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ON FREDHOLM - STIELTJES INTEGRAL EQUATIONS

(Preliminary communication)

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For a real $k \times \ell$ -matrix $A = (a_{ij})$, $i = 1, \dots, k$, $j = 1, \dots, \ell$, we denote by A' its transpose. Let $|A| = \max_{i=1,\dots,k} \sum_{j=1}^{m} |a_{ij}|$. Let R^n be the space of all $n \times 1$ -matrices x, $x' = (x_1, x_2, \dots, x_n)$, $\|x\|$ - for $x \in R^n$ is a norm in R^n. By $\mathcal{L}(R^n \to R^n)$ the space of all $n \times n$ -matrices is denoted, $\|A\|$ for $A \in \mathcal{L}(R^n \to R^n)$ is the corresponding operator norm.

For a given bounded closed interval $\langle a, b \rangle \subset R$, $a < b$ we denote

$$V_n = \{x : \langle a, b \rangle \to R^n ;\ \operatorname{var}_a^b x < +\infty\} .$$

The (total) variation $\operatorname{var}_a^b x$ on $\langle a, b \rangle$ for $x : \langle a, b \rangle \to R^n$ is defined, as usual, by $\sup \sum_i \|x(t_i) - x(t_{i-1})\|$, where the supremum is taken over all finite decompositions of $\langle a, b \rangle$ (similarly for $\operatorname{var}_a^b A$ if $A : \langle a, b \rangle \to \mathcal{L}(R^n \to R^n)$). V_n forms a Banach space with the norm $\|x\|_{V_n} = \|x(a)\| + \operatorname{var}_a^b x$.

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Let $K(s,t): \langle a,b\rangle \times \langle a,b\rangle = \mathcal{I} \to \mathcal{L}(R^n \to R^n)$ be given. For $\mathcal{J} = \langle \alpha, \beta\rangle \times \langle \gamma, \delta\rangle \subset \mathcal{I}$ we set

$m_K(\mathcal{J}) = K(\beta,\delta) - K(\beta,\gamma) - K(\alpha,\delta) + K(\alpha,\gamma) \in \mathcal{L}(R^n \to R^n)$

and define

$$v_{\mathcal{I}}(K) = \sup \sum_i \| m_K(\mathcal{J}_i)\| ,$$

where the supremum is taken over all finite systems of subintervals $\mathcal{J}_i \subset \mathcal{I}$ such that $\mathcal{J}_i^0 \cap \mathcal{J}_j^0 = \emptyset$ when $i \neq j$ ($\mathcal{J}_i^0$ is the interior of $\mathcal{J}_i$). The number $v_{\mathcal{I}}(K)$ is a kind of a twodimensional variation of the matrix function $K(s,t)$ in the interval $\mathcal{I}$. This notion of the variation is considered e.g. in the book [1] of T.H. Hildebrandt (for $n = 1$).

We consider the operator $K: V_n \longrightarrow V_n$ which is for $x \in V_n$ defined by the relation

(1) $$Kx = y ,$$

where

(2) $$y(s) = \int_a^b d_t [K(s,t)] x(t) =$$

$$= \left(\sum_{j=1}^{n} \int_a^b x_j(t) d_t [k_{1j}(s,t)], \ldots, \sum_{j=1}^{n} \int_a^b x_j(t) d_t [k_{nj}(s,t)]\right) .$$

All integrals used in this communication are the Perron-Stieltjes integrals. The following theorem holds:

<u>Theorem 1</u>. If $K(s,t): \mathcal{I} \to \mathcal{L}(R^n \to R^n)$ satisfies

(3) $$v_{\mathcal{I}}(K) < +\infty$$

and

(4) $$\operatorname{var}_a^b K(a,\cdot) < +\infty ,$$

then $K: V_n \longrightarrow V_n$ from (1) is a completely continuous operator.

<u>Remark</u>. In (4) $var_a^b K(a,\cdot)$ means the variation of $K(s,t)$ in the second variable for fixed $s = a$. Since we have

$$var_a^b K(s,\cdot) \le v_y(K) + var_a^b K(a,\cdot)$$

for any $s \in \langle a, b \rangle$, the integral $\int_a^b d_t[K(s,t)]x(t)$ exists for all $s \in \langle a, b \rangle$ and any $x \in V_m$. Further it is

$$\| K x \|_{V_m} \le (v_y(K) + var_a^b K(a,\cdot)) \| x \|_{V_m} \ .$$

Theorem 1 yields immediately a Fredholm type theorem for the Fredholm-Stieltjes integral equation (F.-S.i.e.)

$$(5) \qquad x(s) - \int_a^b d_t[K(s,t)]x(t) = \tilde{x}(s), \ \tilde{x} \in V_m$$

in the terms of the adjoint operator $K^* : V_m' \longrightarrow V_m'$. Unfortunately, we have no satisfactory description of the dual V_m' to V_m which would make it possible to derive the analytic form of K^*, Nevertheless a Fredholm type theorem for Eq. (5) can be proved, where the usual adjoint equation is substituted by an other one whose analytic form is known. This is based on the following

<u>Proposition</u>. Let X, Y be normed spaces with duals X', Y' respectively, and let $K: X \longrightarrow X$, $L: Y \longrightarrow Y$ be the completely continuous operators. Let $\langle x, y \rangle$ be a bilinear form on $X \times Y$ which separates the points of X and Y such that for $x \in X$, $y \in Y$ the inequality

$$|\langle x, y\rangle| \le c \cdot \|x\|_X \; \|y\|_Y \quad (c = const)$$

holds and let

$$\langle Kx, y\rangle = \langle x, Ly\rangle$$

for any $x \in X$, $y \in Y$. Then we have

$$dim\, T^{-1}(0) = dim\, T^{*-1}(0) = dim\, S^{-1}(0) = dim\, S^{*-1}(0) = n$$

and

$$T^{*-1}(0) \subset [Y] \subset X', \; S^{*-1}(0) \subset [X] \subset Y'$$

where n is a nonnegative integer, $T = I_X - K$, $S = I_Y - L$, $T^* = I_{X'} - K^*$, $S^* = I_{Y'} - L^*$, K^*, L^* are the adjoints to K, L respectively, I_X is the identity operator in X (similarly I_Y, $I_{X'}$, $I_{Y'}$), $T^{-1}(0)$ is the null-space of T and $[X]$ is the immersion of X into Y' given by the bilinear form $\langle x, y\rangle$ (similarly for $[Y]$).

This proposition is used to derive the following

<u>Theorem</u>. Let $K(s,t): \mathcal{J} \to \mathcal{L}(R^m \to R^m)$, $v_{\mathcal{J}}(K) < +\infty$, $var_a^b K(a,\cdot) < +\infty$, $var_a^b K(\cdot, a) < +\infty$.

Then either the F.-S.i.e. (5) admits a unique solution for any $\tilde{x} \in V_m$ or the homogeneous F.-S.i.e.

$$(6) \qquad x(s) - \int_a^b d_t [K(s,t)]\, x(t) = 0$$

admits n linearly independent solutions $x_1, x_2, \dots$ $\dots, x_n \in V_m$.

In the first case, the equation

$$(7) \qquad \varphi(t) - \int_a^b K'(s,t)\, d\varphi(s) = \tilde{\varphi}(t), \; \tilde{\varphi} \in V_m$$

has a solution for any $\varphi \in V_n$ (not necessarily unique). In the second case, Eq. (5) has a solution in V_n iff

$$\int_a^b \tilde{x}'(t)\, d\varphi(t) = \sum_{j=1}^{n} \int_a^b \tilde{x}_j(t)\, d\varphi_j(t) = 0$$

for any solution $\varphi \in V_n$ of the equation

$$\varphi(t) - \int_a^b K'(s,t)\, d\varphi(s) = 0$$

and symmetrically Eq. (7) has a solution iff

$$\int_a^b x'(t)\, d\tilde{\varphi}(t) = 0$$

for any solution $x \in V_n$ of Eq. (6).

Note that Eq. (7) is not the adjoint equation to (5).

The complete version of this work will appear in Časopis pro pěstování matematiky, 1972.

R e f e r e n c e s

[1] T.H. HILDEBRANDT: Introduction to the Theory of Integration. Academic Press,New York,London,1963.

[2] A.P. ROBERTSON, W. ROBERTSON: Topological Vector Spaces. Cambridge University Press,1964.

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