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FIXED POINT THEOREMS, NEMYCKII AND URYSON OPERATORS, AND CONTINUITY OF NONLINEAR MAPPINGS

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Introduction. This paper contains three sections. In Section 1 some fixed points theorems are given. The theorems are of two types: the former are characterized by the condition $\inf_{x \in C} \| x - T(x) \| = 0$ (see Belluce-Kirk [1], and, implicitly, Daneš [4,5]), the latter deal with concentrative mappings. Section 2 contains some necessary and sufficient conditions for Nemyckii and Uryson operators to be subadditive, convex, subhomogeneous, Lipschitzian and so on. Also, a sufficient condition for the concentrativity of the Uryson operator is given. In the last section, we give a sufficient condition for a nonlinear Gâteaux-differentiable mapping to be continuous.

§ 1. Fixed point theorems. In this section we derive fixed point theorems generalizing some results of Belluce-Kirk [1] and Daneš [4 - 7].

Let C be a subset of a normed linear space X. Then a mapping $T : C \to Y$ (Y is a normed linear space) is said to be demicontinuous if $T : (C, s) \to (Y, w)$

is continuous (s = the strong topology, w = the weak topology). T is weakly continuous iff $T:(C,w)\rightarrow(Y,w)$ is continuous.

A subset C of a linear space X is said to be star-shaped with respect to a point $x\in X$ if the segment $[x,z]$ is contained in C for each $z\in C$. By the kernel of C we mean the set $K(C)$ of all points $x\in X$ with respect to which C is star-shaped.

Let (X,d) be a pseudometric space and C its subset. We define the set $Q(C)=\{\varepsilon>0$: there exists a finite subset σ of X such that the closed ε-ball about σ contains $C\}$. The function $\chi_d=\chi:2^X\rightarrow\langle 0,+\infty\rangle$, defined by $\chi(C)=\inf Q(C)$, is called the measure of non-compactness in the space (X,d). A continuous mapping T of X into another pseudometric space (Y,e) is called concentrative if for each bounded non-precompact subset C of X, $\chi_e(T(C))<\chi_d(M)$.

If (X,p) is a pseudonormed linear space, then we denote $B_p(x_0,R)=\{x\in X: p(x-x_0)\leq R\}$, the closed R-ball at x_0 ($x_0\in X$, $R>0$), and $B_p=B_p(0,1)$. Further, (X,P) denotes a locally convex Hausdorff linear topological space, where P is a system of pseudonorms on X such that $\{p^{-1}(\langle 0,\varepsilon)) : 0<\varepsilon<1, p\in P\}$ is a basis of neighborhoods at 0 for the topology of X. A continuous mapping T of a subset C of X into X is said to be P-concentrative, if $M\subset C$, $p\in P$, $0<\chi_p(M)<+\infty$

imply $\chi_n(T(M)) < \chi_n(M)$. For some results on the measure of non-compactness and fixed point theorems for concentrative mappings see [2 - 8, 11 - 15].

The following lemma is obvious, but useful.

<u>Lemma 1</u>. Let C be a non-empty compact topological space, $T: C \longrightarrow C$ a mapping; $d: C \times C \longrightarrow \langle 0, +\infty)$ a function such that $d(x,y) = 0$ if and only if $x = y$, for $x, y \in C$. Suppose that the function $f = d \circ (id_C \times T)$, i.e., $f(x) = d(x, T(x))$ for $x \in C$, is lower semi-continuous on C. Then T has a fixed point in C if and only if

$$\inf_{x \in C} d(x, T(x)) = 0 .$$

<u>Theorem 2.</u> Let X be a normed linear space, C a non-empty weakly compact subset of X and $T: C \rightarrow C$ a mapping. Consider the following conditions:

(1) the functional $f(x) = \|x - T(x)\|$ is weakly lower semi-continuous on C;

(2) the mapping T is weakly continuous on C;

(3) the set C and the functional $f(x) = \|x - T(x)\|$ are convex, and the mapping T is demicontinuous on C.

Suppose that one of the conditions (1) - (3) is satisfied. Then T has a fixed point in C if and only if

$$\inf_{x \in C} \|x - T(x)\| = 0 .$$

<u>Proof</u>. In the case (1), the theorem follows immediately from Lemma 1, where we set $d(x,y) = \|x - y\|$ for $x, y \in C$.

Case (2). Since T is weakly continuous, $I - T$

is so. Hence the weak lower semi-continuity of the norm $\|\cdot\|$ in X implies that of the functional $f(x) = \|x-T(x)\|$. Thus, the theorem is reduced to the case (1).

Case (3). Since T is demicontinuous, $I-T$ is also demicontinuous. From the weak lower semi-continuity of the norm $\|\cdot\|$ it follows the (strong) lower semicontinuity of the functional $f(x) = \|x-T(x)\|$. Hence the convex sets $\{x \in C : f(x) \leq c\}$, $c \in \mathbb{R}$, are closed, and therefore weakly closed. Thus, the functional $f(x)$ is weakly lower semi-continuous on C. Now, by the case (1), the theorem follows.

Corollary 3. (Belluce-Kirk [1, Theorem 4.1].) Let C be a non-empty weakly compact convex subset of a normed linear space X, and T a continuous mapping of C into itself, such that the mapping $I-T$ is convex (i.e., the functional $f(x) = \|x-T(x)\|$ is convex) on C. If $\inf_{x \in C} \|x-T(x)\| = 0$, then T has a fixed point in C.

Proof. T is, clearly, demicontinuous, and we can apply Theorem 2,(3).

The following proposition is a simple generalization of Göhde [9, Lemma 3].

Proposition 4. Let X be a normed linear sp^ce, C a non-empty bounded complete subset of X, $T: C \to C$ a nonexpansive mapping (i.e., $\|T(x)-T(y)\| \leq \|x-y\|$, for $x, y \in C$) such that $K(C)$, the kernel of C, intersects $R(T)$, the range of T: $K(C) \cap R(T) \neq \emptyset$.

Then $\inf_{x \in C} \|x-T(x)\| = 0$.

<u>Proof</u>. Let $T(x_0) \in K(C) \cap R(T)$, for some $x_0 \in C$, and $M = \sup_{x \in C} \|T(x) - T(x_0)\| \leq$

$\leq \sup_{x \in C} \|x - x_0\| \leq \operatorname{diam} C < +\infty$. For $0 < \varepsilon < M$, we define

$$T_\varepsilon(x) = T(x_0) + (1 - \frac{\varepsilon}{M}) \cdot (T(x) - T(x_0))$$

($T_\varepsilon(x)$ is well-defined, since C is star-shaped relative to $T(x_0)$). Then

$$\|T_\varepsilon(x) - T_\varepsilon(y)\| = (1 - \frac{\varepsilon}{M}) \cdot \|T(x) - T(y)\| \leq (1 - \frac{\varepsilon}{M}) \cdot \|x - y\| ,$$

for $x, y \in C$, where $1 - \frac{\varepsilon}{M} < 1$; by Banach Contraction Principle, there exists a point $x_\varepsilon \in C$ with $x_\varepsilon = T_\varepsilon(x_\varepsilon)$. Further,

$$\|x_\varepsilon - T(x_\varepsilon)\| \leq \|x_\varepsilon - T_\varepsilon(x_\varepsilon)\| + \|T_\varepsilon(x_\varepsilon) - T(x_\varepsilon)\| = \\ = \frac{\varepsilon}{M} \|T(x_\varepsilon) - T(x_0)\| \leq \frac{\varepsilon}{M} \cdot M = \varepsilon .$$

Hence,

$$\inf_{x \in C} \|x - T(x)\| = 0 .$$

<u>Corollary 5.</u> Let X be a normed linear space, C a non-empty weakly compact subset of X and $T: C \to C$ a nonexpansive weakly continuous mapping (more generally, T is nonexpansive and the functional $f(x) = \|x - T(x)\|$ is weakly lower semi-continuous on C) such that $K(C) \cap R(T) \neq 0$. Then T has a fixed point in C.

<u>Proof</u>. Since C is weakly compact, it is (strongly) complete. Now, we can apply Proposition 4 and Theorem 2,(2) (or (1)).

<u>Proposition 6</u>. Let X be a normed linear space and $B = \{x \in X : \|x\| \leq 1\}$ its unit closed ball.

Then

(1) $\chi(B) = 0$ if and only if X is finite dimensional;

(2) $\chi(B) = 1$ if and only if X is infinite dimensional.

Proof. (1) If $\chi(B) = 0$, then B is precompact, and, by Riesz Theorem, X is finite dimensional. Conversely, if X is finite dimensional, then B is compact.

(2) Clearly, $B(\{0\}, 1) = B$, i.e., $1 \in Q(B)$ and we have $\chi(B) \leqq 1$. Suppose that $\chi(B) < 1$. Let $\chi(B) < a < 1$. Then $a \in Q(B)$; hence there exists a finite subset σ of X (even of B) such that $B(\sigma, a) \supset \supset B$. But $B(\sigma, a) = \sigma + aB$. From this, we can obtain successively,

$$\sigma + a\sigma + a^2 B = \sigma + a(\sigma + aB) \supset \sigma + aB \supset B .$$

By induction, we have

$$\sigma + aB + \dots + a^{n-1}\sigma + a^n B \supset B ,$$

for all positive integers n. The sets $\sigma_n = \sigma + a\sigma + \dots + a^{n-1}\sigma$ $(n \geqq 1)$ are finite and $B(\sigma_n, a^n) \supset \supset B$ for all n, i.e., $a^n \in Q(B)$ for all n. Since $0 < a < 1$, we have $\chi(B) = 0$, and by (1), X is finite-dimensional. From this the assertion (2) follows.

Corollary 7. Let (X, p) be a pseudonormed linear space and B_p its unit closed ball. Then:

(1) $\chi(B_p) = 0$ if and only if $X/p^{-1}(0)$ is finite dimensional;

(2) $\chi(B_p) = 1$ if and only if $X/p^{-1}(C)$ is infinite dimensional.

Theorem 8. Let (X, p) and (Y, q) be two pseudonormed linear spaces and $T: X \to Y$ a linear mapping. Then T is concentrative if and only if

$$\chi_q(T(B_p)) < 1 ,$$

where B_p is the unit closed ball in X .

Proof. Suppose that T is concentrative. If $X/p^{-1}(0)$ is finite dimensional, then $T(B_p)$ is compact, since B_p is so, and hence $\chi_q(T(B_p)) = 0 < 1$. If $X/p^{-1}(0)$ is infinite dimensional, then $0 < 1 = \chi_p(B_p) < +\infty$, and hence $\chi_q(T(B_p)) < \chi_p(B_p) = 1$.

On the other hand, suppose that $\chi_q(T(B_p)) < 1$. Since $\chi_q(T(B_p)) < 1$, $T(B_p)$ is bounded and hence T is continuous. Now, let M be an arbitrary bounded non-precompact subset of X, i.e., $0 < \chi_p(M) < < +\infty$. By the definition of the measure of non-compactness, for each $a > \chi_p(M)$ there exists a finite subset σ of X such that $B_p(\sigma, a) = \sigma + aB_p \supset M$. Hence,

$$T(M) \subset T(\sigma + aB_p) = T(\sigma) + aT(B_p) ,$$

and consequently,

$$\chi_q(T(M)) \leq \chi_q(T(\sigma) + aT(B_p)) = a\chi_q(T(B_p)) .$$

Thus, we have

$$\chi_q(T(M)) \leq \chi_q(T(B_p)) \cdot \chi_p(M) < \chi_p(M) .$$

Therefore, T is concentrative.

Corollary 9. Let (X, P) be a locally convex Hausdorff linear topological space and $T: X \to X$ a a continuous linear mapping. Then T is P -concentrative if and only if

$$(*) \qquad \chi_p(T(B_p)) < 1 \quad \text{for all } p \in P .$$

Remark. The condition $(*)$ is satisfied, if

$$\chi_p(T) = \sup_{p \in P} \chi_p(T(B_p)) < 1 .$$

Theorem 10. Let (X, P) be a locally convex Hausdorff linear topological space, C a non-empty bounded convex subset of X, and $T: X \rightarrow X$ a linear mapping such that $T(C) \prec C$ and

$$\chi_p(T(B_p)) < 1 \quad \text{for all } p \in P .$$

Then T has a fixed point in C .

Proof. See Corollary 9 and [7, Theorem 3].

Corollary 11. Let C be a non-empty complete bounded convex subset of a normed linear space X and $T: X \rightarrow X$ a linear mapping such that $T(C) \subset C$ and $\chi(T(B)) < 1$, where $B = \{x \in X: \|x\| \leq 1\}$ is the unit closed ball in X . Then T has a fixed point in C .

§ 2. Nemyckii and Uryson operators. In this section, we give necessary and sufficient conditions for Nemyckii operator to be subadditive, subhomogeneous, Lipschitzian, and so on. For the Uryson operator, only necessary conditions are given.

Let G be a measurable subset of $\mathbb{R}^n$ with the positive Lebesgue measure, $\operatorname{mes} G > 0$. By a Carathéodory function we mean a function $f(s,u): G \times \mathbb{R} \rightarrow \mathbb{R}$ such that: (a) $f(s,u): G \times \mathbb{R} \rightarrow \mathbb{R}$ is continuous for a.e. $s \in G$; (b) $f(\cdot,u): G \rightarrow \mathbb{R}$ is measurable for each $u \in \mathbb{R}$. It is well-known that $f(s, x(s))$ is measurable on G for any measurable function $x(s)$ on G.

Proposition 12. Let $f(s,u): G \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function. Suppose that $f(s,0) = 0$ for a.e. $s \in G$. Let $T: X \to L_p(G)$ $(1 \leqq p < +\infty)$ be the Nemyckii operator generated by $f(s,u)$, $(T(x))(s) = f(s, x(s))$, where X is a linear space of measurable functions on G, containing characteristic functions x_M of all measurable subsets M of G, $\operatorname{mes} M < +\infty$.

(1) If T is (α, β)-subadditive for some $\alpha, \beta \in \mathbb{R}$, i.e.,

$$\|T(\alpha x_1 + \beta x_2)\| \leqq |\alpha| \cdot \|T(x_1)\| + |\beta| \cdot \|T(x_2)\|$$

for $x_1, x_2 \in X$, then

$$|f(s, \alpha u_1 + \beta_2)| \leqq |\alpha| \cdot |f(s, u_1)| + |\beta| \cdot |f(s, u_2)|$$

for a.e. $s \in G$ and all $u_1, u_2 \in \mathbb{R}$.

(2) If T is α-subhomogeneous, i.e. $\|T(\alpha x)\| \leqq |\alpha| \cdot \|T(x)\|$ for all $x \in X$, then

$|f(s, \alpha u)| \leqq |\alpha| \cdot |f(s,u)|$ for a.e. $s \in G$ and all $u \in \mathbb{R}$.

(3) If T is subadditive, i.e. $\|T(x_1 + x_2)\| \leqq \|T(x_1)\| + \|T(x_2)\|$ for all $x_1, x_2 \in X$, then $|f(s, u_1 + u_2)| \leqq |f(s,u_1)| + |f(s,u_2)|$ for a.e. $s \in G$ and all u_1, $u_2 \in \mathbb{R}$.

(4) If T is α-convex (for some $0 < \alpha < 1$), i.e., $\|T(\alpha x_1 + (1-\alpha) x_2)\| \leqq \alpha \|T(x_1)\| + (1-\alpha) \|T(x_2)\|$, $x_1, x_2 \in X$, then $|f(s, \alpha u_1 + (1-\alpha) u_2)| \leqq \alpha |f(s,u_1)| + (1-\alpha) |f(s, u_2)|$ for a.e. $s \in G$ and all u_1, $u_2 \in \mathbb{R}$.

Proof. First, let us note that T is almost additive: if x_1, x_2 are functions in X with disjoint supports, then $T(x_1 + x_2) = T(x_1) + T(x_2)$.

(1) For any measurable subset M of G,

$\operatorname{mes} M < +\infty$, we have

$$\| T(\alpha u_1 \chi_M + \beta u_2 \chi_M) \| \leq |\alpha| \cdot \| T(u_1 \chi_M) \| +$$
$$+ |\beta| \cdot \| T(u_2 \chi_M) \| \quad (u_1, u_2 \in \mathbb{R}) ,$$

and by the almost additivity of T ,

$$(*) \quad [\int_M |f(s, \alpha u_1 + \beta u_2)|^p ds]^{1/p} \leq |\alpha| \cdot$$
$$\cdot [\int_M |f(s, u_1)|^p ds]^{1/p} + |\beta| \cdot [\int_M |f(s, u_2)|^p ds]^{1/p} .$$

For each measurable subset M of $\mathbb{R}^n$, we denote:

$$\sigma(M) = \int_{G \cap M} |f(s, \alpha u_1 + \beta u_2)|^p ds ,$$

$$\sigma_i(M) = \int_{G \cap M} |f(s, u_i)|^p ds, \quad i = 1, 2 .$$

By $(*)$, we obtain:

$$(**) \quad [\sigma(M)]^{1/p} \leq |\alpha| \cdot [\sigma_1(M)]^{1/p} + |\beta| \cdot [\sigma_2(M)]^{1/p} .$$

The measures σ_1, σ_i possess regular derivatives (see Zaanen [16, § 37]):

$$D_\mu \sigma(s) = |f(s, \alpha u_1 + \beta u_2)|^p =$$
$$= \lim_{m \to +\infty} \frac{1}{\operatorname{mes} K_m(s)} \int_{G \cap K_m(s)} |f(s', \alpha u_1 + \beta u_2)|^p ds' ,$$
$$D_\mu \sigma_i(s) = |f(s, u_i)|^p =$$
$$= \lim_{m \to +\infty} \frac{1}{\operatorname{mes} K_m(s)} \int_{G \cap K_m(s)} |f(s', u_i)|^p ds', \; i = 1, 2,$$

for a.e. $s \in G$, where $K_m(s)$ denotes the closed ball of radius $\frac{1}{m}$ centered at s. The inequality $(**)$ implies that

$$[D_p \sigma(s)]^{1/p} \leq |\alpha| \cdot [D_p \sigma_1(s)]^{1/p} + |\beta| \cdot [D_p \sigma_2(s)]^{1/p}$$

for a.e. $s \in G$, that is

$|f(s, \alpha u_1 + \beta u_2)| \leq |\alpha| \cdot |f(s, u_1)| + |\beta| \cdot |f(s, u_2)|$, for a.e. $s \in G$.

The assertions (2),(3),(4) follow from (1) by setting $(\alpha = \alpha, \beta = 0)$, $(\alpha = \beta = 1)$, $(\alpha = \alpha, \beta = 1 - \alpha)$, respectively.

Remark. A similar proposition holds for (α, β) - superadditivity, α -superhomogeneity, superadditivity, and α -concavity of the Nemyckii operator.

Proposition 13. Let G, f, p be as in Proposition 12. If X is a linear space of measurable functions on G' and $T: X \longrightarrow L_p(G)$ the Nemyckii operator generated by $f(s, u)$, then the converses of (1),(2),(3),(4) of Proposition 12 hold.

Proof. The easy proof is omitted.

Remark. Proposition 13 does not hold, in general, for (α, β)-superadditivity, α-concavity and superadditivity of the Nemyckii operator, as simple examples show. But it is true for α -superhomogeneity of T (the proof is as that of Proposition 13 in the case $\alpha = \alpha$, $\beta = 0$).

Corollary 14. Let $f(s, u): G \times \mathbb{R} \longrightarrow \mathbb{R}$ be a Carathéodory function with $f(s, 0) = 0$ for a.e. $s \in G$. Let T be the Nemyckii operator generated by $f(s, u)$ and suppose that $T: L_q(G) \longrightarrow L_p(G)$ (for some p, q with $1 \leq p, q < +\infty$). Then T is convex, i.e.

$\| T(\alpha x_1 + (1-\alpha)x_2)\| \leq \alpha \| T(x_1)\| + (1-\alpha)\| T(x_2)\|$ for all $\alpha \in \langle 0,1\rangle$, $x_1, x_2 \in L_q(G)$, if and only if

$$(+) \qquad \left| f\left(s, \frac{u_1+u_2}{2}\right)\right| \leq |f(s,u_1)| + \frac{1}{2}|f(s,u_2)|$$

for a.e. $s \in G$ and all $u_1, u_2 \in \mathbf{R}$.

<u>Proof.</u> By Propositions 12 and 13 we have: T is convex if and only if $|f(s,\cdot)| : \mathbf{R} \longrightarrow \mathbf{R}$ is convex for a.e. $s \in G$. It is sufficient to prove that (+) implies the convexity of T. But this follows by the $\frac{1}{2}$ - convexity (Proposition 13) and the continuity of T (see, for example, Krasnoselskii [10, Theorem 17.1]).

<u>Remark</u>. Analogous corollaries hold for (α,β)-subadditivity, α-subhomogeneity, α-superhomogeneity, and subadditivity.

<u>Proposition 15</u>. Let $f(s,u): G \times \mathbf{R} \longrightarrow \mathbf{R}$ be a Carathéodory function and T the Nemyckii operator generated by $f(s,u)$ such that $T: L_p(G) \longrightarrow L_p(G)$ ($1 \leq p < +\infty$). Then T is Lipschitzian with constant K if and only if $f(s,\cdot): \mathbf{R} \longrightarrow \mathbf{R}$ is Lipschitzian with constant K for a.e. $s \in G$.

<u>Proof.</u> If T is Lipschitzian with constant K, then

$$\int_G |f(s,u_1 x_M) - f(s,u_2 x_M)|^p ds \leq K^p \int_G |u_1-u_2|^p x_M \, ds$$

for all $u_1, u_2 \in \mathbf{R}$ and $M \subset G$, $\operatorname{mes} M < +\infty$. If we take $M = K_m(s) = \{\tau \in \mathbf{R}^n : \|\tau - s\| \leq 1/m\}$, then

$$\frac{1}{mes\, K_m(s)} \int_{G\cap K_m(s)} |f(s',u_1)-f(s',u_2)|^p ds' \le K^p |u_1-u_2|^p .$$

Hence by Zaanen [16, § 37], similarly as in the proof of Proposition 12, we have $|f(s,u_1)-f(s,u_2)| \le K|u_1-u_2|$ for a.e. $s \in G$ and all $u_1, u_2 \in \mathbb{R}$.

The converse is trivial.

Corollary 16. Under the hypotheses of Proposition 15, T is k-contractive ($0 \le k < 1$) if and only if $f(s,\cdot): \mathbb{R} \to \mathbb{R}$ is k-contractive for a.e. $s \in G$.

Let $\Gamma \subset \mathbb{R}^m$, $G \subset \mathbb{R}^n$ be measurable sets with positive Lebesgue measures, $K(t,s,u): \Gamma \times G \times \mathbb{R} \to \mathbb{R}$ be a function satisfying the Carathéodory conditions (that is, $K(t,s,u)$ is continuous in $u \in \mathbb{R}$ for a.e. $(t,s) \in \Gamma \times G$ and measurable in $(t,s) \in \Gamma \times G$ for each $u \in \mathbb{R}$) and let $T: L_q(G) \to L_p(\Gamma)$ be the Uryson operator generated by $K(t,s,u)$: $(T(x))(t) = \int_G K(t,s,x(s))ds$, $x \in L(G)$. Let $\{E_m(s)\}$ be a sequence of measurable sets in $\mathbb{R}^n$ tending regularly to s, that is, $mes\, E_m(s) > 0$, $E_m(s) \subset K_m(s)$, $mes\, K_m(s) \le k \cdot mes\, E_m(s)$ for all m (for some $1 \le k < +\infty$), and $\lim_{m\to+\infty} diam\, E_m(s) = 0$, where $K_m(s)$ is the closed ball centered at s with radius $1/m$.

By similar considerations as made above, we can obtain the following two propositions.

Proposition 17. Suppose that $1 \le p, q < +\infty$, $K(t,s,0) = 0$ for a.e. $(t,s) \in \Gamma \times G$. Let there exist, for each $u \in \mathbb{R}$ and a.e. $s \in G$, a function $g_{u,s} \in L_p(\Gamma)$ such that

$$\left| \int_{G \cap E_m(s)} K(t,s',u)\,ds' \right| \leq \operatorname{mes} E_m(s) \cdot g_{u,s}(t)$$

for a.e. $(t,s) \in \Gamma \times G$, all $u \in \mathbb{R}$ and m. If T is (α, β)-subadditive for some $\alpha, \beta \in \mathbb{R}$, then

$\|K(\cdot,s,\alpha u_1+\beta u_2)\|_{L_p} \leq |\alpha| \cdot \|K(\cdot,s,u_1)\|_{L_p} + |\beta| \cdot \|K(\cdot,s,u_2)\|_{L_p}$

for a.e. $s \in G$ and all $u \in \mathbb{R}$. Similarly, for the (α, β)-superadditivity, α-subhomogeneity, and so on.

Proposition 18. Suppose that $1 \leq p = q < +\infty$, $n = m$ and $\Gamma = G$. Let there exist, for each $u_1, u_2 \in \mathbb{R}$ and a.e. $s \in G$, a function $g_{u_1,u_2,s} \in L_p(G)$ such that

$$\left| \int_{G \cap E_m(s)} \{K(t,s',u_1) - K(t,s',u_2)\}\,ds' \right| \leq$$
$$\leq \operatorname{mes} E_m(s) \cdot g_{u_1,u_2,s}(t),$$

for a.e. $(t,s) \in G \times G$ and all $u_1, u_2 \in \mathbb{R}$ and m. If T is Lipschitzian with constant K, then

$$\|K(\cdot,s,u_1) - K(\cdot,s,u_2)\|_{L_p} \leq K|u_1 - u_2|$$

for a.e. $s \in G$ and all $u_1, u_2 \in \mathbb{R}$.

The following proposition is motivated by Krasnoselskii [10, Theorem 19.3].

Proposition 19. Let the Uryson operator $T: L_q(G) \to L_p(\Gamma)$ ($1 \leq p, q < +\infty$) generated by $K(t,s,u)$ be regular (see Krasnoselskii [10, p.378]). Suppose that there is $0 \leq k < 1$ such that

$$\lim_{\operatorname{mes} D \to 0} \sup_{x \in B_q(x_0,R)} \left\| \int_D K(\cdot,s,x(s))\,ds \right\|_p \leq \frac{kR}{2}$$

for all $x \in L_q(G)$ and $R > 0$ ($B_q(\cdot,\cdot)$, $B_p(\cdot,\cdot)$, $\|\cdot\|_q$, $\|\cdot\|_p$ denote the closed balls and norms in $L_q(G)$ and $L_p(\Gamma)$, respectively).

Then T is concentrative.

<u>Proof.</u> T is continuous by Krasnoselskii [10, Theorem 18.5]. Let $R > 0$, $x_0 \in L_q(G) \cap L_\infty(G)$ be given. We can choose $\delta > 0$ such that $mes\, D \leq \delta$ implies

$$\sup_{x \in B_q(x_0,R)} \| \int_D K(\cdot, s, x(s))ds \|_p \leq \frac{2k+1}{3} \cdot \frac{R}{2} \quad .$$

By Krasnoselskii [10, Theorem 19.2], the set $C = T(B_\infty(x_0, R\delta^{-1/q})) \subset T(B_\infty(0, \|x_0\|_\infty + R\delta^{-1/q}))$ is precompact in $L_p(\Gamma)$. Let $x \in B_q(x_0, R)$ and define $\tilde{x} = x_0 + \min\{|x - x_0|, R\delta^{-1/q}\} \cdot sign\,(x - x_0)$. Then

$\tilde{x} \in B_\infty(x_0, R\delta^{-1/q}) \cap B_q(x_0, R)$, because of

$|\tilde{x} - x_0| \leq R\delta^{-1/q}$ and $|\tilde{x} - x_0| \leq |x - x_0|$. Let $\tilde{D} = \{s \in G : x(s) \neq \tilde{x}(s)\}$. From

$$mes\, \tilde{D} \cdot R \cdot \delta^{-1} = [\int_{\tilde{D}} (R\delta^{-1/q})^q ds]^{1/q} \leq \|\tilde{x}\|_q \leq R$$

it follows that $mes\, \tilde{D} \leq \delta$. Then

$$\|T(x) - T(\tilde{x})\|_p = \| \int_{\tilde{D}} \{K(\cdot,s,x(s)) - K(\cdot,s,\tilde{x}(s))\} ds \|_p \leq \| \int_{\tilde{D}} K(\cdot,s,x(s))ds \|_p + \| \int_{\tilde{D}} K(\cdot,s,\tilde{x}(s))ds \|_p \leq 2 \cdot \frac{2k+1}{3} \cdot \frac{R}{2} = \frac{2k+1}{3} \cdot R \ ,$$

and hence $B_p(C, \frac{2k+1}{3} \cdot R) \supset T(B_q(x_0, R))$, that is, $\chi(T(B_q(x_0,R))) \leq \frac{2k+1}{3} \cdot R$ for all $x_0 \in L_q(G) \cap L_\infty(G)$.

Let $x_0 \in L_q(G)$ be arbitrary and $x_1 \in L_q(G) \cap$ $\cap L_\infty(G)$ with $\|x_1 - x_0\|_q \leq R \cdot \frac{1-k}{4k+2}$. Then

$$\chi(T(B_q(x_0,R))) \leq \chi(T(B_q(x_1, \|x_1 - x_0\|_q + R))) \leq$$
$$\leq \frac{2k+1}{3} \cdot (\|x_1 - x_0\|_q + R) \leq \frac{2k+1}{3} \cdot$$
$$\cdot \left[\frac{1-k}{4k+2} + 1 \right] \cdot R = \frac{k+1}{2} \cdot R \quad .$$

Let $M \subset L_q(G)$ be a bounded and non-precompact set. Then there exists a finite subset σ of $L_q(G)$ with

$$\sigma + B_q(0, \frac{3}{k+2} \cdot \chi(M)) = B_q(\sigma, \frac{3}{k+2} \cdot \chi(M)) \supset M \quad .$$

Then
$$\chi(T(M)) \leq \chi(T(\sigma + B_q(0, \frac{3}{k+2} \cdot \chi(M))) \leq \frac{3}{k+2} \cdot \frac{k+1}{2} \cdot$$
$$\cdot \chi(M) < \chi(M) \quad ,$$

and T is concentrative.

Corollary 20. Let $T, p = q, G = \Gamma$ be as in Proposition 19 and let T map a bounded non-empty convex closed subset of $L_p(G)$ into itself. Then T has a fixed point.

Proof. See Proposition 19 and Sadovskii's Theorem [13].

§ 3. Continuity of nonlinear mappings.

Lemma 21. Let X and Y be two normed linear spaces, C a subset of X, $T: C \to Y$ a mapping, demicontinuous at a point $x_0 \in int\, C$ (= the interior of C). Suppose that there is a function $\kappa(t): (0, +\infty) \to$ $\to (0, +\infty)$ such that $\lim_{t \to +\infty} \kappa(t) = +\infty$ and

(*) $\|T(x_0 + th) - T(x_0)\| \geq \kappa(t) \|T(x_0 + h) - T(x_0)\|$, whenever $t > 0$, $h \in X$, $x_0 + h \in C$, $x_0 + th \in C$.

Then T is continuous at x_0.

<u>Proof</u>. Suppose that T is not continuous at x_0. Then there are: a positive number ε and a sequence $\{h_n\} \subset X$ with $h_n \neq 0$, $h_n \to 0$, $x_0 + h_n \in C$, and

$$\| T(x_0 + h_n) - T(x_0) \| \geq \varepsilon > 0 .$$

Setting $t_n = \| h_n \|^{-1/2}$ and $y_n = t_n h_n$, we have $\| y_n \| = t_n \| h_n \| = \| h_n \|^{1/2} \to 0$, i.e. $x_0 + y_n \to x_0$. Without loss of generality, we can assume that $\{x_0 + y_n\} \subset C$. Since T is demicontinuous at x_0, $T(x_0 + y_n) \to T(x_0)$ weakly; hence the sequence $\{\| T(x_0 + y_n) - T(x_0) \|\}$ is bounded. On the other hand, $(*)$ implies

$\| T(x_0 + y_n) - T(x_0) \| \geq \kappa(t_n) \| T(x_0 + h_n) - T(x_0) \| \geq \kappa(t_n) \varepsilon \to +\infty$

as $n \to +\infty$, since $t_n \to +\infty$, and we have $\| T(x_0 + y_n) - T(x_0) \| \to +\infty$. This contradiction proves our lemma.

<u>Theorem 22</u>. Let X and Y be two normed linear spaces, C a subset of X, $T : C \to Y$ a mapping, demicontinuous at a point $x_0 \in \operatorname{int} C$. Suppose that T possesses the Gâteaux differential $VT(x_0, h)$ at x_0, demicontinuous at $h = 0$. Further, suppose that there is a function $\kappa(t) : (0, +\infty) \to (0, +\infty)$ such that

$\lim_{t \to +\infty} \kappa(t) = +\infty$ and

$$\kappa(t) \| \omega(x_0, h) \| \leq \| \omega(x_0, th) \| \quad (t > 0,\ h \in X),$$

where $\omega(x_0, h)$ denotes the Gâteaux remainder of T at x_0.

Then T is continuous at x_0 .

Proof. By the definition of the Gâteaux differential, we have

$$T(x_0+h)-T(x_0)=VT(x_0,h)+\omega(x_0,h) .$$

It is obvious that $\omega(x_0,h)$ is demicontinuous at $h=0$. Clearly, $VT(x_0,h)$ and $\omega(x_0,h)$ satisfy the conditions of Lemma 21 with $\varkappa(t)\equiv t$ and $\varkappa(t)\equiv\varkappa(t)$, respectively. By Lemma 21, $VT(x_0,h)$ and $\omega(x_0,h)$ are continuous at $h=0$, i.e. $T(x)$ is continuous at x_0 .

The following well-known fact is a direct consequence of Lemma 21 or Theorem 22.

Corollary 23. Let X and Y be normed linear spaces and $T: X \to Y$ a linear mapping. Then the following conditions are equivalent:

(1) T is continuous;

(2) T is demicontinuous;

(3) T is weakly continuous.

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