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ON SPECTRA OF NONLINEAR OPERATORS

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Introduction. In the present paper, some properties of spectra of nonlinear operators are studied. Let $A: X \to X$ be a nonlinear operator on a complex Banach space X such that $A(0) = 0$. A complex number λ is called an eigenvalue of the operator A if there is a point $x_\lambda \in X$, $x_\lambda \neq 0$ such that $A(x_\lambda) = \lambda x_\lambda$. Some authors consider the spectrum of the operator A as a set of its eigenvalues. In this sense, the spectrum has been studied by Němyckij [1], Krasnoselskij [3], Vajnberg [4] and others. Neuberger defines (in [2]), at first, the resolvent as follows. A complex number λ is called a point of resolvent of the operator A if there is a Fréchet differentiable operator $(\lambda I - A)^{-1}$ (I is the identity operator on X) satisfying the Lipschitz's condition locally on X. A complex number λ which is not a point of the resolvent is called the point of spectrum of the operator A. We can find a similar definition of the spectrum in [5], but, instead of the assumption on Fréchet differentiability, the author requests the Lipschitz's condition on X.

This paper is divided into three sections. In the first one, we give a general definition of a spectrum with respect to a given set in X and show some properties of this spectrum. In Section two, sufficient conditions for the existence of the spectrum are given. Section three deals with homogeneous operators on a Hilbert space. Some conditions are shown for a symmetric operator to have merely a real spectrum and boundaries of this spectrum are determined. Let us remark that some of our results are related to the results declared by Kačurovskij in [5] (but without proofs).

1. Definition and properties of a spectrum of nonlinear operator with respect to a given set

In this section, let X, Y denote complex Banach spaces and let C be the space of complex numbers.

Definition 1.1: Let $G: X \times C \longrightarrow Y$ be an operator such that $G(0,0) = 0$. Let $M \subset X$ be a given non-empty set. We shall say that $\lambda \in C$ is a point of the spectrum of the operator G with respect to M if there is a sequence $\{x_n\} \in M$, $x_n \neq 0, n = 1, 2, \ldots$ such that

$$\lim_{n \to \infty} \| G(x_n, \lambda) \| = 0 .$$

Let us denote $\mathcal{S}_G(M)$ the set of all points of the spectrum of the operator G with respect to M. The set $\mathcal{S}_G(M)$ is called the spectrum of the operator G with respect to M. We shall say that $\lambda_0 \in C$, $\lambda_0 \neq 0$ is the eigenvalue of the operator G with respect to M if there is an element $x_0 \in M$, $x_0 \neq 0$ such that $G(x_0, \lambda_0) = 0$. The element x_0 is called the eigenvector of the operator G with respect to M (corresponding to the eigenvalue λ_0).

Remark 1.2: Every eigenvalue of the operator G with respect to M belongs to $\mathcal{S}_G(M)$. If $G(x,\lambda) = S(x) - \lambda T(x)$, where $S, T: X \longrightarrow Y$, $\lambda \in C$, then the set

$\mathcal{S}_{S,T}(M) \equiv \mathcal{S}_G(M)$ is called the spectrum of the couple (S, T) with respect to M and the eigenvalues of the operator G with respect to M are called the eigenvalues of the couple (S, T) with respect to M. (In case $X = Y, T = I$ the eigenvalues of the couple (S, I) with respect to X are the eigenvalues of the operator S in the usual sense.) The spectrum "with respect to M" can be useful in the problems of solving equations of the form $G(x, \lambda) = 0$ whose solutions are subjected to some other conditions represented by a given set M.

Proposition 1.3: Let $G: X \times C \longrightarrow Y$ be an operator, $M \subset X$, $N \subset X$, $M_k \subset X$, $k = 1, 2, \dots$ be non-empty sets. Then the following assertions hold:

a) If $M \subset N$, then $\mathcal{S}_G(M) \subset \mathcal{S}_G(N)$.

b) If $M \cap N \neq 0$, then $\mathcal{S}_G(M \cap N) \subseteq \mathcal{S}_G(M) \cap \mathcal{S}_G(N)$.

c) $\mathcal{S}_G(\bigcup_{k=1}^{\infty} M_k) = \bigcup_{k=1}^{\infty} \mathcal{S}_G(M_k)$.

The proof is evident.

We assume further that $M \subset X$ is a given non-empty set and $S, T: X \longrightarrow Y$ are operators such that $S^{-1}(0) \cap T^{-1}(0) \cap M \subseteq \{0\}$.

Proposition 1.4: Let T be a bounded operator on X (i.e., T maps bounded sets in X onto bounded sets in Y). Then it holds:

a) If M is a bounded set in X, then $\mathcal{S}_{S,T}(M)$ is closed in C.

b) If M is an arbitrary set, then $\mathcal{S}_{S,T}(M)$ is a F_σ-set.

Proof: Let M be a bounded set and let $\{\lambda_k\} \in \mathcal{S}_{S,T}(M)$ be a sequence such that $\lambda_k \to \lambda_o$ as $k \to \infty$. Then there is, for any $k = 1, 2, \dots$, a sequence $\{x_n^{(k)}\} \in M$ such that $\lim_{n\to\infty} \| S(x_n^{(k)}) - \lambda_k T(x_n^{(k)}) \| = 0$. If we choose the "diagonal" sequence $\{y_n\} = x_n^{(n)}$, then it holds:

$$\| S(y_n) - \lambda_o T(y_n) \| \leq \| S(y_n) - \lambda_n T(y_n) \| + \| T(y_n) \| \cdot |\lambda_n - \lambda_o| ,$$

hence

$\lim_{n\to\infty} \| S(y_n) - \lambda_o T(y_n) \| = 0$, that is $\lambda_o \in \mathcal{S}_{S,T}(M)$ and the assertion a) is proved. If M is an arbitrary set and n_o the smallest natural number such that $K_{n_o} \cap M \neq 0$, where $K_n = \{x \in X / \| x \| \leq n\}$, then, using Proposition 1.3 c), we obtain

$$\bigcup_{n=n_o}^{\infty} \mathcal{S}_{S,T}(K_n \cap M) = \mathcal{S}_{S,T}\Big[\bigcup_{n=n_o}^{\infty} (K_n \cap M)\Big] = \mathcal{S}_{S,T}(M) .$$

Thus, according to Proposition 1.4 a), $\mathcal{S}_{S,T}(M)$ is a F_σ-set.

Proposition 1.5: Let $M \subset X$ be a bounded set, $S, T : X \to Y$ bounded operators and let $dist(T(M), \{0\}) = d > 0$. Then $\mathcal{S}_{S,T}(M)$ is a compact set in C.

Proof: According to Proposition 1.4 a) $\mathcal{S}_{S,T}(M)$ is closed. We show that $\mathcal{S}_{S,T}(M)$ is a bounded set. Assume, on the contrary, that $\mathcal{S}_{S,T}(M)$ is not bounded. Then for any $K > 0$ there is $\lambda \in \mathcal{S}_{S,T}(M)$ such that $|\lambda| > K$. Denote $\| S \|_M = \sup_{x \in M} \| S(x) \|$ and let $K = \frac{\| S \|_M + 1}{d}$. According to Definition 1.1, there is a sequence $\{x_n\} \in M$ such that $\lim_{n\to\infty} \| S(x_n) - \lambda T(x_n) \| = 0$. But

$$\| S(x_n) - \lambda T(x_n) \| \geq |\lambda| \cdot \| T(x_n) \| - \| S(x_n) \| \geq K \cdot d - \| S \|_M = 1$$

and we come to a contradiction which completes the proof.

Proposition 1.6: Let $M \subset X$ be a non-empty set such that $0 \notin M$ and let $S, T : X \to Y$ be a couple of operators. Then the following assertions hold: If M is a compact and closed (weakly compact and weakly closed) set and the operators S, T are continuous (strongly continuous), then any non-zero element of $\mathcal{S}_{S,T}(M)$ is an eigenvalue of the couple (S,T) with respect to M.

Proof: Let $\lambda_0 \in \mathcal{S}_{S,T}(M)$, $\lambda_0 \neq 0$. Then there is a sequence $\{x_n\} \in M$ such that $\lim_{n \to \infty} \| S(x_n) - \lambda_0 T(x_n) \| = 0$. Using compactness (weak compactness) of M we can choose a subsequence $\{x_{n_k}\}$ which converges (weakly converges) to $x_0 \in M$, $x_0 \neq 0$. Now, according to the triangular inequality, we obtain

$\| S(x_0) - \lambda_0 T(x_0) \| \leq \| S(x_0) - S(x_{n_k}) \| + \| S(x_{n_k}) - \lambda_0 T(x_{n_k}) \| + |\lambda_0| \cdot \| T(x_{n_k}) - T(x_0) \|$. But $\lim_{k \to \infty} \| S(x_0) - S(x_{n_k}) \| = \lim_{k \to \infty} \| T(x_0) - T(x_{n_k}) \| = 0$ because S, T are continuous (strongly continuous) and thus $\| S(x_0) - \lambda_0 T(x_0) \| = 0$. Hence, λ_0 is an eigenvalue of the couple (S, T) with respect to M .

Proposition 1.7: Let $M \subset X$ be a non-empty set and let $S, T: X \longrightarrow Y$ be positive homogeneous operators of the order α, β (i.e., there are $\alpha, \beta > 0$ such that $S(t \cdot x) = t^{\alpha} S(x)$, $T(t \cdot x) = t^{\beta} T(x)$ for any $t > 0$ and any $x \in X$). Then $\mathcal{S}_{S,T}(t \cdot M) = t^{\alpha - \beta} \mathcal{S}_{S,T}(M)$ for any positive real number t .

Proof: If $\lambda \in \mathcal{S}_{S,T}(t \cdot M)$, then there is a sequence $\{x_n\} \in M$ such that $\lim_{n \to \infty} \| S(t \cdot x_n) - \lambda T(t \cdot x_n) \| = 0$. But $\| S(t \cdot x_n) - \lambda T(t \cdot x_n) \| = \| t^{\alpha} S(x_n) - \lambda t^{\beta} T(x_n) \|$ and thus $\lim_{n \to \infty} \| S(x_n) - \lambda t^{\beta - \alpha} T(x_n) \| = 0$. We see that $\lambda t^{\beta - \alpha} \in \mathcal{S}_{S,T}(M)$. Assume, on the contrary, that

$\mu \in t^{\alpha-\beta} \cdot \mathcal{S}_{S,T}(M)$. Then there is $\tilde{\chi} \in \mathcal{S}_{S,T}(M)$ and a sequence $\{\tilde{x}_n\} \in M$ such that $\mu = t^{\alpha-\beta} \cdot \tilde{\chi}$ and $\lim_{n\to\infty} \|S(\tilde{x}_n) - \tilde{\chi}\,T(\tilde{x}_n)\| = 0$. It follows that $\lim_{n\to\infty} \|S(\tilde{x}_n) - \mu t^{\beta-\alpha}\, T(\tilde{x}_n)\| = 0$ and also $\lim_{n\to\infty} \|S(t\tilde{x}_n) - \mu\, T(t\tilde{x}_n)\| = 0$, hence $\mu \in \mathcal{S}_{S,T}(t \cdot M)$.

<u>Remark 1.8</u>: The point $\lambda = 0$ need not generally belong to $\mathcal{S}_{S,T}(M)$. But if at least one of the following conditions a), b) holds:

a) $S^{-1}(0) \cap M$ contains a point $x_0 \neq 0$;

b) $0 \notin M$, $\operatorname{dist}(S^{-1}(0), M) = 0$ and S is a Lipschitzian operator;

then $0 \in \mathcal{S}_{S,T}(M)$.

Indeed: If the condition a) is satisfied, then for $x_n = x_0$, $n = 1, 2, \dots$, we have $\|S(x_n) - 0 \cdot T(x_n)\| = 0$ and thus $0 \in \mathcal{S}_{S,T}(M)$. If the condition b) is satisfied, then there are sequences $x_n \in M$, $y_n \in S^{-1}(0)$ such that $\lim_{n\to\infty} \|x_n - y_n\| = 0$. Finally, we obtain
$$\lim_{n\to\infty} \|S(x_n) - 0 \cdot T(x_n)\| = \lim_{n\to\infty} \|S(x_n) - S(y_n)\| \leq K \cdot \lim_{n\to\infty} \|x_n - y_n\| = 0,$$
where $K > 0$ is a constant. Therefore $0 \in \mathcal{S}_{S,T}(M)$.

<u>Remark 1.9</u>: Let $G: X \times C \to Y$, $G(0, \lambda) = 0$, $\lambda \in C$ be a Lipschitzian operator with respect to the variable λ in some neighbourhood $\mathcal{U}_0 \times \Lambda$ of a bifurcation point $(0, \lambda_0)$ (i.e., $\|G(x,\lambda) - G(x,\mu)\| \leq K(x)\,|\lambda - \mu|$ for any $x \in \mathcal{U}_0$, $\lambda, \mu \in \Lambda$, where $K(x)$ is a bounded functional on $\mathcal{U}_0$). Then $\lambda_0 \in \mathcal{S}_G(\mathcal{U})$ for any sufficiently small neighbourhood $\mathcal{U}$ of the point $0 \in X$.

In fact: There are sequences $\{x_n\} \in X$, $x_n \neq 0$, $\lambda_n \in C$ such that $\lim_{n\to\infty} \|x_n\| = 0$, $\lim_{n\to\infty} \lambda_n = \lambda_0$ and

$G(x_n, \lambda_n) = 0$. Hence, for any sufficiently small neighbourhood $\mathcal{U}$ of the point $0 \in X$, there is a sequence $\{\tilde{x}_n\} \in \mathcal{U}$, $\tilde{x}_n \neq 0$ such that $G(\tilde{x}_n, \lambda_n) = 0$ and

$$\|G(\tilde{x}_n, \lambda_0)\| = \|G(\tilde{x}_n, \lambda_0) - G(\tilde{x}_n, \lambda_n)\| \leqq K(\tilde{x}_n)|\lambda_n - \lambda_0| .$$

That is, $\lim_{n \to \infty} \|G(\tilde{x}_n, \lambda_0)\| = 0$ and thus $\lambda_0 \in \mathcal{S}_G(\mathcal{U})$.

Corollary 1.10: Let $S, T : X \longrightarrow Y$ be operators such that $S(0) = T(0) = 0$ and let T be bounded pn some neighbourhood $\mathcal{U}_0$ of the point $0 \in X$. Then any bifurcation point of the couple (S, T) (with respect to zero) belongs to the spectrum $\mathcal{S}_{S,T}(\mathcal{U})$ with respect to any sufficiently small neighbourhood $\mathcal{U}$ of the point $0 \in X$.

Proposition 1.11: Let $S, T : X \longrightarrow Y$ be positive homogeneous operators of the order $\alpha > 0$ defined and strongly continuous in a reflexive Banach space X. Let $M \subset X$ be a bounded closed convex set such that $0 \notin M$. Then any non-zero point of the spectrum $\mathcal{S}_{S,T}(M)$ of the couple (S, T) with respect to M is a bifurcation point of the couple (S, T). Further, any bifurcation point of the couple (S, T) belongs to the spectrum $\mathcal{S}_{S,T}(S_1)$ of the couple (S, T) with respect to the unit sphere $S_1 = \{x \in X / \|x\| = 1\}$.

Proof: Let $0 \neq \lambda_0 \in \mathcal{S}_{S,T}(M)$. Then, according to Proposition 1.7, it follows that $\mathcal{S}_{S,T}(t.M) = \mathcal{S}_{S,T}(M)$ for any $t > 0$. Choose a sequence of positive real numbers t_n such that $\lim_{n \to \infty} t_n = 0$. Then $\lambda_0 \in \mathcal{S}_{S,T}(t_n . M)$, $n = 1, 2, \dots$ and, according to Proposition 1.6, λ_0

is an eigenvalue of the couple (S,T) with respect to M Let $x_0 \in M$ be an eigenvector corresponding to λ_0. Denoting $x_n = t_n x_0$, we see that $\lim_{n\to\infty} \|x_n\| = 0$ and $S(x_n) - \lambda_0 T(x_n) = t_n^{\alpha}(S(x_0) - \lambda_0 T(x_0)) = 0$. Therefore, x_n are eigenvectors of the couple (S,T) and λ_0 is the bifurcation point. On the other hand, if μ_0 is a bifurcation point of the couple (S,T), then there is a sequence $\{\mu_n\}$ of eigenvalues with eigenvectors x_n such that $\lim_{n\to\infty} \mu_n = \mu_0$ and $\lim_{n\to\infty} \|x_n\| = 0$. If we put $\tilde{x}_n = \frac{x_n}{\|x_n\|}$ then $\tilde{x}_n \in S_1$ and $\tilde{x}_n$ are also eigenvectors of the couple (S,T) corresponding to the eigenvalues μ_n. According to Proposition 1.4, the set $\mathcal{S}_{S,T}(S_1)$ is closed and thus $\mu_0 \in \mathcal{S}_{S,T}(S_1)$.

2. The existence of a spectrum of the couple (S, T) of bounded operators

In this section, let X denote a Banach space, Y a Hilbert space and let $(\cdot,\cdot)$ denote the inner product in Y.

Theorem 2.1: Let $S,T: X \longrightarrow Y$ be bounded operators such that $S(0) = T(0) = 0$ and let $M \subset X$ be a bounded set. Let, further, the following condition hold:

$$(p_1) \qquad 0 < \sup_{x \in M} |(S(x), T(x))| = \|S\|_M \cdot \|T\|_M ,$$

where $\|S\|_M = \sup_{x\in M} \|S(x)\|$; $\|T\|_M = \sup_{x \in M} \|T(x)\|$.

Then the couple of operators (S, T) has a non-empty spectrum $\mathcal{S}_{S,T}(M)$ with respect to M and if, in addition, $dist(T(M), \{0\}) > 0$, then $\mathcal{S}_{S,T}(M)$ is a compact set.

Proof: Assume $\varepsilon > 0$ an arbitrary positive real number. Then there is a point $x_0 \in M$, $x_0 \neq 0$ such that

$$|(S(x_0),T(x_0))| > \|S\|_M \cdot \|T\|_M - \varepsilon \frac{\|T\|_M}{2\|S\|_M} .$$

Denote further

$\lambda_0 = e^{i\theta} \dfrac{\|S\|_M}{\|T\|_M}$, where θ is the argument of the complex number $(S(x_0), T(x_0))$. Then it holds:

$$\|S(x_0) - \lambda_0 T(x_0)\|^2 = \|S(x_0)\|^2 - 2\mathrm{Re}\,[\lambda_0 (T(x_0), S(x_0))] +$$
$$+ |\lambda_0|^2 \|T(x_0)\|^2 \leq \|S\|_M^2 - 2|(S(x_0), T(x_0))| \frac{\|S\|_M}{\|T\|_M} + \|S\|_M^2 < \varepsilon .$$

Now, it is evident that there are sequences $\{x_n\} \in M$, $x_n \neq 0, \lambda_n \in C, |\lambda_n| = \dfrac{\|S\|_M}{\|T\|_M}$ such that $\lim\limits_{n\to\infty} \|S(x_n) - \lambda_n T(x_n)\| = 0$. At the same time we can assume that the sequence λ_n converges to a point $\lambda_0 \neq 0$. Using the triangular inequality we conclude that

$$\|S(x_n) - \lambda_0 T(x_n)\| \leq \|S(x_n) - \lambda_n (x_n)\| + |\lambda_n - \lambda_0| \cdot \|T(x_n)\| ,$$

so that $\lim\limits_{n\to\infty} \|S(x_n) - \lambda_0 T(x_n)\| = 0$ and thus $\lambda_0 \in \mathcal{S}_{S,T}(M)$. Finally, Proposition 1.5 completes the proof.

<u>Remark 2.2</u>: Let $S, T: X \longrightarrow Y$ be bounded operators, $M \subset X$ a bounded set and let for any $x \in M$ the following inclusion hold:

$\{y \in Y / \|y\| = \|T(x)\|\} \subset T(M)$. Then the following condition

$$(p_2) \qquad 0 < \sup_{x \in M} |(S(x), T(x))| = \sup_{\substack{x \in M \\ y \in M}} |(S(x), T(y))|$$

implies the condition (p_1) from Theorem 2.1.

<u>Proof</u>: For any positive real number $\varepsilon > 0$ there are points $x_0, y_0 \in M$ such that

$$\|S(x_0)\| > \|S\|_M - \varepsilon, \quad \|T(y_0)\| > \|T\|_M - \varepsilon .$$

Choose a point $z_0 \in M$ such that $T(z_0) = \frac{S(x_0)}{\|S(x_0)\|} \cdot \|T(y_0)\|$. Then it holds that $(S(x_0), T(z_0)) = \|S(x_0)\| \cdot \|T(y_0)\| > (\|S\|_M - \varepsilon)(\|T\|_M - \varepsilon)$, hence

$\sup_{\substack{x \in M \\ y \in M}} |(S(x), T(y))| \geq \|S\|_M \cdot \|T\|_M$. On the other hand we have

$$\sup_{\substack{x \in M \\ y \in M}} |(S(x), T(y))| \leq \sup_{x \in M} \|S(x)\| \sup_{y \in M} \|T(y)\| = \|S\|_M \cdot \|T\|_M .$$

Clearly, the condition (p_1) from Theorem 2.1 is satisfied.

Remark 2.3: The conditions (p_1) and (p_2) from Theorem 2.1 and Remark 2.2 are equivalent (under the assumptions of Remark 2.2). Especially, if $T = I$ is the identity operator, $X = Y$ is a Hilbert space and $S : X \to X$ is a bounded operator, then the conditions (p_1) and (p_2) are equivalent for $M = \{x \in X / r \leq \|x\| \leq R, \ 0 < r \leq R\}$. If, in addition, the operator S is a homogeneous polynomial and symmetric operator, then the conditions (p_1) and (p_2) are satisfied (see [6], Theorem 4.5). But these conditions can be satisfied even if the operator S is not symmetric as the following examples show.

Example 2.4: Let E_2 be the Euclidean two-dimensional space. Define for $x = (x_1, x_2) \in E_2$, the operator P by

$$P(x) = (x_2^2, x_1^2) .$$

Then

$$\sup_{\|x\| = \sqrt{2}} \|P(x)\| \cdot \sqrt{2} = \sup_{\|x\| = \sqrt{2}} |(P(x), x)| = 2 .$$

Example 2.5: For any $x \in L^2([0,1])$ define the operator P by

$$P(x) = y(s) = \int_0^1 s \cdot x^2(t)\, dt .$$

Then

$$\sup_{\|x\|=\kappa} \|P(x)\| \cdot \kappa = \sup |(P(x), x)| = \frac{\kappa^3}{\sqrt{3}} \text{ for any } \kappa > 0.$$

Theorem 2.6: Let $S: X \to Y$ be completely continuous, $T: X \to Y$ a continuous operator with the Lipschitzian inverse operator T^{-1}. Then any non-zero element of the spectrum $\mathcal{S}_{S,T}(M)$ with respect to a bounded closed set $M \subset X$ such that $0 \notin M$ is an eigenvalue of the couple (S, T) with respect to M.

Proof: Consider $\lambda \in \mathcal{S}_{S,T}(M)$, $\lambda \neq 0$. Then there is a sequence $\{x_n\} \in M$ such that $\lim_{n\to\infty} \|S(x_n) - \lambda T(x_n)\| = 0$ and we can assume that the sequence $\{S(x_n)\}$ is convergent. Denote $z_n = T(x_n)$, so that $x_n = T^{-1}(z_n)$ and for arbitrary natural numbers n, m we obtain

$$\|x_n - x_m\| = \|T^{-1}(z_n) - T^{-1}(z_m)\| \le K\|z_n - z_m\| \le \frac{1}{|\lambda|} K \|S(x_n) - S(x_m)\| +$$

$$+ \frac{1}{|\lambda|} K \|S(x_n) - \lambda T(x_n)\| + \frac{1}{|\lambda|} K \|S(x_m) - \lambda T(x_m)\| .$$

Now, we see that $\{x_n\}$ is a fundamental sequence and thus there is a point $x_0 \in M$, $x_0 = \lim_{n\to\infty} x_n \neq 0$. Clearly, it holds:

$$\|S(x_0) - \lambda T(x_0)\| \le \|S(x_0) - S(x_n)\| + \|S(x_n) - \lambda T(x_n)\| +$$

$+ \|T(x_n) - T(x_0)\| \cdot |\lambda|$. Using continuity of the operators S, T we conclude that $\|S(x_0) - \lambda T(x_0)\| = 0$. Hence, λ is an eigenvalue of the couple (S, T) with respect to M.

Corollary 2.7: Let $S: X \to Y$ be completely continuous, $T: X \to Y$ a continuous operator with an inverse operator T^{-1} and let T^{-1} be a homogeneous polynomial operator of the order $k \ge 1$. Then the conclusion of Theorem

2.6 holds.

Proof: According to [6] (Theorem 3.4), the operator T^{-1} is continuous. Being continuous polynomial operator, T^{-1} is a Lipschitzian operator. Using Theorem 2.6, we complete the proof.

Remark 2.8: If $S, T: X \longrightarrow Y$ are analytical operators in a bounded domain $D \subset X$ which are continuous and bounded on the closure $\bar{D}$ and satisfy the condition (p_1) from Theorem 2.1, then the couple (S, T) has a non-empty spectrum with respect to the boundary ∂D of the domain D.

The proof follows immediately from the well-known "maximum modulus principle" for analytical operators and Theorem 2.1.

3. Spectra of positive homogeneous operators with respect to a sphere

In this section, let X denote a complex Hilbert space.

Definition 3.1: Let $F: X \longrightarrow X$ be a bounded homogeneous operator of the order $\gamma > 0$. Denote

$$\| F \| = \sup_{\|x\| = 1} \| F(x) \| ,$$

$$||| F ||| = \sup_{\|x\| = 1} |(F(x), x)| .$$

We shall call $\| F \|$ the norm of the operator F and $||| F |||$ the absolute norm of the operator F.

Remark 3.2: If F is a linear operator, then the norm and the absolute norm of F are well-known. For a homogeneous operator F of the order $\gamma > 0$ it follows that $\|F(x)\| \leq \|F\| \cdot \|x\|^{\gamma}$ for any $x \in X$ and $|||F||| \leq \leq \|F\|$. If F is a continuous homogeneous polynomial operator of the order $k \geq 1$ and symmetric in X, then $|||F||| = \|F\|$ (see [6], Theorem 4.5).

We consider further the spectrum of the operator F with respect to a given set $M \subset X$ (i.e., the spectrum of the couple (F, I) with respect to M, where I is the identity operator). The general case of the spectrum of a couple (S, T) with positively homogeneous operators S, T of the order α, $\beta > 0$ we can reduce to the above problem assuming that the inverse operator T^{-1} exists. Really, then T^{-1} is a homogeneous operator of the order β^{-1} and the operator $F = T^{-1}S$ is a homogeneous operator of the order $\gamma = \frac{\alpha}{\beta}$. It is evident that $\lambda \in \in \mathscr{S}_{S,T}(M)$ if and only if $\lambda^{\frac{1}{\beta}} \in \mathscr{S}_{F,I}(M)$.

<u>Definition 3.3</u>: Let $F: X \to X$ have the Gâteaux differential $VF(x, h)$ on the set $M \subset X$. We shall say that the operator F is symmetric on M if

$$(VF(x,h),k) = (h, VF(x,k)) \text{ for any } x \in M, h, k \in X.$$

<u>Lemma 3.4</u>: Let $D \subset X$ be a set such that for any $x \in \in D$ and any positive real number t the point $t \cdot x \in D$, $0 \notin D$. Suppose $F: X \to X$ possesses the Gâteaux differential $VF(x, h)$ on D. Then the operator F is homogeneous of the order $\alpha > 0$ on D if and only if

$$VF(x, x) = \alpha F(x) \qquad \text{for any } x \in D.$$

<u>Proof</u>: If F is homogeneous of the order $\alpha > 0$, then for any $x \in D$ it holds

$$VF(x,x)=\lim_{t\to 0}\frac{F(x+tx)-F(x)}{t}=\lim_{t\to 0}\frac{(1+t)^{\alpha}-1}{t}F(x)=\alpha F(x) .$$

On the other hand, if for any $x \in D$ it holds $VF(x,x)=$ $=\alpha F(x)$, then for the abstract function $f(t)=t^{-\alpha}F(t\cdot x)-$ $-F(x)$, $t>0$, $x\in D$, we obtain

$$f'(t)=-\alpha t^{-\alpha-1}F(t\cdot x)+t^{-\alpha}VF(t\cdot x,x)=t^{-\alpha-1}[-\alpha F(t\cdot x)+VF(t\cdot x,t\cdot x)] .$$

Hence $f'(t)\equiv 0$ and $f(1)=0$, so that $f(t)\equiv 0$ and thus $F(t\cdot x)=t^{\alpha}F(x)$.

Theorem 3.5: Let $F: X\rightarrow X$ be a bounded homogeneous operator of the order $\gamma>0$. Let $\|F\|=|||F|||$. Then the operator F has a non-empty compact spectrum $\mathcal{S}_F(S_r)$ with respect to any sphere $S_r=\{x\in X/\|x\|=r,\ r>0\}$, $|\lambda|\le r^{\gamma-1}\|F\|$ for any $\lambda\in\mathcal{S}_F(S_r)$ and there is a $\lambda_r\in\mathcal{S}_F(S_r)$ such that $|\lambda_r|=r^{\gamma-1}\|F\|$. If, in addition, F is completely continuous, then any non-zero element from $\mathcal{S}_F(S_r)$ is an eigenvalue of the operator F with respect to S_r .

Proof: We shall show that the condition $\|F\|=|||F|||$ implies the condition (p_1) from Theorem 2.1: Let r be a positive real number and let $x\in X$, $\|x\|=1$. Then for $y=$ $=r\cdot x$, we have $\|y\|=r$ and

$$\sup_{\|y\|=r}|(F(y),y)|=\sup_{\|x\|=1}|(F(x),x)|\cdot r^{\gamma+1}=|||F|||\cdot r^{\gamma+1}=$$

$$=\|F\|\cdot r^{\gamma+1}=\sup_{\|x\|=1}\|F(x)\|\,r^{\gamma+1}=\sup_{\|y\|=r}\|F(y)\|\cdot r .$$

Now, using Proposition 1.7, Theorem 2.1 and 2.6, we obtain the assertion.

Theorem 3.6: Let $F: X\rightarrow X$ be a bounded symmetric and homogeneous operator of the order $\gamma>0$ satisfying the condition $\|F\| = |||F|||$. Then it holds:

a) The operator F has only a real compact spectrum $\mathcal{S}_F(S_r)$ with respect to any sphere $S_r = \{x \in X / \|x\| = r, r > 0\}$.

b) $\mathcal{S}_F(S_r)$ is contained in the interval $J_r = [r^{\gamma-1}m, r^{\gamma-1}M]$, where $m = \inf_{\|x\|=1} (F(x), x)$, $M = \sup_{\|x\|=1} (F(x), x)$. Both $r^{\gamma-1}m$ and $r^{\gamma-1}M$ are contained in $\mathcal{S}_F(S_r)$.

c) If, in addition, the operator F is completely continuous, then any non-zero point of $\mathcal{S}_F(S_r)$ is an eigenvalue of the operator F with respect to S_r .

Proof: According to Definition 3.3 and Lemma 3.4, we obtain $(VF(x,x),x) = (x, VF(x,x)) = \overline{(VF(x,x),x)} = \alpha(F(x),x)$ for any $x \in S_r$. Now, we see that the expression $(VF(x,x),x)$ is real abd thus also $(F(x),x)$ is real. Assume $\lambda \in C$, $\lambda = a + ib$, $b \neq 0$. Then for $x \in S_r$ and $y = F(x) - \lambda x$, we obtain

$$(y,x) = (F(x),x) - \lambda(x,x) ,$$

$$(x,y) = \overline{(y,x)} = (F(x),x) - \overline{\lambda}(x,x) ,$$

so that $(x,y) - (y,x) = (\lambda - \overline{\lambda})(x,x) = 2ib\|x\|^2 = 2ib \cdot r^2$. It follows that $2|b|r^2 = |(x,y) - (y,x)| \leq 2\|y\| \cdot r$. Hence $\|y\| = \|F(x) - \lambda . x\| \geq |b| r > 0$ and thus $\lambda \notin \mathcal{S}_F(S_r)$ for any $r > 0$. Further, using Theorem 2.1, we obtain the assertion a). To prove b) let us suppose that $\lambda = M \cdot r^{\gamma-1} + d$, where $d > 0$. Then $(F(x) - \lambda x, x) = (F(x),x) - \lambda(x,x) \leq M \cdot \|x\|^{\gamma+1} - \lambda\|x\|^2$, so that for $x \in S_r$ we obtain $(F(x) - \lambda x, x) \leq [M \cdot r^{\gamma-1} - (M \cdot r^{\gamma-1} + d)] r^2 = -r^2 \cdot d < 0$

and thus $|(F(x)-\lambda x, x)| \geq d \cdot r^2$, hence $\|F(x)-\lambda x\| \cdot r \geq$ $\geq |(F(x)-\lambda x, x)| \geq d \cdot r^2$. Finally, we have

$$\|F(x)-\lambda x\| \geq d \cdot r > 0 \text{ and thus } \lambda \notin \mathcal{S}_F(S_r) .$$

The case $\lambda < m \cdot r^{\gamma-1}$ may be examined analogously. Using the proof of Theorem 2.1, we can show that both $m \cdot r^{\gamma-1}$, $M \cdot r^{\gamma-1}$ belong to $\mathcal{S}_F(S_r)$ and the proof of b) is finished. The assertion c) follows immediately from Theorem 2.6.

Remark 3.7: The assumptions of Theorem 3.6 are satisfied if the operator F is a completely continuous symmetric homogeneous polynomial operator of the order $k \geq 1$. Suppose, further, that λ, $\mu \in \mathcal{S}_F(S_r)$ are two different eigenvalues with eigenvectors $x, y \in S_r$. Then the following inequality holds:

$$|(F(x),y)-(F(y),x)| = |\lambda-\mu|\cdot|(x,y)| \leq \|F\|(k-1)\|x-y\|\cdot r^{k-1} .$$

Especially, if $k = 1$, then the eigenvectors x, y are orthogonal.

Proof: If $F(x) = \lambda x$, $F(y) = \mu y$, then

$$(F(x),y)-(F(y),x) = (\lambda-\mu)(x,y) = (F(x), y-x)+(F(x)\cdot$$

$$-(F(y),x) = (\sum_{i=1}^{k-1} F^*(x^{k-i-2}, y^i, x-y), x) ,$$

hence

$$|(F(x),y)-(F(y),x)| = |\lambda-\mu|\cdot|(x,y)| \leq \sum_{i=1}^{k-1} \|F^*\|\cdot\|x\|^{k-i-2}\|y\|^i\|x-y\| =$$

$$= (k-1)\|F^*\|\cdot r^{k-1}\|x-y\| = (k-1)\|F\|\cdot r^{k-1}\|x-y\| ,$$

where F^* is the polar operator to F. The last equality follows from [6] (Lemma 4.2 and Remark 4.3).

R e f e r e n c e s

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