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STABILITY OF THE METHOD OF LEAST SQUARES FOR FINDING THE EIGENVALUES OF A SYMMETRIC OPERATOR

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In [1,2,3], we studied the method of least suqares for approximating the eigenvalues and eigenfunctions of a DS-operator. In this paper, we shall deal with the stability of this method for finding the eigenvalues.

Let A be a DS-operator whose domain $\mathcal{D}(A)$ is dense in the separable Hilbert space H, i.e., A is a symmetric operator in H such that the set of its eigenvalues is of the first category on the real axis and the spectrum $\sigma(A)$ of A is the closure of this set. Let μ be a real number such that the following conditions be satisfied:

(1) 1) $\mu \,\bar{\in}\, \sigma(A)$.

2) There exists an eigenvalue λ_j of A such that

$$\inf_{\lambda \in \sigma(A)} |\lambda - \mu| = |\lambda_j - \mu| \ .$$

For the sake of brevity we shall sometimes write A_μ instead of $A - \mu I$, where I denotes the identity operator in H. The principle of finding λ_j by using the method of least squares can be outlined as follows.

By Theorem 3 of [1] we have

$$\lim_{n \to \infty} q_n = |\lambda_j - \mu| \ ,$$

where

(2) $$q_m = \min_{\substack{u \in \mathcal{L}\{\Psi_i\}_{i=1}^n \\ u \neq 0}} \frac{\|Au - \mu u\|^2}{\|u\|^2}$$

and thus $\mu + q_m$ or $\mu - q_m$ is the approximation to λ_j (cf.[1],[2] and [3]). The number q_m^2 is the smallest eigenvalue of the algebraic eigenvalue problem

(3) $$(\mathcal{A}_n - \sigma \mathcal{B}_n)u = 0 ,$$

where $\mathcal{A}_n$ and $\mathcal{B}_n$ are symmetric matrices

$$\mathcal{A}_n = \{(A_\mu \Psi_i , A_\mu \Psi_j)\}_{i,j=1}^n ,$$

$$\mathcal{B}_n = \{(\Psi_i , \Psi_j)\}_{i,j=1}^n .$$

The matrix $\mathcal{B}_n$ is known as the Gram matrix of $\Psi_1, \dots$ $\dots, \Psi_n$. Since $\Psi_1, \dots, \Psi_n$ are linearly independent, $\mathcal{B}_n$ is positive definite. On the basis of the assumptions relative to μ, it follows that $\mathcal{A}_n$ (as the Gram matrix of linearly independent vectors) is a positive definite matrix.

Let $\sigma_1^{(n)} \leq \sigma_2^{(n)} \leq \dots \leq \sigma_n^{(n)}$ be the eigenvalues of (1). Then $q_n^2 = \sigma_1^{(n)}$. Since $q_{n+1} \leq q_n$, the sequence $\{\sigma_1^{(n)}\}_{n=1}^\infty$ is monotone decreasing. It follows that

(4) $$\inf_n \sigma_1^{(n)} = \lim_{n \to \infty} \sigma_1^{(n)} = (\mu - \lambda_j)^2 .$$

In numerical work, it is usually impossible to carry out all required calculations with unlimited precision. Thus, the actual results $\tilde{\sigma}_k^{(n)}$ one obtains, do not satisfy (3), but rather

(5) $$[(\mathcal{A}_n + \Gamma_n) - \mu(\mathcal{B}_n + \Delta_n)]v = 0 ,$$

where Γ_n and Δ_n are symmetric matrices.

If $\|\Gamma_n\|$ is small, one expects that $\tilde{\sigma}_k^{(n)}$ will also differ only slightly from $\sigma_k^{(n)}$ for $n \geq n_0$. Regarding (5) as a perturbation of (3), one may then say, loosely speaking, that the method for finding λ_j defined by (2) and (3) is stable if $\sigma_1^{(n)}$ is not very sensitive to small perturbances Γ_n and Δ_n.

In Section 1, we shall define the stability of the method of least squares for finding λ_j defined by (2) and (3). We shall formulate a necessary and sufficient condition for the stability. In Section 2, we give conditions which guarantee that the method of least squares for finding λ_j is stable in the sense of Michlin [4].

1. Let $\mu_1^{(n)} \leq \mu_2^{(n)} \leq \dots \leq \mu_n^{(n)}$ be an enumeration of eigenvalues of the problem (5). Denote the smallest eigenvalue of $\mathcal{A}_n$ and $\mathcal{B}_n$ by $\lambda_1^{(n)}$ and $t_1^{(n)}$, respectively. Since $\mathcal{A}_n$ and $\mathcal{B}_n$ are positive definite matrices, it follows that $\lambda_1^{(n)} > 0$ and $t_1^{(n)} > 0$ for any positive integer n.

The definition of the stability of the method of least squares for finding the eigenvalues is as follows.

<u>Definition 1</u>. Let Γ_n and Δ_n be symmetric matrices. Let $\mu_1^{(n)}$ and $\sigma_1^{(n)}$ be the smallest eigenvalues of (3) and (5), respectively. The method of least squares for finding λ_j defined by (2) and (3) will be said to be stable, if there exist positive constants p, κ, s, q which do not depend on n such that

$$|\sigma_1^{(n)} - \mu_1^{(n)}| \le p \cdot \|\Gamma_n\| + q \cdot \|\Delta_n\|$$

for every Γ_n, Δ_n satisfying the following inequalities

$$\|\Gamma_n\| \le r, \quad \|\Delta_n\| \le s,$$

where $\|\Gamma_n\|$ denotes the Euclidean norm of Γ_n.

Let us now recall from [4] the terminology about the strongly minimal system in H.

Definition 2 (cf. Michlin [4], p.20). Let there be given a linearly independent system $\{\varphi_i\}_{i=1}^{\infty}$. Denote the smallest eigenvalue of the Gram matrix of $\varphi_1, \dots, \varphi_n$ by $c_1^{(n)}$. This system $\{\varphi_i\}_{i=1}^{\infty}$ will be said strongly minimal in H if $\inf\limits_n c_1^{(n)} > 0$.

Remark 1. Let $\{\varphi_i\}_{i=1}^{\infty}$ be an orthonormal system. Then $\inf\limits_n c_1^{(n)} = 1$ and consequently $\{\varphi_i\}_{i=1}^{\infty}$ is a strongly minimal system in H.

The following theorem is needed.

Theorem 1. Let there be given two Hilbert spaces $\mathcal{H}_1$, $\mathcal{H}_2$ with the following properties:

a) $u \in \mathcal{H}_1 \implies u \in \mathcal{H}_2$.

b) There exists a constant K such that

$$\|u\|_2 \le K \cdot \|u\|_1 \quad \text{for any } u \in \mathcal{H}_1,$$

where $\|u\|_1$ and $\|u\|_2$ are norms on $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively.

Let $\{\psi_i\}_{i=1}^{\infty}$, $\psi_i \in \mathcal{H}_1$ be a strongly minimal system in $\mathcal{H}_2$. Then the system $\{\psi_i\}_{i=1}^{\infty}$ is also strongly minimal in $\mathcal{H}_1$. If we denote the smallest eigenvalue of the matrices $\{(\psi_i, \psi_j)_1\}_{i,j=1}^{n}$ and $\{(\psi_i, \psi_j)_2\}_{i,j=1}^{\infty}$ by $\kappa_1^{(n)}$ and $t_1^{(n)}$, respectively, then we have

$$r_1^{(n)} \geq \frac{1}{k^2} s_1^{(n)} .$$

Proof: Michlin [4],p.25.

Since $\mu \overline{\in} \sigma(A)$, we see that $(A_\mu u, A_\mu v)$ is a scalar product on $\mathfrak{D}(A)$. Consequently, $\|u\|_1^2 = (A_\mu u, A_\mu u)$ is a norm on $\mathfrak{D}(A)$. The complete hull of $\mathfrak{D}(A)$ with the norm $\|u\|_1$ will be denoted by H_1. From the definition of λ_j, it follows that

$$\|u\|_1 = \|A_\mu u\| \geq |\mu - \lambda_1| \cdot \|u\| .$$

If we apply Theorem 1 with H in place of $\mathcal{H}_2$ and H_1 in place of $\mathcal{H}_1$, we obtain the following

Lemma 1. Let $\{\Psi_i\}_{i=1}^{\infty}$, $\Psi_i \in \mathfrak{D}(A)$ be a strongly minimal system in H. Then the system $\{A_\mu \Psi_i\}_{i=1}^{\infty}$ is strongly minimal in H and the smallest eigenvalues $\lambda_1^{(n)}$ and $t_1^{(n)}$ satisfy the following relation

$$\lambda_1^{(n)} \geq (\mu - \lambda_j)^2 \cdot t_1^{(n)} .$$

As an application of the result of Theorem 1, we have

Lemma 2. Let $\{\Psi_i\}_{i=1}^{\infty}$, $\Psi_i \in \mathfrak{D}(A)$ be a linearly independent system such that the system $\{A_{\mu_0} \Psi_i\}_{i=1}^{\infty}$ is strongly minimal in H for some $\mu_0 \overline{\in} \sigma(A)$. Then the system $\{A_\mu \Psi_i\}_{i=1}^{\infty}$ is strongly minimal in H for every $\mu \overline{\in} \sigma(A)$.

Proof: Choose any $\mu \overline{\in} \sigma(A)$. Let $\|u\|_1 = \|A_\mu u\|$ and $\|u\|_2 = \|A_{\mu_0} u\|$. Since $\mu_0 \overline{\in} \sigma(A)$, $\|u\|_1$ and $\|u\|_2$ are the norms on $\mathfrak{D}(A)$. The complete hull of $\mathfrak{D}(A)$ with $\|u\|_1$ and $\|u\|_2$ will be denoted by $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively. Obviously

$$A_{\mu_0} u = A_\mu u + (\mu - \mu_0) \cdot A_\mu^{-1} A_\mu u$$

and hence

$$\|u\|_2 \leq (1 + |\mu - \mu_0| \cdot \|A_\mu^{-1}\|) \cdot \|u\|_1 \ .$$

From this it follows that $\mathcal{H}_1 \subset \mathcal{H}_2$. The conclusion of this lemma now follows from Theorem 1.

The following fundamental result gives us a necessary and sufficient condition for the stability.

<u>Theorem 2</u>. Let A be a DS-operator in Hilbert space H and let $\{\Psi_i\}_{i=1}^\infty$ be a totally complete system. Then, with the assumptions of (1) the method of least squares for finding λ_j defined by (2) and (3) is stable if and only if the system $\{\Psi_i\}_{i=1}^\infty$ is strongly minimal in H . When this condition is fulfilled, we can choose p, q, r, s in Definition 1 as follows:

$$p = \frac{1}{t-s} \ ,$$

$$q = \frac{1}{t-s} \cdot \frac{\|A_\mu \Psi_1\|^2}{\|\Psi_1\|^2} \ ,$$

where r and s are arbitrary numbers in $(0, (\lambda_j - \mu)^2 \cdot t)$ and $(0,t)$, respectively, $t = \inf_n t_1^{(n)}$.

<u>Proof</u>: Suppose that the method of least squares for finding λ_j is stable but that the system $\{\Psi_i\}_{i=1}^\infty$ is not strongly minimal in H . Then, by Definition 1, there exist positive constants p, q, r, s such that

(6) $$|\sigma_1^{(n)} - \mu_1^{(n)}| \leq p \cdot \|\Gamma_n\| + q \cdot \|\Delta_n\|$$

for symmetric matrices Γ_n and Δ_n satisfying

$$\|\Gamma_n\| \leq r \ , \quad \|\Delta_n\| \leq s \ .$$

Since $\inf_{m} t_1^{(m)} = 0$, there is an infinite set M of positive integers such that $\inf_{m \in M} t_1^{(m)} = 0$ and $s > 2 t_1^{(m)} > 0$ for $m \in M$. Choose $\Gamma_m = 0$ and $\Delta_m = - 2 t_1^{(m)} \cdot E_m$ for every $m \in M$, where E_m is the identity matrix. In this case it follows from (6) that

$$| \sigma_1^{(m)} - \mu_1^{(m)} | \leq 2 q \cdot t_1^{(m)} , \quad m \in M .$$

Hence, since $\lim_{m \to \infty} \sigma_1^{(m)} = (\mu - \lambda_j)^2 > 0$ and $\inf_{m \in M} t_1^{(m)} = 0$, there exists some $m_0 \in M$ such that

$$\mu_1^{(m_0)} > \frac{1}{2} (\mu - \lambda_j)^2 > 0 .$$

This last inequality shows that the eigenvalues of the problem

(7) $$A_{m_0} u - \mu (B_{m_0} + \Delta_{m_0}) u = 0$$

are positive.

Since the A_{m_0} is a positive definite matrix, we see that every eigenvalue of the problem (10) is one of the problem

$$v - \mu A_{m_0}^{-\frac{1}{2}} (B_{m_0} + \Delta_{m_0}) A_{m_0}^{-\frac{1}{2}} v = 0$$

and conversely.

Note that

$$Q_{m_0} = A_{m_0}^{-\frac{1}{2}} (B_{m_0} + \Delta_{m_0}) A_{m_0}^{-\frac{1}{2}}$$

is a symmetric matrix. Consequently, Q_{m_0} is positive definite. It follows that

(8) $$f(u) = (Q_{m_0} u, u) > 0$$

for every $u \neq 0$.

We know that $t_1^{(m_0)}$ is the smallest eigenvalue of B_{m_0}. If we denote a normalized eigenfunction of B_{m_0} corresponding to $t_1^{(m_0)}$ by v, we have

$$(\mathcal{B}_{m_0} + \Delta_{m_0}) v = - t_1^{(m_0)} v \quad .$$

Let $u = \mathcal{A}_{m_0}^{\frac{1}{2}} v$. Then

$$Q_m u = \mathcal{A}_{m_0}^{-\frac{1}{2}} (\mathcal{B}_{m_0} + \Delta_{m_0}) v = - t_1^{(m_0)} \cdot \mathcal{A}_{m_0}^{-\frac{1}{2}} v \; .$$

Therefore it follows

$$f(u) = (Q_{m_0} u, u) = - t_1^{(m_0)} \cdot \| v \|^2 < 0 \quad .$$

This contradicts the inequality of (8).

Conversely, suppose now that $\{ \Psi_i \}_{i=1}^{\infty}$ is a strongly minimal system in H . For the sake of simplicity, we shall use the notation

$$F(C, D, u) = \frac{(Cu, u)}{(Du, u)} \quad ,$$

where C, D are $(n \times n)$ real matrices and u is a vector. It follows from Lemma 1 that

(9) $$\inf \lambda_1^{(n)} \geq (\mu - \lambda_j)^2 \cdot t \quad ,$$

where $t = \inf_n t_1^{(n)} > 0$.

Choose κ and s so that

(10) $$0 < \kappa < (\mu - \lambda_j)^2 \cdot t , \; 0 < s < t \; .$$

Suppose that Γ_n and Δ_n are symmetric matrices such that

(11) $$\| \Gamma_n \| \leq \kappa , \; \| \Delta_n \| \leq s \quad .$$

From (9),(10) and (11) it is easy to see that $(\Gamma_n + \mathcal{A}_n)$ and $(\mathcal{B}_n + \Delta_n)$ are positive definite matrices.

Consequently,

$$\sigma_1^{(n)} = \min_{\|u\| = 1} F(\mathcal{A}_n, \mathcal{B}_n, u) \quad ,$$

(12) $$\mu_1^{(n)} = \min_{\|u\|=1} F(\mathcal{A}_n + \Gamma_n, \mathcal{B}_n + \Delta_n, u) .$$

Choose v and w such that $\|v\| = 1$, $\|w\| = 1$ and

$$\sigma_1^{(n)} = F(\mathcal{A}_n, \mathcal{B}_n, v) ,$$

$$\mu_1^{(n)} = F(\mathcal{A}_n + \Gamma_n, \mathcal{B}_n + \Delta_n, v) .$$

Hence, by (12)

$$\mu_1^{(n)} - \sigma_1^{(n)} \le \frac{(\mathcal{A}_n v, v) + \|\Gamma_n\|}{(\mathcal{B}_n v, v) - \|\Delta_n\|} - \sigma_1^{(n)} ,$$

whence

$$\mu_1^{(n)} - \sigma_1^{(n)} \le \frac{\|\Gamma_n\| + \sigma_1^{(n)} \cdot \|\Delta_n\|}{(\mathcal{B}_n v, v) - \|\Delta_n\|} .$$

Since $(\mathcal{B}_n v, v) \ge t_1^{(n)} > t$ and $\sigma_1^{(n)} \le \sigma_1^{(1)}$, it follows from (10) and (11) that

(13) $$\mu_1^{(n)} - \sigma_1^{(n)} \le \frac{\|\Gamma_n\| + \sigma_1^{(1)} \cdot \|\Delta_n\|}{t - s} .$$

It is easily verified that

$$\mu_1^{(n)} = \frac{F(\mathcal{A}_n, \mathcal{B}_n, w) + F(\Gamma_n, \mathcal{B}_n, w)}{1 + F(\Delta_n, \mathcal{B}_n, w)} .$$

Since

$$F(\mathcal{A}_n, \mathcal{B}_n, w) \ge \sigma_1^{(n)} ,$$

$$F(\Gamma_n, \mathcal{B}_n, w) \ge - \frac{\|\Gamma_n\|}{t_1^{(n)}} ,$$

$$|F(\Delta_n, \mathcal{B}_n, w)| \le \frac{\|\Delta_n\|}{t_1^{(n)}} < \frac{s}{t} < 1$$

and $\|\Gamma_n\| < (\mu - \lambda_j)^2 \cdot t_j \le \sigma_1^{(n)} \cdot t_1^{(n)}$,

we find that

$$\mu_1^{(n)} \ge \frac{\sigma_1^{(n)} t_1^{(n)} - \|\Gamma_n\|}{t_1^{(n)} + \|\Delta_n\|} > 0 .$$

Therefore

$$(14)\quad \sigma_1^{(n)} - \mu_1^{(n)} \leq \frac{\sigma_1^{(n)} \| \Delta_n \| + \| \Gamma_n \|}{t_1^{(n)} + \| \Delta_n \|} \leq \frac{\sigma_1^{(1)} \cdot \| \Delta_n \| + \| \Gamma_n \|}{t - s} .$$

This, together with (13), leads to the second assertion of the theorem.

2. In this section we shall investigate the stability of the method of least squares in the sense of Michlin. As to the mathematical formulation and to some assumptions we shall use the book of S.G. Michlin [4].

For each $n = 1, 2, \ldots$ let H_n be a separable complex Hilbert space with the scalar product (x, y), x, $y \in H_n$. Let A_n and B_n be two operators in the Hilbert space H_n with the following properties:

α) A_n and B_n are self-adjoint and positive definite operators on H_n into H_n.

β) A_n^{-1} and $\bar{B}_n = A_n^{-\frac{1}{2}} B_n A_n^{-\frac{1}{2}}$ are compact operators.

Since the operator $A_n^{-1} B_n$ is compact on H_n, we can arrange the eigenvalues $\{ \sigma_k^{(n)} \}_{k=1,2,\ldots}$ of the problem

$$(15)\qquad (A_n - \sigma B_n) x = 0$$

in an increasing order

$$0 < \sigma_1^{(n)} \leq \sigma_2^{(n)} \leq \ldots .$$

We will consider "the approximation eigenvalue problem"

$$(16)\qquad [(A_n + \Gamma_n) - \mu (B_n + \Delta_n)] x = 0 ,$$

where Γ_n and Δ_n are bounded and self-adjoint operators on H_n. Let $\mu_1^{(n)} \leq \mu_2^{(n)} \leq \ldots$ be an enumeration of eigenvalues of (16). S.G. Michlin defines the stability of the processes for finding the k-th eigenvalue of (15) as follows:

Definition 3 (Michlin [4],p.260). Let Γ_n and Δ_n be bounded and self-adjoint operators on H_n with the following properties:

1) $\Delta_n + B_n$ and $A_n + \Gamma_n$ are positive definite operators on H_n.

2) $T_n = (I + \bar{\Gamma}_n)^{-\frac{1}{2}} (\bar{B}_n + \bar{\Delta}_n)(I_n + \bar{\Gamma}_n)^{-\frac{1}{2}}$

is compact on H_n, where I_n is the identity operator on H_n and

$$\bar{\Gamma}_n = A_n^{-\frac{1}{2}} \Gamma_n A_n^{-\frac{1}{2}} , \quad \bar{\Delta}_n = A_n^{-\frac{1}{2}} \Delta_n A_n^{-\frac{1}{2}} .$$

Then the process for finding the k-th eigenvalue of (15) will be said to be stable, if there exist three positive numbers p, q, r which do not depend on n such that, if $\| \Gamma_n \| \leq \leq r$, $n = 1, 2, \ldots$, then

$$\left| \frac{\sigma_k^{(n)}}{\mu_k^{(n)}} - 1 \right| \leq p \cdot \| \Gamma_n \| + q \cdot \| \Delta_n \| , \quad n = 1, 2, \ldots .$$

We now state - without proof - the following basic result.

Theorem 3 (Michlin [4],p.260). Let C_1, C_2, C_3 be constants (which do not depend on n)such that

1) $\| A_n^{-1} \| \leq C_1$,

2) $\dfrac{(B_n u, u)}{(A_n u, u)} \leq C_3$ for $u \in H_n$, $u \neq 0$,

3) $\sigma_k^{(n)} \leq C_2$.

Then the process for finding the k-th eigenvalue $\sigma_k^{(n)}$ of the problem (15) is stable. The first condition is necessary for the stability.

Remark 2. It follows from the proof of Theorem 3 (in [4], p.263) that

$$r = \frac{\beta}{C_1}, \; p = \frac{C_1 \cdot C_2 \cdot C_3}{1-\beta}, \; q = \frac{C_1 \cdot C_3}{1-\beta},$$

where β is an arbitrary real number in $(0,1)$.

Let the assumptions of Section 1 be satisfied. In the case that H_n is an n-dimensional Euclidean space we can represent A_n and B_n by $(n \times n)$ matrices $\mathcal{A}_n$ and $\mathcal{B}_n$, respectively. Thus, we can consider the problems (3) and (5) in place of (15) and (16), respectively.It is now easily verified that the above conditions α) and β) are satisfied. Then the method of least squares for finding λ_j defined by (2) and (3) will be said to be stable in the sense of Michlin, if the process for finding the first eigenvalue $\sigma_1^{(n)}$ of (15) is stable in the sense of Definition 3.

From Theorem 3 we obtain the following result.

Theorem 4. Under the hypotheses of Theorem 2 the method of least squares for finding λ_j defined by (3) and (5) is stable in the sense of Michlin, if and only if the system $\{A_{\mu_0} \varphi_i\}_{i=1}^{\infty}$ is strongly minimal in H for some real $\mu_0 \bar{\in} \sigma(A)$.

Proof: Firstly, we verify that the conditions of Theorem 3 are satisfied for $k = 1$. Then, it follows that the condition 1) of Theorem 3 is necessary and sufficient for the stability.

Since $\sigma_1^{(n)} = q_n^2$, the sequence $\{\sigma_1^{(n)}\}_{n=1}^{\infty}$ is monotone decreasing. Consequently,

$$\sup_n \sigma_1^{(n)} = \sigma_1^{(1)} = \frac{\|A_\mu \Psi_1\|^2}{\|\Psi_1\|^2}$$

and $$\inf_n \sigma_1^{(n)} = \lim_{n\to\infty} \sigma_1^{(n)} = (\mu - \lambda_j)^2 \quad .$$

Letting

$$C_2 = \frac{\|A_\mu \Psi_1\|^2}{\|\Psi_1\|^2} \quad ,$$

we see that the condition of 3) is fulfilled.

Now, $\mathcal{A}_n$ and $\mathcal{B}_n$ are positive definite matrices. Therefore it is easily verified that

$$(17) \quad \frac{(\mathcal{B}_n u, u)}{(\mathcal{A}_n u, u)} = \frac{(\overline{\mathcal{B}}_n v, v)}{(v, v)} \qquad \text{for } u \in \mathcal{H}_n,\ u \neq 0 ,$$

where $v = \mathcal{A}_n^{\frac{1}{2}} u$ and $\overline{\mathcal{B}}_n = \mathcal{A}_n^{-\frac{1}{2}} \mathcal{B}_n \mathcal{A}_n^{-\frac{1}{2}}$.

It follows from 3) that $\frac{1}{\sigma_1^{(n)}}$ is the largest eigenvalue of $\overline{\mathcal{B}}_n$. Hence, by (17)

$$\frac{(\mathcal{B}_n u, u)}{(\mathcal{A}_n u, u)} \leq \frac{1}{\sigma_1^{(n)}} \quad \text{for } u \in \mathcal{H}_n,\ u \neq 0 .$$

Since $\inf_n \sigma_1^{(n)} = (\mu - \lambda_j)^2 > 0$, the condition 2) of Theorem 3 is satisfied with

$$C_3 = \frac{1}{(\mu - \lambda_j)^2} \quad .$$

Now, $\|\mathcal{A}_n^{-1}\| = \frac{1}{\lambda_1^{(n)}}$, where $\lambda_1^{(n)}$ is the smallest eigenvalue of the symmetric and positive definite matrix $\mathcal{A}_n$. The assertion of our Theorem follows at once from Lemma 2.

As a consequence of Theorem 4, Lemma 1 and Remark 2 we have

Remark 3. With the assumptions of Theorem 4, let $\{\Psi_i\}_{i=1}^{\infty}$ be a strongly minimal system in H . Then the method of least squares for finding λ_j defined by (3) and (5) is stable in the sense of Michlin. In this case we can choose p, q, κ as follows:

$$p = \frac{1}{(\mu-\lambda_j)^4} \cdot \frac{1}{t} \cdot \frac{1}{1-\beta} \cdot \frac{\|A_\mu \Psi_1\|^2}{\|\Psi_1\|^2} \quad ,$$

$$q = (\mu-\lambda_j)^2 \cdot p \quad ,$$

$$\kappa = t \cdot \beta \quad ,$$

where $t = \inf_m t_1^{(m)}$ and β is an arbitrary number in $(0,1)$.

Remark 4. From the assumptions α), β) and 2) of Theorem 3 it follows that $\mathfrak{D}(A_m) = H_m$ and A_m is a self-adjoint and bounded operator on H_m. Since A_m^{-1} is a compact operator on H_m, it follows that the eigenvalues of A_m form a finite set. Consequently, H_m is finite dimensional.

References

[1] K. NAJZAR: On the method of least squares of finding eigenvalues of some symmetric operators, Comm. Math.Univ.Carolinae 9(1968),311-323.

[2] K. NAJZAR: On the method of least squares of finding eigenvalues and eigenfunctions of some symmetric operators, Comm.Math.Univ.Carolinae 11 (1970),449-462.

[3] K. NAJZAR: Error-estimates for the method of least squares of finding eigenvalues and eigenfunctions, Comm.Math.Univ.Carolinae 11(1970),463-479.

[4] S.G. MICHLIN: Čislennaja realizacija variacionnych metodov, 1966.

[5] N.I. ACHIEZER - I.M. GLASMANN: Theorie der linearen Operatoren in Hilbert-Raum, 1960.

[6] A.E. TAYLOR: Introduction to functional analysis,1958.

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