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PROJECTIVE PURITIES

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In this note we shall give a necessary and sufficient condition for a projectively closed purity to be projective. Further, two dual criteria for the projectivity or the injectivity of a purity ω are proved. Finally, one of the possible solutions of Problem 55 from [2] is given.

Throughout this paper, Λ stands for an associative ring with unity and all the modules are assumed to be unitary Λ -modules. Concerning the terminology and notation, we refer to [4] and [5] or [1]. For the sake of simplicity we shall write $Ext(B, A)$ instead of $Ext^1_\Lambda(B, A)$. Further, if $E : 0 \longrightarrow A \longrightarrow X \longrightarrow B \longrightarrow 0$ is a short exact sequence then there is no danger of confusions in writing $E \in Ext(B, A)$. Recall that for a purity ω we denote $\omega Ext(B, A)$ the subset of $Ext(B, A)$ formed by all short exact sequences $0 \longrightarrow A \xrightarrow{i} X \longrightarrow B \longrightarrow 0$ with $i \in \mathfrak{F}_\omega$. Finally, a homomorphism $f : P \longrightarrow B$ induces the homomorphism $Ext(B, A) \longrightarrow Ext(P, A)$ which we denote f^* (see [3]).

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Lemma 1. Let ω be a projectively closed purity. Then

(1) $$\omega \, Ext\,(B,A) = \bigcap Ker\, f^*$$

where f ranges all over the $f : P \longrightarrow B$ for all $P \in \mathfrak{P}_\omega$.

Proof. Consider the commutative diagram

(2) $$\begin{array}{lccccccccc} E f^* : & 0 & \longrightarrow & A & \xrightarrow{j} & Y & \xrightarrow{\sigma} & P & \longrightarrow & 0 \\ & & & \| & & \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle f} & & \\ E \;\; : & 0 & \longrightarrow & A & \xrightarrow{i} & X & \xrightarrow{\pi} & B & \longrightarrow & 0 \end{array}$$

with exact rows and $P \in \mathfrak{P}_\omega$. If $E \in \omega\, Ext\,(B,A)$, then $j \in \mathfrak{H}_\omega$, ω being triangular. Therefore $E f^*$ splits and $\omega\, Ext\,(B,A) \subseteq \bigcap Ker\, f^*$. Conversely, for $E f^* = 0$ let $\sigma' : P \longrightarrow Y$ be a homomorphism with $\sigma' \sigma = 1_P$. Then $(\sigma' \varphi)\pi = \sigma' \sigma f = f$ and the assertion follows by the definition of the projectively closed purity.

From (1) it immediately follows that to any pair A, B of modules there exists a subset $\mathfrak{M}_{AB}$ of $\mathfrak{P}_\omega$ such that

(3) $$\omega\, Ext\,(B,A) = \bigcap_{\substack{f \in Hom(P,B) \\ P \in \mathfrak{M}_{AB}}} Ker\, f^* \quad .$$

Lemma 2. Let A, B be two modules and let ω be a projectively closed purity. Then $\mathfrak{P}_\omega$ contains a module

P' such that

$$(4) \qquad \omega \operatorname{Ext}(B,A) = \bigcap_{f \in Hom(P',B)} \operatorname{Ker} f^* .$$

Proof. Putting $P' = \sum_{P \in \mathfrak{M}} P$ we get (4), since $\bigcup_{P \in \mathfrak{M}_{AB}} Hom(P,B) \subseteq Hom(P',B)$. Here $P' \in \mathfrak{P}_\omega$ by (1.5) from [1].

Lemma 3. Let A, B be two modules and let ω be a projectively closed purity. Then there exist $P \in \mathfrak{P}_\omega$ and $\eta' : P \longrightarrow B$ such that

$$(5) \qquad \omega \operatorname{Ext}(B,A) = \operatorname{Ker} \eta'^* .$$

Proof. Let $P' \in \mathfrak{P}_\omega$ be a module satisfying (4). Now we put $P = \sum_{f \in Hom(P',B)} P'_f$, where $P'_f = P'$ for any $f \in Hom(P',B)$ and we define the homomorphism $\eta' : P \longrightarrow B$ by the formula $\{ p'_f \} \eta' = \sum_{f \in Hom(P',B)} (p'_f) f$ (this can be made since only a finite number of p'_f's is non-zero). Let $E : 0 \longrightarrow A \xrightarrow{i} X \xrightarrow{\pi} B \longrightarrow 0$ be an exact sequence. Supposing $E \eta'^* = 0$, we have the commutative diagram

$$
(6)\qquad
\begin{array}{rcccccccccc}
E\eta'^{*}: & 0 & \longrightarrow & A & \xrightarrow{\ j\ } & A \dotplus P & \xrightarrow{\ \sigma\ } & P & \longrightarrow & 0 \\
 & & & \Big\| & & \Big\downarrow{\scriptstyle \varphi} & & \Big\downarrow{\scriptstyle \eta'} & & \\
E\ : & 0 & \longrightarrow & A & \xrightarrow{\ i\ } & X & \xrightarrow{\ \pi\ } & B & \longrightarrow & 0
\end{array}
$$

with exact rows. Now for an arbitrary homomorphism $f : P' \longrightarrow B$ we consider the following diagram

$$
(7)\qquad
\begin{array}{ccccccccc}
0 & \longrightarrow & A & \xrightarrow{\ j'\ } & A \dotplus P' & \xrightarrow{\ \sigma'\ } & P' & \longrightarrow & 0 \\
 & & \Big\| & & \Big\downarrow{\scriptstyle \psi} & & \Big\downarrow{\scriptstyle f} & & \\
0 & \longrightarrow & A & \xrightarrow{\ i\ } & X & \xrightarrow{\ \pi\ } & B & \longrightarrow & 0
\end{array}
$$

with exact rows and $\{a, p'\}\psi = \{a, 0, \dots\ 0, p', 0, \dots\}\varphi$, where p' is on the f-th place. We have $\{a, p'\}\psi\pi =$

$= \{a, 0, \dots, 0, p', 0, \dots\}\varphi\pi = \{a, 0, \dots, 0, p', 0, \dots\}\sigma\eta' = p'f = \{a, p'\}\sigma' f$

and the right square of (7) commutes. Further, $a j'\psi =$

$= \{a, 0\}\psi = \{a, 0, 0, \dots\}\varphi = a j\varphi = a i$ and the left square of (7) commutes. Since $P \in \mathfrak{P}_\omega$ (by (4.5) from [1]), we

have $\operatorname{Ker}\eta'^{*} \subseteq \bigcap_{f \in Hom(P', B)} \operatorname{Ker} f^{*} = \omega \operatorname{Ext}(B, A) \subseteq \operatorname{Ker}\eta'^{*}$ and

we are ready.

Lemma 4. Let A, B be two modules and let ω be a projectively closed purity. Then there exist $P \in \mathfrak{P}_\omega$ and an epimorphism $\eta : P \longrightarrow B$ such that

$$\omega \operatorname{Ext}(B, A) = \operatorname{Ker} \eta^* \quad . \tag{8}$$

Proof. Let P', η' be a module and a homomorphism the existence of which was stated in the preceding lemma and let $\eta'' : F \longrightarrow B$ be an epimorphism of a free module F on B. Putting $P = P' \dotplus F$ and defining $\eta : P \longrightarrow B$ by the formula $\{p', q\}\eta = p'\eta' + q\eta''$, $p' \in P'$, $q \in F$, we can apply the same method as in the proof of the preceding lemma for obtaining (8).

Any couple (P, η) satisfying (8), where $P \in \mathfrak{P}_\omega$ and $\eta : P \longrightarrow B$ is an epimorphism, we call the characteristic couple of the pair A, B of modules and we denote it $(P, \eta)_{AB} = (P_{AB}, \eta_{AB})$. It may happen that there exists a couple (P, η) which is characteristic of the pair A, B for all modules A. In this case we say that (P, η) is the characteristic couple of B.

Theorem 1. A projectively closed purity ω is projective if and only if any module has a characteristic couple.

Proof. Let the condition be satisfied and let B be an arbitrary module. If (P, η) is a characteristic couple of B, then the exact sequence $0 \longrightarrow A \xrightarrow{i} P \xrightarrow{\eta} B \longrightarrow 0$ clearly belongs to $\omega \operatorname{Ext}(B, A)$.

Conversely, if ω is projective and B an arbitrary module, then there exists an exact sequence

$0 \longrightarrow K \xrightarrow{j} P \xrightarrow{\eta} B \longrightarrow 0$ with $j \in \mathfrak{P}_\omega$ and $P \in \mathfrak{P}_\omega$. We are going to show that (P, η) is the characteristic couple of B . It clearly suffices to show that

$Ker\ \eta^* \subseteq \omega\, Ext\,(B, A)$ for any module A . Let E : $: 0 \longrightarrow A \xrightarrow{i} X \xrightarrow{\pi} B \longrightarrow 0$ be an exact sequence satisfying $E\eta^* = 0$ and let $P' \in \mathfrak{P}_\omega$ be an arbitrary module and $f : P' \longrightarrow B$ an arbitrary homomorphism. From $E\eta^* = 0$ it follows the existence of $\varphi : P \longrightarrow X$ with $\varphi\pi = \eta$. Further, there exists a homomorphism $\psi : P' \longrightarrow P$ with $\psi\eta = f$, since $j \in \mathfrak{f}$. Therefore $(\psi\varphi)\pi = \psi\eta = f$ and $E \in \omega\, Ext\,(B, A)$.

Lemma 5. Let ω be a projectively closed purity, m a cardinal and B an arbitrary module. Then there exists a couple (P, η) , where $P \in \mathfrak{P}_\omega$ and $\eta : P \longrightarrow B$ is an epimorphism such that

(9) $\omega\, Ext\,(B,A) = Ker\ \eta^*$ for any A of power $\leq m$.

Proof. Let $\mathfrak{M}$ be the set of all pair-wise non-isomorphic modules of powers $\leq m$. By Lemma 4, to any $A \in \mathfrak{M}$ there exists a characteristic couple $(P', \eta')_{AB}$. Let us put $P = \dot{\sum}_{A \in \mathfrak{M}} P'_{AB}$ and let us define the homomorphism $\eta : P \longrightarrow B$ by the formula $\{p'_{AB}\}\eta = \sum p'_{AB}\, \eta'_{AB}$. Now the proof runs on the same lines as that of Lemma 3 and we therefore omit it.

Let us note that all the above results can be easily dualised for injectively closed purities. In [5] we have introduced the notion of a basis (a $\mathfrak{P}$-basis) of an injectively (projectively) closed purity. In the present paper, we shall deal with purities that are both projectively and injectively closed. We shall therefore use the words injective (projective) basis for the basis of ω considered as the injectively (projectively) closed purity.

<u>Theorem 2</u>. Let ω be a projectively closed purity. If ω is injective with a set as an injective basis, then ω is projective.

<u>Proof</u>. By Theorem 15 from [5] there exists a module $\mathcal{Q}$ which forms an injective basis for ω. If the power of $\mathcal{Q}$ is m, then let (P, η) be the couple the existence of which is guaranteed by Lemma 5. Thus we have an exact sequence

$$0 \longrightarrow K \xrightarrow{i} P \xrightarrow{\eta} B \longrightarrow 0 \tag{10}$$

and we are going to show $i \in \mathcal{B}_\omega$. η induces the homomorphism $\eta^* : Ext(B, \mathcal{Q}) \longrightarrow Ext(P, \mathcal{Q})$ the kernel of which is $\omega Ext(B, \mathcal{Q})$. Since $\mathcal{Q}$ is ω-injective, we have $\omega Ext(B, \mathcal{Q}) = 0$ and η^* is a monomorphism. Furthermore, the sequence (10) induces the exact sequence

$$Hom(P, \mathcal{Q}) \xrightarrow{i^*} Hom(K, \mathcal{Q}) \longrightarrow Ext(B, \mathcal{Q}) \xrightarrow{\eta^*} Ext(P, \mathcal{Q}) .$$

Since η^* is a monomorphism, one easily gets that i^* is an epimorphism. Hence $i \in \mathfrak{H}_{\omega}$, since $\mathcal{Q}$ is the injective basis of ω and we are ready.

The proof of the following theorem is dual and we therefore omit it.

<u>Theorem 3</u>. Let ω be an injectively closed purity. If ω is projective with a set a projective basis, then ω is injective.

Now we are going to give one of the possible solutions of Problem 55 from [2], namely a characterization of $Ext(B, A, m)$. For a cardinal m we denote by ω^m the purity having the set of all pair-wise non-isomorphic modules of powers $< m$ as a projective basis. The corresponding class of monomorphisms we denote by $\mathfrak{H}^m$ (see [1],[4],[5]). Instead of $\omega^m Ext(B,A)$ we shall write $Ext(B, A, m)$. By Theorem 4 from [5] the purity ω^m is projective. Therefore Theorem 1 yields that to any module B there exists a characteristic ooup-le (P, η). We have therefore proved:

<u>Theorem 4</u>. Let m be a cardinal. Then

$Ext(B, A, m) = Ker\, \eta^*$, where (P, η) is a characteristic couple of B.

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