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A CHARACTERIZATION OF THE EIGENVALUES OF A COMPLETELY  
CONTINUOUS SELFADJOINT OPERATOR

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1. Introduction. In the present paper we give a characterization of the eigenvalues of a completely continuous selfadjoint operator which acts in a Hilbert space by a variational principle. Our arguments are based on the variant of Ljusternik-Schnirelmann-theory in [4] without the explicit use of the notion category. This procedure makes it possible to dispense with the oddness of the operator and therefore to handle the problem of existence and bifurcation of nontrivial solutions for nonlinear operator equations with not necessary odd operators. This has a great importance in the study of some problems in nonlinear elasticity.

In [2] M.S. Berger has given a similar characterization with an explicit use of the category arguments based on a result about the dimension of the critical set of a functional. Our formulations are, roughly speaking, a "curved" variant with respect to the formulation in [5] and with respect to the well-known variational formulation

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(see e.g. [8])<sup>1)</sup>. But it is not the aim of the present paper to give a treatment of some connections between these various formulations.

The advantage of our results and the variational formulations consist in the immediate applicability to bifurcation theory for nonlinear equations of the type  $\lambda A\mu = B\mu$ . This will be done in a forthcoming paper about a generalization of the bifurcation procedure in [7].

2. Preliminaries. In this section, we recall some wellknown facts about the spectral analysis of completely continuous selfadjoint operators in a Hilbert space and give some inequalities for the use in the next sections (see [4]).

Let  $H$  be a real Hilbert space with the scalar product  $(\cdot, \cdot)$  and the norm  $\|\cdot\|$ . In the whole paper we suppose that  $L$  is a completely continuous selfadjoint positive operator (i.e.  $(L\mu, \mu) > 0$  for  $\mu \neq \theta$ ) which acts in  $H$ . It holds ([1]): There exists a finite or infinite sequence of orthogonal in pairs and normalized eigenvectors

$$e_1, e_2, \dots, e_n, \dots$$

which belong to the eigenvalues

$$\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n \geq \dots$$

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 1) Particularly for Courant-Weinstein-characterization see: Dunford, N. and Schwartz, J.T.: Linear operators, part II. New York, London 1963.

For each  $\mu \in H$  the expansion holds

$$(1) \quad L\mu = \sum_{j=1}^{\infty} \lambda_j^* (\mu, e_j) e_j .$$

We denote by

$$\lambda_1^* > \lambda_2^* > \dots > \lambda_m^* > \dots$$

the sequence of all distinct eigenvalues of  $L$  (each  $\lambda_m^*$  has a certain finite multiplicity  $r_m$ , e.g.  $\lambda_1^* = \lambda_1 \equiv \dots \equiv \lambda_{r_1}$ ) and by  $E_m$  ( $m = 1, 2, \dots$ ) the eigenpaces of  $\lambda_m^*$ . Further we denote by  $H_m$  ( $m = 1, 2, \dots$ ) the closed linear hull of all eigenvectors which belong to the eigenvalues  $\lambda_1^*, \dots, \lambda_m^*$ . Clearly  $H_1 = E_1$ .

Let  $P_m$  be the projections onto  $H_m$  and  $P_m^\perp$  the projection onto  $H \ominus H_m$ , respectively. From (1) follows

$$\begin{aligned} (LP_1^\perp \mu, \mu) &= \sum_{j=r_1+1}^{\infty} \lambda_j^* (P_1^\perp \mu, e_j) (\mu, e_j) \\ &= \sum_{j=r_1+1}^{\infty} \lambda_j^* (P_1^\perp \mu, e_j)^2 , \end{aligned}$$

therefore for each  $\mu \in H$

$$(2) \quad (LP_1^\perp \mu, \mu) \leq \lambda_2^* \|P_1^\perp \mu\|^2 .$$

Let be  $\mu \in H \ominus H_m$ . Then  $(\mu, e_j) = 0$  for all eigenvectors  $e_j$  which belong to the eigenvalues  $\lambda_1^*, \dots, \lambda_m^*$  and we obtain

$$(3) \quad (L\mu, \mu) \leq \lambda_{m+1}^* \|\mu\|^2 , \quad \mu \in H \ominus H_m .$$

From this and (1) it follows that

$$(4) \quad (L\mu, \mu) = (LP_m \mu, P_m \mu) + (LP_m^\perp \mu, P_m^\perp \mu) \leq$$

$$\begin{aligned} &\leq \lambda_1^* \|P_m \mu\|^2 + \lambda_{m+1}^* \|P_m^\perp \mu\|^2 \\ &= \lambda_{m+1}^* \|\mu\|^2 + (\lambda_1^* - \lambda_{m+1}^*) \|P_m \mu\|^2 \end{aligned}$$

for arbitrary  $\mu \in H$ . By a similar consideration we obtain

$$(5) \quad (L\mu, \mu) \geq \lambda_m^* \|P_m \mu\|^2, \quad \mu \in H.$$

3. The first eigenvalue. For arbitrary but fixed real  $R > 0$  we define

$$\begin{aligned} S_R &= \{ \mu \mid \mu \in H, \frac{1}{2} \|\mu\|^2 = R \}, \\ B_R &= \{ \mu \mid \mu \in H, \frac{1}{2} \|\mu\|^2 \leq R \}. \end{aligned}$$

The functional

$$\phi(\mu) = \frac{1}{2} (L\mu, \mu), \quad \mu \in H$$

has  $L$  as its gradient (in the Fréchet sense) on  $H$ .

The following result is wellknown, however the proof is given in a nontraditional way by the aid of an argument from [3].

Theorem 1. The variational problem

$$\text{maximize } \phi(\mu) \text{ over } B_R$$

has for each fixed  $R > 0$  a solution  $\mu_1 = \mu_1(R) \in S_R$  such that  $\phi(\mu_1(R)) = \max_{B_R} \phi(\mu)$ . There exists a  $\lambda_1 = \lambda_1(R) \in \mathbb{R}^1$  with  $\lambda_1 \mu_1 = L\mu_1$ . Furthermore

$$(i) \quad \phi(\mu_1(R)) = \lambda_1(R) R \quad \text{for each } R > 0.$$

$$(ii) \quad \text{For each } R > 0 \text{ holds } \lambda_1(R) = \lambda_1^*.$$

Proof of Theorem 1. The functional  $\phi$  is weakly continuous on  $H$  and after a wellknown theorem of functional analysis there exists an element  $u_1 = u_1(R) \in B_R$  such that  $\phi(u_1(R)) = \max_{B_R} \phi(u)$ . Suppose  $\frac{1}{2} \|u_1(R)\|^2 < R$ . There would exist a real number  $t > 1$  with  $\frac{1}{2} \|u_1(R)\|^2 < \frac{1}{2} \|tu_1(R)\|^2 \leq R$ . On account of the maximality of  $u_1(R)$  it follows

$$\begin{aligned} \frac{t^2}{2} (Lu_1(R), u_1(R)) &= \phi(tu_1(R)) \leq \phi(u_1(R)) = \\ &= \frac{1}{2} (Lu_1(R), u_1(R)) \end{aligned}$$

a contradiction, since  $u_1(R) \neq \theta$ . Therefore  $u_1(R) \in S_R$  and a trivial consideration shows

$$\phi(u_1(R)) = \max_{B_R} \phi(u) = \max_{S_R} \phi(u).$$

After a slight modification of the proof of Theorem 4 in [3] we obtain the existence of a  $\lambda_1 = \lambda_1(R) \in \mathbb{R}^1$  with  $\lambda_1 u_1 = Lu_1$ . From the last statement it follows simply

$$\phi(u_1(R)) = \frac{1}{2} (Lu_1(R), u_1(R)) = \frac{1}{2} \lambda_1(R) \|u_1(R)\|^2 = \lambda_1(R) R$$

which proves (i).

Let  $u \in E_1 \cap S_R$  be arbitrary. We obtain

$$(6) \lambda_1^* R = \frac{\lambda_1^*}{2} \|u\|^2 = \phi(u) \leq \max_{S_R} \phi(u) = \lambda_1(R) R.$$

On the other hand, let  $u \in H$  be arbitrary. With  $u = P_1 u + P_1^\perp u$  it follows from (2) that

$$\phi(u) = \frac{\lambda_1^*}{2} (P_1 u, u) + \frac{1}{2} (LP_1^\perp u, u) \leq$$

$$\leq \frac{\lambda_1^*}{2} \|P_1 u\|^2 + \frac{\lambda_2^*}{2} \|P_1^\perp u\|^2 \leq \frac{\lambda_1^*}{2} \|u\|^2 .$$

Therefore, for arbitrary  $u \in S_R$

$$\lambda_1(R)R = \max_{S_R} \phi(u) = \max_{S_R} \phi(u) \leq \lambda_1^* R$$

and together with (6) finally  $\lambda_1(R) = \lambda_1^*$  , which completes the proof.

4. Lemma. In this section and the next we prepare the characterization of the eigenvalues  $\lambda_m$  with  $m \geq 2$  . We begin with a result about ordinary differential equations in Hilbert spaces.

Let  $h \in H$  with  $\|h\| = 1$  be chosen arbitrary, but fixed. Now let us consider the following initial value problem

$$(7) \quad \frac{du(t)}{dt} = h - \frac{(u(t), h)}{2R} u(t), \quad u_0 \in S_R .$$

From [7], [9] we obtain

Lemma 1. There exists a real number  $t_0 > 0$  which depends neither from  $u_0$  nor from  $h$  such that

(i) in the interval  $0 \leq t \leq t_0$  there exists a unique solution  $u(t)$  of (7);

(ii) there exists a constant  $c = c(R)$  which depends only on  $R$  but neither on  $u_0$  nor on  $h$  with

$$\|u(t) - u_0\| \leq c(R)t \quad \text{for all } 0 \leq t \leq t_0 ;$$

(iii)  $u(t) \in S_R$  holds for all  $0 \leq t \leq t_0$  .

The crucial importance of this lemma is the uniformity of  $t_0$  with respect to  $u_0$  and  $h$  . We call the solution of (7) a trajectory on the sphere  $S_R$  .

For arbitrary  $\mu \in S_R$  and  $v \in H$ , we introduce the following operators

$$S(v, \mu) = Lv - \frac{(Lv, \mu)}{2R} \mu, \quad Q\mu = L\mu - \frac{(L\mu, \mu)}{2R} \mu.$$

Lemma 2. There exists a constant  $c_1$  (independently of  $R$ ) such that for all  $\mu, \mu_0 \in S_R$  and  $v \in B_R$  there holds

$$(8) \quad \|S(v, \mu) - Q\mu_0\| \leq c_1 (\|\mu - \mu_0\| + \|v - \mu_0\|).$$

Proof of Lemma 2. First of all we obtain

$$\|(Lv, \mu)\mu - (L\mu_0, \mu_0)\| \leq 2R \|L\| (2\|\mu - \mu_0\| + \|v - \mu_0\|).$$

Therefore

$$\begin{aligned} \|S(v, \mu) - Q\mu_0\| &\leq \|Lv - L\mu_0\| + \frac{1}{2R} \|(Lv, \mu)\mu - (L\mu_0, \mu_0)\mu_0\| \\ &\leq c_1 (\|\mu - \mu_0\| + \|v - \mu_0\|) \end{aligned}$$

with  $c_1 = 2\|L\|$ , q.e.d.

For arbitrary  $\mu, v \in H$  it holds that

$$\phi(\mu) - \phi(v) = \int_0^1 (L(v + s(\mu - v)), \mu - v) ds.$$

Let  $\mu(t)$  be a solution of the initial value problem (7).

For a certain mean value  $\xi(t) \in (0, 1)$  it follows from the last equation

$$\phi(\mu(t)) - \phi(\mu_0) = (L(\mu_0 + \xi(t)(\mu(t) - \mu_0)), \mu(t) - \mu_0)$$

and with the notation  $\mu_\xi = \mu_0 + \xi(t)(\mu(t) - \mu_0)$  and after change of integration and scalar product (see [6]) therefore

$$(9) \quad \phi(\mu(t)) - \phi(\mu_0) = \int_0^t (S(\mu_\xi, \mu(s)), \mu(s)) ds$$



for all  $0 \leq t \leq t_0$ .

5. Lemmas (continued). Let us consider the space  $H_m$  with  $m \geq 2$  (see Section 2). Then we define

$$\mathcal{O}(H_m) = \{ \mu \mid \mu \in H, \| P_m \mu \| > 0 \}.$$

Clearly  $(S_R \cap H_m) \subset \mathcal{O}(H_m)$ . The following two lemmas are contained in [4] (see [4] for definitions, too).

Lemma 3. The set  $S_R \cap H_m$  is noncontractible in  $\mathcal{O}(H_m)$ .

Lemma 4. Let  $\tilde{H}$  be a proper subspace of  $H_m$ . If for a set  $V \subset H$  it holds that  $P_m V \cap \tilde{H} = \emptyset$  then it follows that  $V \subset \mathcal{O}(H_m)$  and  $V$  is contractible in  $\mathcal{O}(H_m)$ .

Proof of Lemma 4. Suppose there would exist  $\mu \in V$  with  $P_m \mu = \theta$ . Then  $\theta \in P_m V$  and therefore  $\theta \in P_m V \cap \tilde{H}$  which contradicts our assumption. The proof of the second part of Lemma 4 corresponds to the proof of Lemma 2.7 of [4], p.331, q.e.d.

Our next lemma provides an explicit condition for the belonging of an element  $\mu \in H$  to  $\mathcal{O}(H_m)$ .

Lemma 5. Let  $0 \leq \sigma < \lambda_m^* - \lambda_{m+1}^*$ . If  $\phi(\mu) \geq (\lambda_m^* - \sigma)R$  for  $\mu \in S_R$  then

$$\| P_m \mu \|^2 \geq 2R \frac{\lambda_m^* - \lambda_{m+1}^* - \sigma}{\lambda_1^* - \lambda_{m+1}^*} > 0.$$

Proof of Lemma 5. Suppose the contrary holds. Then we obtain with aid of (4)

$$\phi(\mu) = \frac{1}{2} (L\mu, \mu) \leq \frac{\lambda_{m+1}^*}{2} \|\mu\|^2 + \frac{\lambda_1^* - \lambda_{m+1}^*}{2} \|P_m \mu\|^2 <$$

$$\begin{aligned}
&< \frac{\lambda_{m+1}^*}{2} \|\mu\|^2 + R(\lambda_1^* - \lambda_{m+1}^*) \frac{\lambda_m^* - \lambda_{m+1}^* - \sigma}{\lambda_1^* - \lambda_{m+1}^*} \\
&= \lambda_{m+1}^* R + (\lambda_m^* - \lambda_{m+1}^* - \sigma)R = (\lambda_m^* - \sigma)R .
\end{aligned}$$

This contradiction proves the assertion of the lemma.

Corollary 1. For  $0 \leq \sigma < \lambda_m^* - \lambda_{m+1}^*$  it holds that

$$\{\mu \mid \mu \in S_R, \phi(\mu) \geq (\lambda_m^* - \sigma)R\} \subset \sigma(H_m) .$$

6. The eigenvalues  $\lambda_m^*$  with  $m \geq 2$ . In this section we proceed to the characterization of the eigenvalues  $\lambda_m^*$  with  $m \geq 2$ .

Definition.  $[V]_{H_m}$  denotes the class of all compact subsets  $V \subset S_R$  which lie in  $\sigma(H_m)$  and which are non-contractible in  $\sigma(H_m)$ .

Consequently  $[V]_{H_m}$  contains with a set  $V$  all sets which are the result of  $V$  by a continuous deformation, remaining in  $S_R$ . Lemma 3 provides  $(S_R \cap H_m) \in [V]_{H_m}$ .

Now we formulate the following variational problem

$$(*) \quad c_m = c_m(R) = \sup_{[V]_{H_m}} \inf_V \phi(\mu) .$$

Obviously it holds

$$(1Q) \quad c_m(R) \geq \min_{S_R \cap H_m} \phi(\mu) = \lambda_m^* R .$$

It is the aim of our following considerations to show that in the last inequality the equal sign must hold always, which provides the desired characterization by the variatio-

nal problem (\*).

For  $\varepsilon > 0$  we introduce

$$W_\varepsilon = \{ \mu \mid \mu \in S_R, |\phi(\mu) - c_m(R)| \leq \varepsilon \}.$$

We prove now

Lemma 6. For each  $\varepsilon > 0$  there exists a  $\mu \in W_\varepsilon$  with  $\|Q\mu\| < \varepsilon$ .

Proof of Lemma 6. Suppose the contrary holds, i.e. it holds that  $\|Q\mu\| \geq \varepsilon_0$  for a certain  $\varepsilon_0 > 0$  and all  $\mu \in W_{\varepsilon_0}$ .

Let  $\mu_0 \in W_{\varepsilon_0}$  be arbitrarily chosen. There exists a  $h = h_{\mu_0}$  with  $\|h\| = 1$  and  $(Q\mu_0, h) \geq \frac{1}{2} \|Q\mu_0\|$ . Further, let  $u(t)$  be the unique solution of the initial value problem (7) in the interval  $0 \leq t \leq t_0$  with the initial value  $\mu_0$  and  $h = h_{\mu_0}$  (see Lemma 1). Now we take

$$t_1 = \min \left\{ t_0, \frac{\varepsilon_0}{8c_1 c(R)} \right\}.$$

From Lemma 1 (ii) it follows

$$\|u(t) - \mu_0\| \leq \frac{\varepsilon_0}{8c_1} \quad \text{for all } 0 \leq t \leq t_1$$

and from this

$$\|u_{t_1} - \mu_0\| \leq \frac{\varepsilon_0}{8c_1} \quad \text{for all } 0 \leq t \leq t_1.$$

Moreover, it holds that  $\|u_{t_1}\| \leq (2R)^{\frac{1}{2}}$  and (8) provides for all  $0 \leq t \leq t_1$  the estimation

$$\|S(u_{t_1}, u(t)) - Q\mu_0\| \leq \frac{\varepsilon_0}{4}.$$

We obtain

$$\begin{aligned}
 (S(u_{\frac{1}{2}}, u(s)), h) &\cong \frac{1}{2} \|Q u_0\| - |(S(u_{\frac{1}{2}}, u(s)) - Q \dot{u}_0, h)| \\
 &\cong \frac{\varepsilon_0}{2} - \frac{\varepsilon_0}{4} = \frac{\varepsilon_0}{4}, \quad 0 \leq s \leq t_1.
 \end{aligned}$$

From (9) it follows therefore

$$(11) \quad \phi(u(t_1)) - \phi(u_0) = \int_0^{t_1} (S(u, u(s)), h) ds \geq \frac{1}{4} \varepsilon_0 t_1.$$

This estimation holds uniformly for all  $u_0 \in W_{\varepsilon_0}$ .

We take  $\varepsilon_1 = \min(\frac{1}{8} \varepsilon_0 t_1, \varepsilon_0)$ . For this  $\varepsilon_1$  there exists a  $V_{\varepsilon_1} \in [V]_{H_m}$  such that  $\min_{V_{\varepsilon_1}} \phi(u) \geq c_m(R) - \varepsilon_1$ .

Now we consider the decomposition

$$\begin{aligned}
 V_{\varepsilon_1}^{(1)} &= \{u \mid u \in V_{\varepsilon_1}, c_m(R) + \varepsilon_1 > \phi(u) \geq c_m(R) - \varepsilon_1\}, \\
 V_{\varepsilon_1}^{(2)} &= \{u \mid u \in V_{\varepsilon_1}, \phi(u) \geq c_m(R) + \varepsilon_1\}.
 \end{aligned}$$

Obviously  $V_{\varepsilon_1} = V_{\varepsilon_1}^{(1)} \cup V_{\varepsilon_1}^{(2)}$ . Let  $u \in V_{\varepsilon_1}^{(1)}$ , i.e.

$|\phi(u) - c_m(R)| \leq \varepsilon_1$ . Since  $\varepsilon_1 \leq \varepsilon_0$ , it follows

that  $u \in W_{\varepsilon_0}$ . Then we obtain from (11) (take  $u = u_0 = u(0)$ )

$$\begin{aligned}
 (12) \quad \phi(u(t_1)) &\geq \phi(u) + \frac{1}{4} \varepsilon_0 t_1 \geq \phi(u) + 2\varepsilon_1 \\
 &\geq c_m(R) + \varepsilon_1.
 \end{aligned}$$

Now we displace the element  $u \in V_{\varepsilon_1}^{(1)}$  along the trajectory  $u(t)$  with an initial point  $u = u_0 = u(0)$  and  $h = h_{u_0}$  up to a point  $u(t)$  with  $\phi(u(t)) \geq c_m(R) + \varepsilon_1$ . The inequality (12) shows that this is always possible.

Therefore it is possible to displace the whole set  $V_{\varepsilon_1}^{(1)}$  continuously. The length of the displacement of a  $\mu \in V_{\varepsilon_1}^{(1)}$  tends to zero, if  $(c_m(R) + \varepsilon_1 - \phi(\mu))' \rightarrow 0+$ . The points in  $V_{\varepsilon_1}^{(2)}$  remain fixed.

We denote the set which is displaced in such a way by  $\tilde{V}_{\varepsilon_1}$ . Obviously  $\tilde{V}_{\varepsilon_1} \subset S_R$  (see Lemma 1 (iii)) and  $\tilde{V}_{\varepsilon_1}$  is the result of a continuous deformation of  $V_{\varepsilon_1}$ . Since  $\phi(\tilde{\mu}) \geq c_m(R) + \varepsilon_1 \geq \lambda_m^* R + \varepsilon_1$  for all  $\tilde{\mu} \in \tilde{V}_{\varepsilon_1}$  it follows from Corollary 1 that  $\tilde{V}_{\varepsilon_1} \subset \mathcal{C}(H_m)$ . After the supposition  $V_{\varepsilon_1}$  is noncontractible in  $\mathcal{C}(H_m)$  and since  $\tilde{V}_{\varepsilon_1}$  is the result of a continuous deformation, it follows that  $\tilde{V}_{\varepsilon_1}$  is noncontractible in  $\mathcal{C}(H_m)$ . Therefore  $\tilde{V}_{\varepsilon_1} \in [V]_{H_m}$  and

$$c_m(R) = \sup_{[V]_{H_m}} \inf_V \phi(\mu) \geq \inf_{\tilde{V}_{\varepsilon_1}} \phi(\mu) \geq c_m(R) + \varepsilon_1.$$

This contradiction completes the proof.

The next theorem presents our main result.

**Theorem 2.** For each natural integer  $m \geq 2$  it holds:

(i) The variational problem

$$(*) \quad c_m = c_m(R) = \sup_{[V]_{H_m}} \inf_V \phi(\mu)$$

has a solution  $\mu_m(R) \in S_R$  for each  $R > 0$  such that

$$\phi(\mu_m(R)) = c_m(R) = c_m.$$

(ii) There exists a real number  $\lambda_m(R)$  with

$$\lambda_m(R) \mu_m(R) = L \mu_m(R).$$

(iii) It holds  $\lambda_n(R) = \lambda_n^*$  for all  $R > 0$ .

Corollary 2. For each  $R > 0$  there holds  $c_n(R) = \lambda_n^* R$ .

Proof of Corollary 2. From Theorem 2 (ii) it follows that

$$\phi(u_n(R)) = \frac{1}{2} (L u_n(R), u_n(R)) = \frac{1}{2} \lambda_n(R) \|u_n(R)\|^2$$

and since  $u_n(R) \in S_R$  we obtain with (i) and (iii)

$$c_n = c_n(R) = \phi(u_n(R)) = \lambda_n^* R, \quad \text{q.e.d.}$$

Proof of Theorem 2 (i) and (ii). We choose a sequence  $\{\varepsilon_j\}$  of real numbers with  $\varepsilon_j \rightarrow 0$  for  $j \rightarrow \infty$ . From Lemma 6 we can conclude the existence of a sequence  $\{\mu_{j,n}\} \in W_{\varepsilon_j}$  with  $\|Q \mu_{j,n}\| \leq \varepsilon_j$  (in the whole proof we denote  $u_{j,n}(R)$  simply by  $\mu_{j,n}$ ). Without loss of generality we can assume that  $\mu_{j,n} \rightarrow \mu_n$  (weak convergence). Since

$\phi$  is weakly continuous, it follows that  $\phi(\mu_n) = c_n$ .

Suppose now  $j_0$  is sufficiently large such that  $0 < \varepsilon_j \leq \sigma R$  for all  $j \geq j_0$ , while  $\sigma$  is chosen as in Lemma 5. We obtain for  $j \geq j_0$

$$\phi(\mu_{j,n}) \geq c_n(R) - \varepsilon_j \geq (\lambda_n^* - \sigma) R$$

and from the inequality (5) and Lemma 5 it follows that

$$\begin{aligned} \phi(\mu_{j,n}) &= \frac{1}{2} (L \mu_{j,n}, \mu_{j,n}) \geq \frac{\lambda_n^*}{2} \|P_n \mu_{j,n}\|^2 \\ &\geq \lambda_n^* R \frac{\lambda_n^* - \lambda_{n+1}^* - \sigma}{\lambda_1^* - \lambda_{n+1}^*}, \quad j \geq j_0. \end{aligned}$$

The tending of  $j \rightarrow \infty$  provides

$$(13) \quad (L u_n, u_n) > 0 .$$

We define

$$\lambda_{j,m} = \lambda_{j,m}(\mathbb{R}) = \frac{1}{2\mathbb{R}} (L u_{j,m}, u_{j,m}) .$$

Again we can assume without loss of generality that  $\lambda_{j,m} \rightarrow \lambda_m$ . A simple argument which makes use of (13) shows that  $\lambda_m > 0$ . Therefore without loss of generality

$$\frac{1}{\lambda_{j,m}} L u_{j,m} \longrightarrow \frac{1}{\lambda_m} L u_m .$$

Now suppose that for  $j \geq j_1$  there holds  $\lambda_{j,m} \geq \frac{1}{2} \lambda_m$ . We obtain for  $j \geq j_1$

$$\begin{aligned} & \| u_{j,m} - \frac{1}{\lambda_m} L u_m \| \leq \\ & \leq \frac{2}{\lambda_m} \| Q u_{j,m} \| + \| \frac{1}{\lambda_{j,m}} L u_{j,m} - \frac{1}{\lambda_m} L u_m \| \xrightarrow{j \rightarrow \infty} 0 \end{aligned}$$

and from this follows

$$\begin{aligned} & \| u_{j,m} - u_m \|^2 = (u_{j,m}, u_{j,m}) - 2(u_{j,m}, u_m) + \| u_m \|^2 \rightarrow \\ & \rightarrow \frac{1}{\lambda_m} (L u_m, u_m) - \frac{1}{\lambda_m} (L u_m, u_m) - (u_m, u_m) + \| u_m \|^2 = 0 . \end{aligned}$$

We conclude  $u_{j,m} \rightarrow u_m$  (strong convergence) and furthermore

$$u_m \in S_{\mathbb{R}} \quad \text{and} \quad u_m = \frac{1}{\lambda_m} L u_m , \quad \text{q.e.d.}$$

Proof of Theorem 2 (iii).  $E_m$  is a proper subspace of  $H_m$ . By Lemma 4 for each  $V \in [V]_{H_m}$  there must exist  $\bar{u} \in V$  with  $P_m \bar{u} \in E_m$ . From (3) it follows that

$$\begin{aligned} \Phi(\bar{u}) &= \frac{1}{2} (L P_m \bar{u}, P_m \bar{u}) + \frac{1}{2} (L P_m^\perp \bar{u}, P_m^\perp \bar{u}) \\ &= \frac{\lambda_m^*}{2} \| P_m \bar{u} \|^2 + \frac{1}{2} (L P_m^\perp \bar{u}, P_m^\perp \bar{u}) \leq \end{aligned}$$

$$\leq \frac{\lambda_m^*}{2} \|P_m \bar{u}\|^2 + \frac{\lambda_{m+1}^*}{2} \|P_m^\perp \bar{u}\|^2 \leq \lambda_m^* R .$$

Therefore

$$\min_V \phi(u) \leq \phi(\bar{u}) \leq \lambda_m^* R$$

and since  $V \in [V]_{H_m}$  was arbitrary,

$$c_m = c_m(R) = \sup_{[V]_{H_m}} \inf_V \phi(u) \leq \lambda_m^* R .$$

In consequence

$$\begin{aligned} \lambda_m(R) R &= \frac{1}{2} \lambda_m(R) \|\mu_m(R)\|^2 = \frac{1}{2} (\langle \mu_m(R), \mu_m(R) \rangle) \\ &= \phi(\mu_m(R)) = c_m(R) \leq \lambda_m^* R . \end{aligned}$$

We combine this with the inequality (10), and the proof is complete.

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