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PRODUCTS OF QUOTIENTS AND OF k'-SPACES

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It follows from results of Whitehead [11], Cohen [3] and Michael 6 that a regular space P is locally compact if and only if $1_P \times q$ is a quotient map for any quotient map q and if and only if $P \times Q$ is a k-space for any k-space Q. We shall try to give here similar results concerning k'-spaces instead of k-spaces (Theorems 1 and 2).

The k'-spaces are usually defined as uniformizable spaces (the terminology of [2] will be used throughout this paper) continuous real-valued functions of which coincide with the functions continuous on compact subsets (i.e., uniformizable spaces for which the set of all real-valued continuous functions is complete in the uniformity of uniform convergence on compact sets (see [10]). Evidently, k'-spaces are just uniformizable modifications of k-spaces, hence, uniformizable quotients of sums of compact regular spaces (thus each mapping on a k'-space into a uniformizable space being continuous on compact subsets is continuous). It is clear that the category of k'-spaces is coreflective in the category of uniformizable spaces. If P is a uniformizable space, then its coreflection (k-modification)

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is the space projectively generated by all real-valued functions on P continuous on compact subsets (i.e., the uniformizable modification of $\mathcal{k}$-modification of P). We prefer the term " $\mathcal{k}'$-space" (see e.g. [4]) to " $\mathcal{k}_R$-space" (e.g. in [5],[9]).

Without loss of generality we confine ourselves to uniformizable spaces in the sequel; moreover, we shall confine our investigation to the category of uniformizable spaces (therefore, quotients of spaces are uniformizable modifications of quotients formed in the category of all topological spaces).

First we shall prove an auxiliary proposition being itself of some interest. By a mapping μ we mean the canonical bijection on the set $M(P \times Q)$ of all mappings defined on $P \times Q$ onto the set $M(P, M(Q))$ of all mappings on P onto $M(Q)$. The function spaces considered are endowed with compact-open topologies.

Proposition. A product $P \times Q$ is a $\mathcal{k}'$-space if and only if P and Q are $\mathcal{k}'$-spaces and $\mu[C(P \times Q)] = C(P, C(Q))$.

Proof. It was shown by Brown in [1] that if P, Q are $\mathcal{k}'$-spaces, then $\mu[C(\mathcal{k}'(P \times Q))] = C(P, C(Q))$, where $\mathcal{k}'(P \times Q)$ denotes the $\mathcal{k}'$-modification of $P \times Q$ (the same result for $\mathcal{k}$-spaces was proved by Morita in [7]). If $P \times Q$ is a $\mathcal{k}'$-space then, evidently, P and Q are $\mathcal{k}'$-spaces, $\mathcal{k}'(P \times Q) = P \times Q$ and, hence by the Brown's theorem, $\mu[C(P \times Q)] = C(P, C(Q))$. Conversely, if P, Q are $\mathcal{k}'$-spaces and $\mu[C(P \times Q)] = C(P, C(Q))$, then, again by the Brown's theorem,

$\mathcal{C}(P \times Q) = \mathcal{C}(k'(P \times Q))$ and consequently $P \times Q = k'(P \times Q)$. The proof is complete.

Before stating the main theorem we recall the concept of relatively pseudocompact subsets: a subset A of a space P is said to be relatively pseudocompact if any continuous real-valued function on P is bounded on A, i.e. the restriction of the uniformity on P projectively generated by all continuous functions on P to the subset A is totally bounded (or equivalently, the closure of A in the Hewitt realcompactification υP of P is compact). Noble in [8] has defined this concept by the following property: if $\mathcal{a}$ is a disjoint locally finite family of open sets in P, then only finitely many members of $\mathcal{a}$ meet $\mathcal{a}$ (of course, the word "disjoint" is inessential here). It can be easily proved that A is relatively pseudocompact in P if and only if the restriction of the fine uniformity on P to the subset A is totally bounded (i.e., each uniformizable covering of P has a uniformizable refinement, only finite number of its elements meet A). Thus in paracompact spaces, relatively pseudocompact subsets coincide with relatively compact subsets. As usual, a space P is defined to be locally relatively pseudocompact if each of its points has a neighborhood which is relatively pseudocompact in P.

<u>Theorem 1</u>. Let P be a space. Then $P \times Q$ is a k'-space for any k'-space Q (or any sequential space Q with a unique cluster point) if and only if P is a k'-space which is locally relatively pseudocompact.

Proof. Let $\mathcal{P}$ be a k'-space being locally relatively pseudocompact at the same time and Q be a k'-space. We wish to prove that $P \times Q$ is a k'-space, i.e. by the Brown's theorem mentioned in the proof of our Proposition, that $\mu[C(P \times Q)] = C(P, C(Q))$.

In any case $\mu[C(P \times Q)] \subset C(P, C(Q))$. Assume that $f \in C(P, C(Q))$; we are to prove that $\mu^{-1}[f] \in C(P \times Q)$. It suffices to show that for each relatively pseudocompact subset U in P the function $\mu^{-1}[f]$ is continuous on $U \times Q$. Since f is continuous on P into $C(Q)$ and $C(Q)$ has a complete uniformity, the closure $\overline{f[U]}$ in $C(Q)$ is compact and, hence, f can be continuously extended onto βU into $C(Q)$. Since $\mu[C(\beta U \times Q)] = C(\beta U, C(Q))$ and $C(\beta U \times Q) \subset C(U \times Q)$, the mapping $\mu^{-1}[f]$ must be continuous on $U \times Q$. The proof of sufficiency is complete.

Assume that P is a space which is not locally relatively pseudocompact at a point $x_0 \in P$. We shall find a paracompact sequential space (hence a k'-space) Q with just one cluster point such that $P \times Q$ is not a k'-space. The construction is similar to that of Michael in [6]. Let $\{U_i \mid i \in I\}$ be a base of neighborhoods of x_0 in P. Since no U_i is relatively pseudocompact in P there exist discrete families $\{V_n^i \mid n \in N\}$, $i \in I$, in P such that $V_n^i \cap U_i \neq \emptyset$ for every n and every i. Take for Q a quotient of $I \times T_{\omega_0+1}$ (I has the discrete topology and T_{ω_0+1} the order-topology) along a mapping g which is one-to-one on $I \times T_{\omega_0}$

and constant with a value c on $I \times (\omega_0)$. For each pair $\langle i, m \rangle \in I \times N$ choose a point $x_{i,m} \in V_m^i \cap U_i$. Then for any i there is a continuous function $f_i : P \times T_{\omega_0+1} \to [0,1]$ such that $f_i[P \times (\omega_0)] = (0)$ and $f_i \langle x_{i,m}, m \rangle = 1$ for each $m \in N$. The corresponding function $f : P \times Q \to [0,1]$ $(f \circ g = \Sigma\{f_i\})$ is not continuous but it is continuous on any compact subset of $P \times Q$. Indeed, if K is a compact subset of $P \times Q$, then $(1_P \times g)^{-1}[K]$ meets at most finitely many copies of $P \times T_{\omega_0}$; consequently, the restriction of $1_P \times g$ to $(1_P \times g)^{-1}[K]$ is a quotient mapping and hence f is continuous on K. The mapping f is not continuous on $P \times Q$ because $\langle x_0, c \rangle$ is a cluster point of the set $X = \{\langle x_{i,m}, m \rangle \mid i \in I, m \in N\}$ and f is 0 at $\langle x_0, c \rangle$ and 1 on X. This concludes the proof.

We have proved more in the second part of the preceding proof. Since f is not continuous on $P \times Q$ and its composition with g is continuous, the product-mapping $1_P \times g$ is not quotient. Thus we have one half of the following

<u>Theorem 2</u>. A space P is locally relatively pseudocompact if and only if for any quotient mapping g onto a $\mathscr{k}'$-space (or onto a sequential space with a unique cluster point) the product $1_P \times g$ is quotient.

<u>Proof</u>. It remains to prove that if P is locally relatively pseudocompact and $g : Q' \to Q$ is a quotient mapping onto a $\mathscr{k}'$-space, then $1_P \times g$ is a quotient mapping; this assertion is equivalent to the following one:

if f is a real-valued function defined on $P \times Q$ such that $f \circ (1_P \times q)$ is continuous then f is continuous, too. Since $f \circ (1_P \times Q)$ is continuous, $\mu(f \circ (1_P \times q)) \in C(P, C(Q'))$ and, consequently, $\mu f \in C(P, C(Q))$. Now, we can infer from the completeness of $C(Q)$ and the local relative pseudocompactness of P that $f \in C(P \times Q)$ in the same way as in the first part of the proof of Theorem 1.

By means of the preceding theorem and the Michael's theorem quoted in the very beginning of this paper, we can construct many examples of mappings between k'-spaces which are quotient in the category of uniformizable spaces and are not quotient in the category of all topological spaces. Indeed, if P is locally relatively pseudocompact, then $1_P \times q$ is quotient for any quotient mapping q; if moreover P is not locally compact, then there is a quotient mapping q between paracompact k'-spaces such that $1_P \times q$ is not quotient in the category of all topological spaces[6] - hence, for any locally relatively pseudocompact space P which is not locally compact there is a quotient mapping q between k'-spaces such that $1_P \times q$ is quotient in the category of uniformizable spaces and not in the category of all topological spaces. A nice class of such spaces P (being k'-spaces in addition) was constructed by Noble in [9]: if X is a space then, for α large enough, the space $P = \beta X \times T_{\omega_\alpha + 1} - ((\beta X - X) \times (\omega_\alpha))$ is a pseudocompact k'-space which is not locally compact whenever X is not locally compact (X is a closed subspace of P).

As a corollary of Theorem 2 we get

Corollary. If f and g are quotient mappings onto k'-spaces and such that the domain of g and the range of f are locally relatively pseudocompact, then $f \times g$ is a quotient mapping.

Proof. It suffices to realize that $f \times g = (1_P \times g) \circ (f \times 1_Q)$, where P is the range of f and Q is the domain of g, and that any composition of quotient mappings is quotient.

R e f e r e n c e s

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