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CLUSTER SETS OF ARBITRARY FUNCTIONS IN EUCLIDEAN SPACES

(Preliminary communication)

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Let T be a topological space, $f: E_m \to T$ a mapping, $M \subset E_m$, $x \in E_m$. By the cluster set of f at x relative to M, we mean the set of all $y \in T$ for which $x \in (f^{-1}(V) \cap M)'$ for each neighbourhood V of y. The set of all $y \in T$ for which the set $M \cap f^{-1}(V)$ has a positive, upper exterior density at x - for each neighbourhood V of y - is the essential cluster set of f at x relative to M. These sets are denoted by $C(f, x, M)$ and $W(f, x, M)$, respectively.

If $\mathcal{U}$ is an open cone with vertex at the origin and $x \in E_m$, then we denote by $\mathcal{U}_x$ the image of $\mathcal{U}$ under the translation taking the origin into x. If $f: E_m \to T$ is a mapping, then we put

$$A(f) = \{x : C(f, x, E_m) \neq \bigcap \{C(f, x, \mathcal{U}_x) : \mathcal{U}\}\},$$

$$A_e(f) = \{x : W(f, x, E_m) \neq \bigcap \{W(f, x, \mathcal{U}_x) : \mathcal{U}\}\}.$$

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Let $x_1', x_2', \ldots, x_m'$ be a system of Cartesian coordinates in E_m $(m > 1)$, and let $f : E_{m-1} \to E_1$ be a Lipschitz function. Then the set of all points $x \in E_m$ such that the coordinates $x_1', x_2', \ldots, x_m'$ of the point x fulfil the equation $x_m' = f(x_1', \ldots, x_{m-1}')$, is called a Lipschitz surface. If a set $M \subset E_m$ is contained in the countable union of Lipschitz surfaces, then the set M is called a sparse set.

The open sphere of the center $x \in E_m$ and radius $r > 0$ is denoted by $K(x, r)$. A point $x \in E_m$ is termed a P-point of a set $M \subset E_m$, if there exists $\delta > 0$ such that for any $\varepsilon > 0$ there exist spheres $K(x, h), K(y, r)$ such that

$$K(y, r) \subset K(x, h) - M, \quad h \leq \varepsilon, \quad \delta < r/h \quad .$$

A set $M \subset E_m$ is termed a P-set, if an arbitrary point $x \in M$ is a P-point of the set M. A subset of E_m is termed a P_σ-set, if it is the union of a sequence of P-sets. An arbitrary P_σ-set is a set of the first category and of measure zero, but there exists a set of the first category and of measure zero which is not a P_σ-set. This assertion is stated in [3].

The following theorems hold.

Theorem 1. Let P be an infinite separable locally compact metric space and let $A \subset E_m$, $(m > 1)$. Then there exists a mapping $f : E_m \to P$ such that $A = A(f)$ iff the set A is a sparse set of type F_σ.

Theorem 2. Let P be a locally compact topological space having a countable basis of open sets. Let $f: E_m \to P$ be an arbitrary mapping. Then the set $A_e(f)$ is a P_σ-set of type $F_{\sigma\delta\sigma}$.

Theorem 3. Let P be a topological space having a countable basis of open sets and let $f: E_m \to P$ be an arbitrary mapping. Then the set of all points $x \in E_m$ for which

$$W(f, x, E_m) \neq \bigcap \{W(f, x, Z) : Z \text{ is a measurable set, } \underline{D}\, Z(x) > 0\}$$

is a set of the first category and of measure zero.

Theorem 4. Let T be a compact topological space having a countable basis of open sets. Let $f: E_1 \to T$ be an arbitrary mapping. Then the set $\{x : W(f, x, (x, \infty)) \cap \cap W(f, x, (-\infty, x)) = \emptyset\}$ is countable.

Theorem 5. Let T be a compact topological space having a countable basis of open sets. Let $f: E_m \to T$, $(m > 1)$ be an arbitrary mapping. Denote by D the set of all $x \in E_m$ for which there exist cones U, V in E_m such that $W(f, x, U_x) \cap W(f, x, V_x) = \emptyset$. Then D is a sparse set of type $F_{\sigma\delta\sigma}$.

Theorem 6. Let T be a compact topological space having a countable basis of open sets and let $f: E_2 \to T$ be an arbitrary mapping. Denote by D the set of all points $x \in E_2$ for which there exists an angle U with vertex at x less than π and Jordan arcs φ, ψ issuing from the point x such that

$$\varphi \subset \mathcal{U}\ ,\ \psi \subset \mathcal{U}\ ,\ C(f,x,\varphi) \cap C(f,x,\psi) = \emptyset\ .$$

Then the set D is a sparse set.

<u>Theorem 7</u>. Let $H \subset E_2$ be the open half-plane $\{y > 0\}$ and let P be a locally compact topological space having a countable basis of open sets. Denote by A the set of all points $x \in E_1$ such that there exists an angle $\mathcal{U} \subset H$ with vertex at the point $(x, 0)$ for which $W(f,x,H) \neq W(f,x,\mathcal{U})$. Then A is a P_σ-set of type $F_{\sigma\delta\sigma}$.

Theorem 2 and Theorem 3 improve the Hunter's theorem from [6] which asserts that for an arbitrary function $f: E_2 \longrightarrow E_1$ the set $A_e(f)$ is of the first category and of measure zero. Theorem 4 generalizes a theorem from [7]. Theorem 6 generalizes the Bagemihl's theorem on "crookedly ambiguous points of function" from [1]. Theorem 7 improves both the theorem from [2] which asserts that the set A is of the first category and the theorem from [4] which asserts that the set A is of measure zero.

These theorems can be proved by means of two general theorems on cluster sets which are analogous to the Hunter's theorem from [5]. These two theorems and the proofs of Theorems 1 - 7 will be published later on.

R e f e r e n c e s

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