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THE NONEXISTENCE OF A WEAK SOLUTION OF DIRICHLET'S PROBLEM FOR THE FUNCTIONAL OF MINIMAL SURFACE ON NONCONVEX DOMAINS

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§ 1. Introduction. In this paper, I will be concerned with the problem if there exists the minimum of the functional

$$\Phi(u) = \int_{\Omega} \sqrt{1 + |\nabla u|^2}\, dx$$

on the set of functions $u \in W_1^{(1)}(\Omega)$, $\Omega \subset E_2$ with the boundary condition $\varphi \in C(\partial\Omega)$.

It is well known that we have the existence theorem for a classical solution of this problem only if the domain Ω is convex, for all nonconvex domains Ω we are able to find $\varphi \in C(\partial\Omega)$ such that there exists no classical solution of this problem ([1]). In this paper, it will be shown that the situation is different for weak solutions:

1) If almost all points of the boundary $\partial\Omega$ are convex points (Def. 2), then there exists a weak solution for all $\varphi \in C(\partial\Omega)$ (see § 3). An interesting situation is, for example, in the well known classical counterxample of T. Radó ([4],p.204). There exists a classical parametric

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solution, but this solution has no singlevalued projection onto (x, y)-plane. There exists a weak (nonparametric) solution $u \in W_1^{(1)}(\Omega)$ which is even from $C^{(2)}(\Omega) \cap C(\bar{\Omega} \setminus \{S\})$ (where S is the only nonconvex point of Ω). These two solutions are different. I mean that the Radon's example is in fact rather a counterexample of regularity of the solution on the boundary $\partial\Omega$ than the counterexample of the existence. On the other hand, the example of Bernstein ([4],p.201) is indeed a counterexample of the existence of the solution.

2) If the nonconvexity of the boundary $\partial\Omega$ is essential (for example, a part of the boundary is a part of the circle which has a positive one-dimensional Lebesgue measure), then we can find a boundary condition $\varphi \in C(\partial\Omega)$ such that there exists no weak solution of our problem (see § 2).

Remark. There is a possibility to extend the functional Φ on the larger space of functions, the space $W_{\mu}^{(1)}(\Omega) \supset \supset W_1^{(1)}(\Omega)$ and ask for

$$\min_{\substack{u \in W_{\mu}^{(1)}(\Omega) \\ tr\, u = \varphi}} \Phi(u) \ ; \quad \varphi \in L_1(\partial\Omega) \ .$$

Then we have an existence theorem for this ultraweak solution for each domain Ω (also nonconvex) with the $C^{(1)}$-boundary and for all $\varphi \in L_1(\partial\Omega)$ ([2],[3]).

§ 2. Nonexistence of the minimum.

Definition 1. Each function $u \in W_1^{(1)}(\Omega)$ for which

$$\Psi(u) = \min_{\substack{v \in W_1^{(1)}(\Omega) \\ v - u \in \mathring{W}_1^{(1)}(\Omega)}} \Phi(v)$$

holds, will be called a weak solution of our problem.

<u>Theorem 1</u>. Let $\Omega \in E_N$ be a bounded domain with the Lipschitz boundary, let u_1, u_2 be two weak solutions and let

$$u_1(x) \leq u_2(x)$$

a.e. on $\partial\Omega$ (in the sense of traces).

Then

$$u_1(x) \leq u_2(x) \quad \text{a.e. in } \Omega .$$

<u>Proof</u>. There exists a measurable set $\Omega_0 \subset \Omega$ such that

$$u_1(x) \leq u_2(x) \quad \text{a.e. in } \Omega \setminus \Omega_0 ,$$

$$u_1(x) > u_2(x) \quad \text{a.e. in } \Omega_0 .$$

We can define the functions

$$u_3(x) = \min [u_1(x), u_2(x)] ,$$

$$u_4(x) = \max [u_1(x), u_2(x)] ,$$

then from the Beppo-Levi definition of the space $W_1^{(1)}(\Omega)$ it follows that $u_3, u_4 \in W_1^{(1)}(\Omega)$.

From the inequality $u_1 \leq u_2$ a.e. in $\partial\Omega$ we have $u_3 = u_1$; $u_4 = u_2$ a.e. in $\partial\Omega$.

Then

$$\Phi(u_1) \leq \Phi(u_3) ,$$

i.e.

$$\int_{\Omega-\Omega_0}\sqrt{1+|\nabla u_1|^2}\,dx + \int_{\Omega_0}\sqrt{1+|\nabla u_1|^2}\,dx \leq$$
$$\leq \int_{\Omega-\Omega_0}\sqrt{1+|\nabla u_1|^2}\,dx + \int_{\Omega_0}\sqrt{1+|\nabla u_2|^2}\,dx$$

and

$$\Phi(u_2) \leq \Phi(u_4) ,$$

i.e.

$$\int_{\Omega-\Omega_0}\sqrt{1+|\nabla u_2|^2}\,dx + \int_{\Omega_0}\sqrt{1+|\nabla u_2|^2}\,dx \leq$$
$$\leq \int_{\Omega-\Omega_0}\sqrt{1+|\nabla u_2|^2}\,dx + \int_{\Omega_0}\sqrt{1+|\nabla u_1|^2}\,dx .$$

From this we obtain

$$\int_{\Omega_0}\sqrt{1+|\nabla u_1|^2}\,dx \leq \int_{\Omega_0}\sqrt{1+|\nabla u_2|^2}\,dx ,$$
$$\int_{\Omega_0}\sqrt{1+|\nabla u_2|^2}\,dx \leq \int_{\Omega_0}\sqrt{1+|\nabla u_1|^2}\,dx ,$$

hence

$$\Phi(u_3) = \Phi(u_1) .$$

The functional Φ is strictly convex on $\{u \in W_1^{(1)}(\Omega),$ $u - u_1 \in \mathring{W}_1^{(1)}\}$ hence

$$u_3(x) = u_1(x) \quad \text{a.e. in } \Omega .$$

<u>Lemma</u>. Let $u \in C^{(2)}(\Omega) \cap C(\overline{\Omega})$ be the classical solution of the equation of a minimal surface in $\Omega \subset$ $\subset E_2$. Then u is a weak solution (of our problem) over $W_1^{(1)}(\Omega)$.

<u>Proof</u>. If $u \in C^{(2)}(\overline{\Omega})$ then the assertion of this lemma holds because the functional Φ is continuous and convex on $W_1^{(1)}(\Omega)$. There exist the domains Ω_j ; $j = 1, 2, \ldots$ such that

$$\bar{\Omega}_j \subset \Omega \, ; \; \Omega_j \subset \Omega_{j+1} \, ; \; \bigcup_{j=1}^{\infty} \Omega_j = \Omega$$

and $\int_{\partial\Omega_j} dS$ are uniformly bounded.

The function $u|_{\Omega_j}$ is a classical solution in $W_1^{(1)}(\Omega_j)$, hence the apriori estimate (see [4])

$$\int_{\Omega_j} \sqrt{1+|\nabla u|^2}\,dx \leq \operatorname{meas}(\Omega_j) + \int_{\partial\Omega_j} |u|\,ds \leq K$$

is valid, where K is independent of j.

By the limit $j \longrightarrow \infty$ we obtain

$$\int_{\Omega} \sqrt{1+|\nabla u|^2}\,dx < +\infty ,$$

hence $u \in W_1^{(1)}(\Omega)$. Because Φ is continuous and convex on $W_1^{(1)}(\Omega)$, we have that u is a weak solution over $W_1^{(1)}(\Omega)$. Q.E.D.

If the function u_2 from Theorem 1 is a special auxiliary minimal surface, for example,

$$u_2(x) = -R \cdot \operatorname{arccosh} \frac{|x|}{R} \, , \; |x| \geq R \, , \; x \in E_2 \, ,$$

we can prove then another maximum principle; we can suppose that the inequality $u_1 \leq u_2$ holds only on some part of $\partial\Omega$. This allows us to construct a counterexample for some nonconvex domains Ω - to take such a boundary condition φ that there exists no weak solution of our problem.

__Theorem 2__. Let $\Omega \subset E_2$ be a bounded domain with the Lipschitz boundary, let $\Gamma(R_0)$ be a part of the circle $K(x_0, R_0) = , \{x \in E_2 \, , |x - x_0| = R_0\}$, let which have a positive one-dimensional Lebesgue measure, let all points Ω be outside of the circle $K(x_0, R_0)$ and

$$\partial\Omega \cap K(x_0, R_0) = \Gamma(R_0) \quad ;$$

let us suppose that there exists $d > 0$ such that the open set

$$\Omega_R = \{x \in \Omega \; ; \; |x - x_0| > R\}$$

is the domain with the Lipschitz boundary for all $R \in \in (R_0, R_0 + d)$.

Further let

(i) $u(x)$ be a weak solution over $W_1^{(1)}(\Omega)$,

(ii) $u(x) \leq - R_0 \operatorname{arccosh} \dfrac{|x - x_0|}{R_0}$ a.e. on $\partial\Omega - \Gamma(R_0)$,

(iii) there exists $c_0 > 0$ such that $|u(x)| \leq c_0$ a.e. on $\Gamma(R_0)$.

Then there holds $u(x) \leq 0$ a.e. on $\Gamma(R_0)$, hence

$$u(x) \leq - R_0 \operatorname{arccosh} \frac{|x - x_0|}{R_0} \quad \text{a.e. in } \Omega .$$

<u>Proof</u>. let us denote (for $R \in \langle R_0, R_0 + d \rangle$)

$\Gamma(R) = \{x \in \Omega \; ; \; x^2 + y^2 = R\}$,

$R_1 = \sup\{R \; ; \; \Gamma(R) \neq 0\}$

and

$$\varphi_R(t) = - R \operatorname{arc\,cosh} \frac{t}{R}$$

for all $t \geq R > 0$.

By Lemma the function

$$\eta(x) = \varphi_{R_0}(|x - x_0|) + c_0$$

is a weak solution over $W_1^1(\Omega)$, we have $u \leq \eta$

a.e. on $\partial\Omega$, hence from Theorem 1 it follows that

$$u(x) \leq \eta(x)$$

a.e. in Ω and we can define the real function

$$\Psi(R) = \sup_{x \in \Gamma(R)} \operatorname{ess} u_R(x) \; ; \; R_0 \leq R < R_0 + d \; ,$$

where $u_R(x)$ is the trace of $u|_{\Omega_R}$ on $\Gamma(R)$ $(\Gamma(R) \subset \partial\Omega_R)$.

It is sufficient to prove that

$$(1) \qquad u_{R_0}(x) \leq \varphi_{R_0}(|x - x_0|) = 0$$

holds a.e. in $\Gamma(R_0)$ and then to use Theorem 1.

Let us assume, on the contrary, that

$$\Psi(R_0) > 0 \; ;$$

then we can denote

$$\varepsilon = \frac{1}{2}\Psi(R_0) > 0 \, .$$

There exists $\delta > 0$, $\delta < d$ such that for all $R \in \langle R_0, R_0 + \delta \rangle$, $\rho \in \langle R, R_1 \rangle$

$$(2) \qquad \varphi_R(\rho) + \varepsilon \geq \varphi_{R_0}(\rho)$$

holds.

<u>Part I</u>: Let $\kappa_0 \in (R_0, R_0 + \delta)$ be fixed; we will prove that

$$\Psi(\kappa_0) \leq \varepsilon \quad .$$

1. From Lemma it follows that the function

$$\eta_0(x) = \varphi_{R_0}(|x - x_0|) + \psi_0 \; ; \quad \psi_0 = \Psi(R_0) > 0$$

is a weak solution over $W_1^1(\Omega)$, it is clear that

$$\eta_0(x) \geq u(x) \quad \text{a.e. on } \Gamma(R_0) \; ,$$

$$\eta_0(x) \geq \varphi_{R_0}(|x - x_0|) \geq u(x) \text{ a.e. on } \partial\Omega - \Gamma(R_0),$$

hence from Theorem 1 we have

$$\eta_0(x) \geq u(x) \quad \text{a.e. on } \Omega$$

and also

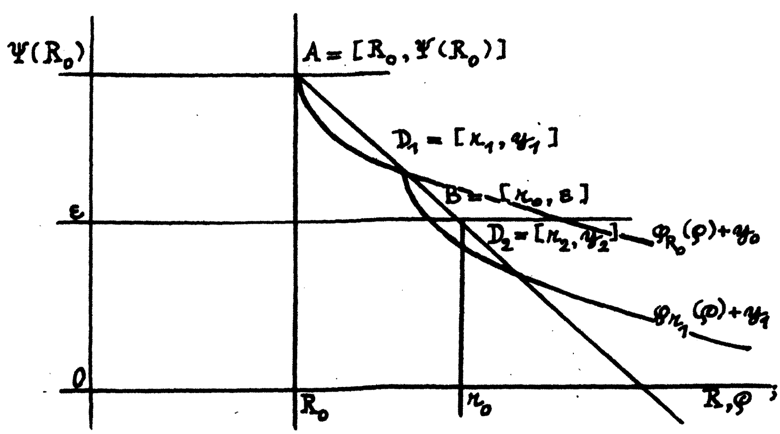
(3) $$\Psi(\varphi) \leq \varphi_{R_0}(\varphi) + \psi_0$$

for all $\varphi \in \langle R_0, R_1 \rangle$.

Because

$$\frac{d}{d\varphi}[\varphi_{R_0}(\varphi)] = - \frac{1}{\left(\frac{\varphi^2}{R_0^2} - 1\right)^{1/2}}$$

we have (see the figure)



the graph of the function $\varphi_{R_0}(\varphi) + \psi_0$ on $\langle R_0, \infty)$ is beginning in the point A, there exists $\kappa_1 > R_0$ such that the graph of the function $\varphi_{R_0}(\varphi) + \psi_0$ intersects the halfline $\overrightarrow{AB}$ in one point $D_1 = [\kappa_1, \psi_1]$, where

(4) $$\psi_1 = \varphi_{R_0}(\kappa_1) + \psi_0$$

and in (R_0, κ_1) the graph of the function $\varphi_{R_0}(\varphi) + \psi_0$

lies more below than the halfline $\overrightarrow{AB}$.

a) If $\kappa_1 \geq \kappa_0$, then clearly

$$\varphi_{R_0}(\kappa_0) + \psi_0 < \varepsilon$$

and from (3) it follows

$$\Psi(\kappa_0) \leq \varepsilon \quad ,$$

what we want to prove.

b) If $\kappa_1 < \kappa_0$, then we will do the second step.

2. The function

$$\eta_1(x) = \varphi_{\kappa_1}(|x - x_0|) + \psi_1$$

is by Lemma a weak solution over $W_1^1(\Omega_\kappa)$, from (3), (4) we have

$$\Psi(\kappa_1) \leq \varphi_{R_0}(\kappa_1) + \psi_0 = \psi_1 = \varphi_{\kappa_1}(\kappa_1) + \psi_1 \quad ,$$

i.e.

$$u(x) \leq \eta_1(x) \quad \text{a.e. on } \Gamma_{\kappa_1}$$

and by (2) we have

$$u(x) \leq \varphi_{R_0}(|x-x_0|) \leq \varphi_{\kappa_1}(|x-x_0|) + \varepsilon < \varphi_{\kappa_1}(|x-x_0|) + \psi_1$$

a.e. in $\{x \in \partial\Omega \,;\, |x - x_0| \geq \kappa_1\}$.

Hence from Theorem 1 we have

$$u(x) \leq \eta_1(x) \quad \text{a.e. in } \{x \in \Omega, |x - x_0| > \kappa_1\},$$

i.e.

$$(5) \quad \Psi(\rho) \leq \varphi_{\kappa_1}(\rho) + \psi_1 \,, \quad \rho \in \langle \kappa_1, R_1 \rangle \ .$$

Again, the graph of the function $\varphi_{\kappa_1}(\rho) + \psi_1$ is beginning in the point D_1 and there exists $\kappa_2 > \kappa_1$ such that the graph of the function $\varphi_{\kappa_1}(\rho) + \psi_1$ intersects the halfline $\overrightarrow{D_1 B}$ in the point $D_2 \equiv [\kappa_2, \psi_2]$, where

$$\psi_2 = \varphi_{\kappa_1}(\kappa_2) + \psi_1 \ .$$

For $\varphi \in (\kappa_1, \kappa_2)$ the graph of $\varphi_{\kappa_1}(\varphi) + \psi_1$ lies more below than the halfline $\overrightarrow{D_1 B}$.

a) If $\kappa_2 \geq \kappa_0$, then

$$\varphi_{\kappa_1}(\kappa_0) + \psi_1 < \varepsilon$$

and from (5)

$$\Psi(\kappa_0) < \varepsilon \quad ,$$

what we need.

b) If $\kappa_2 < \kappa_0$, we can continue, we can do further steps, but because

$$\frac{d}{d\varphi} \varphi_R(\varphi) = - \frac{1}{\sqrt{\frac{\varphi^2}{R^2} - 1}} \quad ,$$

there must exist $\Delta > 0$ such that for all

$R \in \langle R_0, \kappa_0 \rangle$, $\varphi \in \langle R, R + \Delta \rangle$

$$\frac{1}{\sqrt{\frac{\varphi^2}{R^2} - 1}} > \frac{\varepsilon}{\kappa_0 - R_0}$$

holds. Because $- \dfrac{\varepsilon}{\kappa_0 - R_0}$ is the direction of the half-lines $\overrightarrow{AB}$, $\overrightarrow{D_1 B}, \dots$, it is clear now that the numbers

$$\Psi(R_0) - \psi_1 \; ; \; \psi_2 - \psi_1 \; ; \; \psi_3 - \psi_2 \; \dots$$

are bounded below by the number Δ. Hence after a finite number of the same steps we obtain $\kappa_j \geq \kappa_0$ and

$$\Psi(\kappa_0) \leq \varepsilon \quad .$$

Part II. Because $\Psi(R) \leq \varepsilon$ for all $R \in (R_0, R_0 + \delta)$, it follows from Theorem 1 that

$$u(x) \leq \varepsilon \quad \text{a.e. in } \Omega$$

and hence

$$u(x) \leq \varepsilon = \frac{\Psi(R_0)}{2} \quad \text{a.e. in } \partial\Omega \quad ,$$

which is a contradiction with the definition of $\Psi(R_0)$.

Example. Let Ω be the domain from Theorem 2, let us consider $\varphi(x) \in C(\partial\Omega)$ such that:

(i) $\varphi(x) = -R_0 \operatorname{arccosh} \dfrac{|x - x_0|}{R_0}$ for all $x \in \partial\Omega - \Gamma(R_0)$,

(ii) $\max\limits_{x \in \Gamma(R_0)} \varphi(x) > 0$.

If there exists $u \in W_1^1(\Omega)$ such that

$$\Phi(u) = \min_{\substack{v \in W_1^1(\Omega)\\ tr\, v = \varphi}} \Phi(v), \quad tr\, u = \varphi \quad ,$$

then by means of Theorem 2 we have

$$tr\, u(x) \leq -R_0 \operatorname{arccosh} \frac{|x-x_0|}{R_0} = 0 \text{ a.e. in } \Gamma(R_0)$$

which is a contradiction with $u = \varphi$ a.e. on $\Gamma(R_0)$. So there exists no weak solution $u \in W_1^1(\Omega)$ with this boundary condition φ.

Remark. I think that Theorem 2 can be proved for more kinds of domains which contain the part of ellipse, parabola, cycloida and so on in the nonconvex part of the boundary. For these kinds of curve there exist similar auxiliary functions which we need to prove Theorem 2 (see [4],p.202). Hence some counterexamples can be constructed for this kind of domains, too. I mean that form of non-convexity of the domain is not important, only the non-convexity of the domain must be "essential", i.e. a part of any curve must be contained in the nonconvex part of the boundary Ω.

§ 3. The existence theorem for weak solution.

Definition 2. Let $\Omega \subset E_2$ be a bounded domain with the Lipschitz boundary. We say that $x \in \partial\Omega$ is a convex point of boundary, if there exists a neighborhood $\mathcal{U}(x)$ such that $\mathcal{U}(x) \cap \Omega$ is a convex set.

Theorem 3. Let $\Omega \subset E_2$ be a bounded domain with the Lipschitz boundary. Let almost all points of $\partial\Omega$ be convex points of the boundary, let $\varphi \in C(\partial\Omega)$.

Then there exists the point of a minimum of Φ on

$$\{ u \in W_1^1(\Omega) ; u = \varphi \text{ on } \partial\Omega \}$$

and in fact $u \in C^2(\Omega)$.

Proof. Let A be a set of all points of $\partial\Omega$ which are not convex points of boundary. In Serrin's paper ([5]) it is proved by the Perron's method of subfunctions that there exists $u \in C^2(\Omega)$ such that

(i) u is a solution of the equation of minimal surface in Ω ,

(ii) $u \in C(\bar{\Omega} - A)$,

(iii) $u = \varphi$ for all $x \in \partial\Omega - A$.

So I need to prove only:

1) $u \in W_1^1(\Omega)$ and it is a weak solution over $W_1^1(\Omega)$,

2) $u = \varphi$ a.e. on $\partial\Omega$ in the sense of traces.

1) Because $|\varphi| \leq C$ in $\partial\Omega$, we can see from the Perron construction that

$$|u| \leq C \quad \text{in } \Omega .$$

The next part of the proof is the same as the proof of Lemma.

2) If $x \in \partial\Omega$ is a convex point of boundary, there exists the neighborhood $\mathcal{U}(x)$ such that

$$u \in C(\mathcal{U} \cap \overline{\Omega})\,;\; u = \varphi \qquad \text{on } \mathcal{U} \cap \partial\Omega\,,$$

hence

$$tr\, u = \varphi \quad \text{on } \mathcal{U} \cap \partial\Omega\,.$$

We have then

$$tr\, u = \varphi \quad \text{for all } x \in \partial\Omega - A\,.$$

From $u \in W_1^1(\Omega)$ it follows that $tr\, u \in L_1(\partial\Omega)$ and

$$tr\, u = \varphi \quad \text{in } L_1(\partial\Omega)\,.$$

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