F_2 \in \Phi'$, then $f^{-1}[F_1]$ or $f^{-1}[F_2]$ belongs to Φ, and hence F_1 $(= f[f^{-1}[F_1]])$ or F_2 belongs to Φ'. If F meets each $F' \in \Phi'$, then $f^{-1}[F]$ meets each element of Φ and hence $F = f[f^{-1}[F]]$ belongs to Φ'. Finally, by part a), if Φ has CIP then Φ' has CIP. Again by the part a), it follows that if $\mathcal{F}'$ is prime-complete or maximal-complete then so is $\mathcal{F}$.

c) To prove that if $\mathcal{F}$ is prime-complete or maximal-complete then so is $\mathcal{F}'$, take a prime or maximal $\mathcal{F}'$-filter Φ' with CIP and an ultrafilter $\mathfrak{X}'$ in X' such that $\mathfrak{X}' \cap \mathcal{F}' = \Phi'$ (Proposition 1.2). If Φ' is prime, put $\Phi = \mathfrak{X} \cap \mathcal{F}$, where $\mathfrak{X}$ is an arbitrary ultrafilter in X containing $f^{-1}[\mathfrak{X}']$ and if Φ' is maximal put Φ to be a maximal $\mathcal{F}$-filter containing $f^{-1}[\Phi']$. In both cases $f[\Phi] = \Phi'$ and Φ is prime or maximal whenever Φ' is. Hence, by part a), if $\mathcal{F}$ is prime-complete or maximal complete then so is $\mathcal{F}'$. This concludes the proof.

The reasonings of parts b) and c) of the proof of Theorem B are of certain interest in itself, and therefore the results are formulated in the following proposition. If $\mathfrak{X} = \langle X, \mathcal{F} \rangle$ is a paved space we denote by $p\mathfrak{X}$ the set of prime $\mathcal{F}$-filters, by $m\mathfrak{X}$ the set of maximal $\mathcal{F}$-filters, and by $cm\mathfrak{X}$ the set of all comaximal $\mathcal{F}$-filters.

Proposition. Let f be a perfect map of $\mathfrak{X}$ onto $\mathfrak{X}'$. For each $\mathcal{F}$-filter Φ let $pf\Phi$ be the $\mathcal{F}'$-filter which consists of all $f[F]$, $F \in \Phi$. Then $pf : p\mathfrak{X} \to p\mathfrak{X}'$ is onto, $pf[m\mathfrak{X}] = m\mathfrak{X}'$. In addition, $f[\cap \Phi] = \cap\, pf\Phi$, and Φ has the m-intersection property (in particular, CIP) if and only if $pf\Phi$ has the m-intersection property.

Remark. In the set $p\mathfrak{X}$ there is a natural paving which consists of all sets

$$F^* = \{\Phi \mid \Phi \in p\mathfrak{X}, F \in \Phi\}, \quad F \in \mathcal{F} .$$

This space will be studied in a subsequent paper. We just recall a construction of the projective resolution of a topological space in 2.7.

2.6. Normal pavings and extremally disconnected spaces.

Normal pavings are introduced in 1.4. Here we want to add one more characterization which will be used in connection with extremally disconnected spaces which are usually defined by the property that the closure of each open set is open, or equivalently, that the paving of all open sets is normal in the sense of our definition.

If $\mathfrak{M}$ is a filter of subsets of a set X, denote by $\mathrm{filter}_X(\mathfrak{M})$, X is usually omitted, the filter on X which has $\mathfrak{M}$ for its basis.

Proposition. A paving $\mathcal{F}$ of X is normal if and only if for each comaximal $\gamma\mathcal{F}$-filter Ψ

$$\mathit{filter}_X((\mathit{filter}_X \Psi) \cap \mathcal{F}) \cap \gamma\mathcal{F} = \Psi ,$$

i.e. for each $U \in \Psi$ there exist a $V \in \Psi$ and an $F \in \mathcal{F}$ such that $\mathfrak{U} \supset F \supset V$.

Proof. Assume that $\mathcal{F}$ is normal and $\mathfrak{U} \in \Psi$. Since Ψ is comaximal, there exists an F' in $\tilde{\Psi}$ with $F' \subset \mathfrak{U}$. Since $\mathcal{F}$ is normal there exist disjoint V and W in $\gamma\mathcal{F}$ such that $F' \subset V$, $W \supset X - \mathfrak{U}$. Thus $F = X - W$ is in $\mathcal{F}$, $V \in \Psi$, $V \subset F \subset \mathfrak{U}$.

Conversely, assume the condition, take a prime $\mathcal{F}$-filter Φ, consider maximal $\mathcal{F}$-filters Φ_1, Φ_2 such that $\Phi \subset \Phi_1 \cap \Phi_2$; we must prove that $\Phi_1 = \Phi_2$. If $\Phi_1 \neq \Phi_2$, then $F_1 \cap F_2 = \emptyset$ for some $F_i \in \Phi_i$. Thus $X - F_1 = \mathfrak{U} \supset F_2$, and hence $\mathfrak{U} \in \tilde{\Phi}_2$. By the condition applied to Φ_2, we get $\mathfrak{U} \supset F \supset V \supset F'$ for some $\mathfrak{U}, V \in \tilde{\Phi}_2$, F, $F' \in \Phi_2$. Thus

$$F_1 \subset X - F,\ F' \subset V \quad (X - F) \cap V = \emptyset .$$

Thus $F_1 \in \Phi_1$ and $F' \in \Phi_2$ are separated by costones, and this contradicts our assumption that $\Phi_1 \cap \Phi_2$ contains a prime filter.

Corollary. Assume that X is a topological space. Then X is extremally disconnected if and only if for each comaximal closed filter Φ open elements of Φ (i.e. closed-open elements of Φ) form a basis for Φ , i.e. if for each comaximal closed filter Φ the maximal $\mathcal{F} \cap \gamma \mathcal{F}$ -filter $\Phi \cap \mathcal{F} \cap \gamma \mathcal{F}$ is a basis for Φ , where $\mathcal{F} =$ closed $\mathcal{X}$.

Proof. If open $\mathcal{X}$ is normal, thus by Proposition, if Φ is a maximal closed filter then for each $F \in \Phi$ there is an open V and an $F' \in \Phi$ such that $F \supset V \supset F'$; thus the closure of V is a closed-open set which belongs to Φ and is contained in F .

Conversely, if the condition is fulfilled, then the condition in Proposition is fulfilled, and by Proposition, open $\mathcal{X}$ is normal.

We shall need a little bit more:

Theorem. If $\mathcal{X}$ is an extremally disconnected space, then for each maximal clopen filter Γ the closed filter $\Phi_\Gamma = \{F \mid F \in$ closed $\mathcal{X}, F \supset \mathcal{U} \in \Gamma$ for some $\mathcal{U}\}$ is a comaximal closed filter, and the relation $\{\Gamma \longrightarrow \Phi_\Gamma\}$ is bijective.

Proof. It remains to show that Φ_Γ is prime. If $F_1 \cup \cup F_2 \supset \mathcal{U} \in \Gamma$, $F_i \in$ closed $\mathcal{X}$, then the interiors $\mathcal{U}'_i$ of F_i are clopen, and hence they should cover $\mathcal{U}$, and since Γ is prime, either $\mathcal{U}_1 \in \Gamma$ or $\mathcal{U}_2 \in \Gamma$, and hence $F_1 \in \Phi$ or $F_2 \in \Phi$.

2.7. Application to topological spaces.

Proposition. If $\mathcal{X}$ is a regular extremally disconnected space then the following three properties are equivalent:

a) Closed $\mathcal{X}$ is prime-complete (= $\aleph_1$ -ultracompact Slot [1]).

b) Closed X is comaximal-complete (= almost realcompact Frolík [1,2]).

c) X is realcompact (i.e. the following equivalent conditions hold: $\mathit{zero}\, X$ is complete, $\mathit{zero}\, X$ is prime-complete, and $\mathit{zero}\, X$ is comaximal-complete).

d) Clopen X is complete.

Proof. The equivalence of the conditions a) and b) follows from 2.2. The equivalence of d) and b) follows from 2.6. The equivalence of c) to other conditions follows from Theorem A in 2.5.

Theorem. The following properties of a regular topological space X are equivalent:

a) Closed X is prime-complete.

b) Closed X is comaximal-complete.

c) X is the perfect image of a realcompact space.

d) X is the perfect image of an extremally disconnected realcompact space.

It is enough to show that for every regular space X there exists a regular extremally disconnected space Y and a perfect mapping f of Y onto X. The proof of Iliadis (see Fomin-Iliadis [1]): we take $Y' = m$ (open (X)) with the topology of 2.5 and prove that Y' is a compact extremally disconnected space. Let Y be a subspace of Y' which consists of all $\mathcal{U} \in Y'$ such that $\bigcap \widetilde{\mathcal{U}} \neq \emptyset$. The sets $\bigcap \widetilde{\mathcal{U}}$ are singletons (since X is regular), denote the element of $\widetilde{\mathcal{U}}$ by $f\mathcal{U}$. The mapping $f = \{\mathcal{U} \rightarrow f\mathcal{U}\} : Y \rightarrow X$ is perfect).

Remark. The completeness is defined by means of filters with CIP. Almost all results are true for m-completeness where m is an infinite cardinal; complete is then $\aleph_1$-complete. In particular, all results of 2.5, 2.6 and 2.7 hold; the proofs in these paragraphs were formulated to emphasise this fact.

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