Commentationes Mathematicae Universitatis Carolinae

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Commentationes Mathematicae Universitatis Carolinae, Vol. 13 (1972), No. 1, 137--147

Persistent URL: http://dml.cz/dmlcz/105401

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13,1 (1972)

BAIRE SETS AND UNIFORMITIES ON COMPLETE METRIC SPACES

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R.W. Hansel [H 1] proved a lemma, see § 2, which relates measurable discreteness to the topological discreteness in completely metrizable spaces. In this note we want to derive from this lemma several consequences, which all are deep results on completely metrizable spaces. In § 1 the main results are stated. The proofs are given in §'s 2 and 3 . In § 4 we prove two general results to show the relationship between algebras and uniformities. Most of the results have been stated in [F 2].

1. Main results

If X is a topological space we denote by $\mathrm{Ba}\, X$ the σ-algebra of all Baire sets in X as well as the corresponding measurable space. Recall that $\mathrm{Ba}\, X$ is the smallest σ-algebra which makes all continuous functions (real valued) on X measurable. We need to know that $\mathrm{Ba}\, X$ is the smallest collection $\mathcal{C}$ which contains all the zero sets and is closed under countable unions and countable intersections. Hence we can write $\mathrm{Ba}\, X = \bigcup \{\mathcal{B}_\alpha \mid \alpha < \omega\}$, where $\mathcal{B}_0$ is the collection of all zero sets in X ; and $\mathcal{B}_\alpha$ consists of all countable unions or intersections of

AMS, Primary: 54H05, 54E15 Ref.Ž. 3.969.5, 3.962
Secondary: 28A05

elements of $\bigcup \{ \mathcal{B}_\beta \mid \beta < \alpha \}$ according to as α is odd or even.

By an absolute Souslin space we mean a metrizable space X which is Souslin in each metrizable $Y \supset X$. In the proofs we assume that the reader is familiar with the basic properties of Souslin sets in a space or derived from a collection of sets. However, the following four theorems are non-trivial if the assumptions that some spaces are absolute Souslin are replaced by the stronger assumption that the spaces are completely metrizable.

Theorem 1. Every Baire isomorphism of two absolute Souslin spaces is a generalized homeomorphism. In other words, if $f: Ba\,X \longrightarrow Ba\,Y$ is an isomorphism, and if X, Y are absolute Souslin spaces, then there exist countable ordinals α, β such that if U is open in X and V is open in Y then $f[U]$ is of Baire class $\leqq \alpha$ in Y, and $f^{-1}[V]$ is of the Baire class $\leqq \beta$ in X.

Remark. The result is trivial if X and Y are separable (without any additional assumptions on X and Y). The validity of Theorem 1 has been a problem for a long time, see Kuratowski.

Theorem 1 is an immediate consequence of the following

Theorem 2. If f is a Baire measurable mapping of an absolute Souslin space X into a metrizable space Y then f is of certain Baire class $\alpha < \omega_1$ i.e. $f^{-1}[V]$ is of class $\leqq \alpha$ for each open V in Y.

By an ultrauniformity we mean a uniformity which has a base for the uniform covers which consists from disjoint covers (= partitions).

Theorem 3. Let X be an absolute Souslin space. Among all ultrauniformities which induce the proximity induced by all bounded Baire measurable functions, there exists the finest one.

The symbols in the next result will be explained in § 3.

Theorem 4. Let X be an absolute Souslin space. Then

$$H\,\mathrm{Ba}(X) = h\,\mathrm{Ba}(X) = \textit{bi-Souslin}\,\mathrm{Ba}(X) = \textit{bi-Souslin}(X).$$

2. Proofs of Theorems 1 and 2

Following F. Hansell [H 1] a family $\{X_a \mid a \in A\}$ in a topological space is called σ-discretely decomposable, abb. σ-d.d., if there exists a family $\{X_{am} \mid a \in A, m \in N\}$ such that $X_a = \cup\{X_{am} \mid m\}$ for each a and $\{X_{am} \mid a \in A\}$ is discrete in X for each m. Hence $\{X_a \mid a \in A\}$ is σ-discrete if and only if $\{X_a\}$ is σ-d.d. with $\{X_{am}\}$ such that each X_{an} is either X_a or the empty set. The following lemma due to Hansell [H 1] is the basic stone in the proofs of all results mentioned above.

Hansell lemma. Let X be a metrizable space, and let $\{X_a\}$ be a disjoint family of subsets of X such that the union of each subfamily of $\{X_a\}$ is an absolute Souslin set. Then $\{X_a\}$ is σ-d.d.

We need two more lemmas.

<u>Lemma. 1</u>. Let $\{X_a \mid a \in A\}$ be a topologically discrete collection of subsets of a space X, and let

$$\mathcal{B} = \cup \{ \mathcal{B}_\alpha \mid \alpha < \omega_1 \}$$

be defined as follows: $\mathcal{B}_0$ is the collection of closed sets in X, and $\mathcal{B}_\alpha$ is the collection of all countable unions or intersections of elements of $\cup \{ \mathcal{B}_\beta \mid \beta < \alpha \}$ according to as α is odd or even (includes limit ordinals). Thus $\mathcal{B}$ is the smallest collection which contains the closed sets and which is closed under countable unions and countable intersections.

Then the following conditions are equivalent.

A) The family $\{X_a\}$ ranges in some $\mathcal{B}_\alpha$.

B) The union of each subfamily of $\{X_a\}$ belongs to $\mathcal{B}$.

C) The union X' of $\{X_a\}$ belongs to $\mathcal{B}$.

<u>Proof</u>. It follows by the transfinite induction that A implies B. The implication B $\Longrightarrow$ C is self-evident. If the union X' of $\{X_a\}$ belongs to $\mathcal{B}_\alpha$ then $X_a = \mathrm{cl}\, X_a \cap X'$ belongs to $\mathcal{B}_\alpha$ by transfinite induction ($[\mathcal{B}_\alpha] \cap$ closed $= \mathcal{B}_\alpha$). This proves that C implies A.

<u>Lemma 2</u>. Let $\{X_a \mid a \in A\}$ be a disjoint family in an absolute Souslin space X. The following two conditions are equivalent:

A) The family $\{X_a\}$ is σ-d.d. and ranges in some $\mathcal{B}_\alpha$ (see Lemma 1).

B) The union of each subfamily of $\{X_a\}$ belongs to $\mathcal{B}$ (= Baire sets in X).

<u>Proof</u>. It follows easily from Lemma 1 that Condition A implies Condition B. Assume B. By Hansell's lemma we can write $X_a = \cup \{ X_{am} \}$ such that $\{X_{am} \mid a \}$ is discrete for each m. We may and shall assume that X_{am} is closed in X_a for each a and m. Let X'_m be the union of $\{ X_{am} \mid a \}$, and X' be the union of $\{X_a\}$. Let α be the class of X'. Since X'_m is closed in X', X'_m is of class $\leqq \alpha$ for each m, by Lemma 1 all X_{am} are of class $\leqq \alpha$, and hence each X_a is of class $\leqq \alpha + 1$.

<u>Proof of Theorem 2</u>. Let $\mathcal{C} = \cup \{ \mathcal{C}_m \}$ be an open base for Y such that each collection $\mathcal{C}_m$ is discrete in Y. Since the union of each subcollection of $\mathcal{C}$ is a Baire set in Y, the collection $\mathcal{D}_m = f^{-1} [\mathcal{C}_m]$ has the corresponding property in X. By Lemma 2 there is a countable ordinal α_m such that the class of each element of $\mathcal{D}_m$ is $\leqq \alpha_m$. Let $\mathcal{D}$ be the union of all $\mathcal{D}_m$; then each element of $\mathcal{D}$ has the class $\leqq \alpha$, where α is $\sup \alpha$. We shall prove that for each open set U the class of $f^{-1}[U]$ is $\leqq \alpha + 1$. Let U_m be the union of all $C \in \mathcal{C}_m$ with $C \subset U$. Since $\mathcal{C}$ is an open base, U is the union of U_m, and hence $f^{-1} [U]$ is the union of all $f^{-1} [U_m]$. All $f^{-1} [U]$ are of class $\leqq \alpha + 1$ and hence $f^{-1} [U]$ is of class $\leqq \alpha + 2$. This concludes the proof.

<u>Proof of Theorem 1</u>. A corollary to Theorem 2.

3. <u>Proofs of Theorems 3 and 4</u>. Let $\mathcal{B}$ be an algebra of subsets of a set X. A family $\{X_a\}$ of subsets of X is said to be completely $\mathcal{B}$-additive if the union of each subfamily of $\{X_a\}$ belongs to $\mathcal{B}$, and $\{X_a\}$ is said to be $\mathcal{B}$-discrete if there exists a disjoint cover (= partition) $\{B_a\}$ of X, which is completely $\mathcal{B}$-additive and dominates $\{X_a\}$ (i.e. $B_a \supset X_a$ for each a). Clearly a partition of X is $\mathcal{B}$-discrete if and only if it is completely $\mathcal{B}$-additive.

Denote by $u\mathcal{B}$ or $u\langle X, \mathcal{B}\rangle$ the uniform space which has $\mathcal{B}$-discrete partitions for a subbase of uniform covers; this subbase need not be a base because the meet of two $\mathcal{B}$-discrete partitions need not be $\mathcal{B}$-discrete. It follows from Lemma 2 that

<u>Theorem 5</u>. If X is an absolute Souslin space then the $\mathcal{B}$-discrete partitions form a base for the uniform covers of $u\,Ba(X)$. Indeed, let $\{X_a\}$ and $\{Y_b\}$ be $\mathcal{B}$-discrete partitions of X where $\mathcal{B}$ is the σ-algebra of Baire sets in X. By Lemma 2 all elements of $\{X_a\}$ are of class $\leqq \alpha$, all elements of $\{Y_b\}$ are of class $\leqq \beta$, and hence, all elements of $\{X_a \cap Y_b\}$ are of the class $\leqq \gamma = max(\alpha, \beta)$. By Hansell's lemma the partitions $\{X_a\}$ and $\{Y_b\}$ are σ-d.d., and so $\{X_a \cap Y_b\}$ is σ-d.d. Finally, by Lemma 2, the partition $\{X_a \cap Y_b\}$ is $\mathcal{B}$-discrete. The proof is finished.

If m is an infinite cardinal, denote by $u_m\mathcal{B}$ or $u_m\langle X, \mathcal{B}\rangle$ the uniform space such that the

$\mathcal{B}$ -discrete partitions of cardinal less than m form a subbase for the uniform covers. If $m = \aleph_0$ or $m = \aleph_1$ then these covers form a base (if $\mathcal{B}$ is an σ-algebra), and in addition, it is easy to check that the corresponding uniformities are projectively induced by all bounded or all (not necessarily bounded) $\mathcal{B}$ -measurable (real valued) functions.

Proof of Theorem 3. By Theorem 5 the $\mathcal{B}$ -discrete covers form a base of the uniform covers of $u\mathcal{B}$. It follows that $u\mathcal{B}$ is proximally equivalent to $u_{\aleph_0}\mathcal{B}$. Let $\mathcal{U}$ be any ultra-uniformly which is proximally coarser than $u_{\aleph_0}\mathcal{B}$. If $\{X_a\}$ is any uniform partition then $\{X_a\}$ is $\mathcal{B}$ -discrete, and hence $\{X_a\}$ is a uniform cover of $u\mathcal{B}$. This concludes the proof of Theorem 3.

Remark. By an S -uniformity we mean a uniformity which has the point-finite (one gets the same notion if he takes uniformly locally finite) uniform covers for the basis for uniform covers. In Theorem 3 one can replace "ultrauniformity" by " S -uniformity". I don't know the answer in general.

The statement of Theorem 4 requires explanation.

Following [F 2], if $\mathcal{B}$ is a σ-algebra on X , denote by $h\mathcal{B}$ the smallest σ-algebra $\mathcal{C} \supset \mathcal{B}$ which is closed under the unions of arbitrary large $\mathcal{B}$ -discrete families, and denote by $H\mathcal{B}$ the smallest σ-algebra $\mathcal{C} \supset \mathcal{B}$ which has the property $h\mathcal{C} = \mathcal{C}$. For the properties of h and H we refer to [F 3].

Denote by bi-Souslin $\mathcal{B}$ and call bi-Souslin sets over $\mathcal{B}$, the collection of all $Y \subset X$ such that the two sets Y and $X - Y$ are Souslin sets.

Lemma 3. Let $\mathcal{B} = Ba\,X$ where X is an absolute Souslin space. Then

$$h\,\mathcal{B} = H\,\mathcal{B} \subset \text{bi-Souslin}\ \mathcal{B} .$$

Proof. One proves that $h\,\mathcal{B}$ is the smallest collection $\mathcal{D} \supset \mathcal{B}$ which is closed under countable unions, countable intersections, and $\mathcal{B}$-discrete unions. The collection Souslin X is also closed under these operations, and hence

$$h\,\mathcal{B} \subset \text{Souslin}\ \mathcal{B} ,$$

and since $h\,\mathcal{B}$ is an σ-algebra,

$$h\,\mathcal{B} \subset \text{bi-Souslin}\ \mathcal{B} .$$

Since $h\,\mathcal{B} \subset$ Souslin $\mathcal{B}$, every $h\,\mathcal{B}$-discrete partition of X is σ-d.d. by Hansell's lemma, and hence, every $h\,\mathcal{B}$-discrete family in $h\,\mathcal{B}$ is σ-d.d., and finally, the union of every $h\,\mathcal{B}$-discrete family in $h\,\mathcal{B}$ belongs to $h\,\mathcal{B}$ as a countable union of topological discrete unions of elements of $h\,\mathcal{B}$. It follows that $h\,h\,\mathcal{B} = h\,\mathcal{B}$, and hence

$$H\,\mathcal{B} = h\,\mathcal{B} .$$

This concludes the proof.

Lemma 4. Let $\mathcal{B} = Ba\,X$ where X is an absolute Souslin space. Then

$\text{bi-Souslin}\ \mathcal{B} \subset h\,\mathcal{B}$.

Proof. This follows from the Hansell's First Separation Principle for non-separable absolute Souslin sets [H 3]: If X is a metrizable space, and if S_1 and S_2 are disjoint absolute Souslin sets in X, then $S_1 \subset \subset B \subset X - S_2$ for some $B \in h\,\mathcal{B}$. For another proof see [F 4].

Proof of Theorem 4. From Lemmas 3 and 4.

4. Algebras and uniformities

The two results of this sections are developed in [F 3].

Theorem 6. Let $\mathcal{B}$ be a σ-algebra on X. Then $u\,\mathcal{B}$ is locally finite and proximally equivalent to $u_{\aleph_0}\mathcal{B}$ if and only if $h\,\mathcal{B} = \mathcal{B}$.

Proof. We shall prove "if", "only if" is checked similarly. Assume $h\,\mathcal{B} = \mathcal{B}$. If $\{X_a\}$ and $\{Y_b\}$ are two $\mathcal{B}$-discrete partitions then $\{X_a \cap Y_b\}$ is completely $h\,\mathcal{B}$-additive, and hence $\mathcal{B}$-discrete. Thus $u\,\mathcal{B}$ is proximally equivalent to $u_{\aleph_0}\mathcal{B}$. Let $\{X_a\}$ be a $\mathcal{B}$-discrete partition of X, and let $\{X_{ab}\}$ be a $\mathcal{B}$-discrete partition of X_a for each a; then $\{X_{ab}\}$ is $h\,\mathcal{B}$-discrete, hence $\mathcal{B}$-discrete, and hence $\{X_{ab}\}$ is a uniform partition of $u\,\mathcal{B}$. This shows that $u\,\mathcal{B}$ is locally fine.

Theorem 7. The following conditions on a σ-algebra $\mathcal{B}$ on X are equivalent:

1) $u\mathcal{B}$ is proximally equivalent to $u_{\aleph_0}\mathcal{B}$.

2) $u\mathcal{B}$ is the finest ultrauniformity proximally equivalent to $u_{\aleph_0}\mathcal{B}$.

3) The $\mathcal{B}$ -discrete partitions form a base for the uniform covers of $u\mathcal{B}$.

Proof. Clearly Condition 1 is equivalent to Condition 3. If an ultrauniformity is proximally coarser than $u_{\aleph_0}\mathcal{B}$, then it has a base consisting from $\mathcal{B}$ -discrete partitions, and hence Condition 1 implies Condition 2. The implication 2) $\Longrightarrow$ 1) is self-evident.

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(Oblatum 20.1.1972)

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