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GRAPHS WITH SMALL ASYMMETRIES x)

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§ 1. Introduction. With every graph X (here undirected, loopless, without multiple edges) we can associate the group $G(X)$ of its automorphisms, i.e., the group of all permutations of $V(X)$ which preserve the adjacency relation (for notation we follow [5]). If $G(X)$ is the trivial group, then the graph is called asymmetric.

The structure of asymmetric graphs was studied by Erdös and Rényi in [1] using the notion of the asymmetry $A[X]$ of the graph X; this is defined as the smallest number of edges necessary for symmetrization of the graph. In [1] bounds are given for the asymmetry of a graph in terms of the number of its vertices and edges; using probability methods, it is shown that the bounds are asymptotically best possible.

Let us define the function $A(p,q)$: $Card \times Card \rightarrow Card$ by $A(p,q) = max \{A[X];\ |V(X)| = p,\ |E(X)| = q\}$ if $p > 0 \binom{p}{2} \geq$ $\geq q \geq 0$, $A(p,q) = 0$ otherwise. The introduction

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of the function A is motivated by [4] in which an extremal problem for asymmetric graphs is investigated; Theorem 1 in [4] says that if $p \leq 5$ then $A(p,q) = 0$ and further the numbers m_p, M_p are found such that if $p \geq 6$, and $q < m_p$ or $q > M_p$ then $A(p,q) = 0$ while $A(p,m_p) > 0$ $A(p, M_p) > 0$ (p finite).

Here we obtain (in § 3) a characterization of the support of the function A using a result about asymmetric extensions of a graph; then (in § 4) we résolve the analogous question concerning the set $\{(p,q); A(p,q)=1\}$.

In § 2 we summarize basic observation about function A and completely determine $A(p,q)$ where p is infinite.

In § 5, we relate the results of §§ 3, 4 to those of the papers [1],[4], in particular we determine the numbers $F(p, 2)$ which is proposed in [2] .

§ 2. Asymmetry of a graph (infinite case).

Let $p, q \leq \binom{p}{2}$ be cardinals. Denote by $\mathcal{A}_p$ ($\mathcal{A}_{p,q}$ respectively) the class of all asymmetric graphs with p vertices (p vertices and q edges respectively). Let $\mathcal{A} = \bigcup_{p \in \mathrm{Card}} \mathcal{A}_p$.

A graph X with $|V(X)| = p$ $|E(X)| = q$ is shortly called a p,q-graph.

Definition. Let X be a graph $|V(X)| = p$. We define $A[X] = \min\{|\Delta(E(X), E(Y))|;\ V(X) = V(Y),\ Y \notin \mathcal{A}_p\}$

$A^+[X] = \min\{|\Delta(E(X), E(Y))|;\ V(X) = V(Y),\ X \subseteq Y \notin \mathcal{O}\!\ell_p\}$,

$A^-[X] = \min\{|\Delta(E(X), E(Y))|;\ V(X) = V(Y),\ X \supseteq Y \notin \mathcal{O}\!\ell_p\}$,

where $\Delta(A, B)$ is the symmetric difference of the sets A, B and $X \subseteq Y$ means $V(X) \subseteq V(Y)$, $E(X) \subseteq E(Y)$. (This definition coincides with the definition of $A[X]$ in [1] and definitions $A^+[X]$, $A^-[X]$ in [4].) Let us define analogously as in the introduction the functions $A^+(p,q)$ $A^-(p,q)$.

We will investigate these functions simultaneously; if the same statement holds for A, A^-, A^+, we shall, for the sake of brevity, use the symbol $A^o(p,q)$.

Finally, let ${}^k\mathcal{O}\!\ell$ denote the class of all graphs for which $A[X] \geq k$.

Clearly $\mathcal{O}\!\ell = {}^1\mathcal{O}\!\ell \supset {}^2\mathcal{O}\!\ell \supset {}^3\mathcal{O}\!\ell \ldots$.

The following lemma gives us first information about the functions A^+, A^-, A :

<u>Lemma 1.</u> (i) $A[X] \leq \min(A^+[X], A^-[X])$, hence

$$A(p,q) \leq \min(A^+(p,q), A^-(p,q)) \quad ;$$

(ii) $X = \cup X_i \Rightarrow A^o[X] \leq \min_i A^o[X_i]$

($\cup X_i$ means disjoint union),

(iii) $A^+(p,q) = A^-(p, \binom{p}{2} - q)$,

$$A^-(p,q) = A^+(p, \tbinom{p}{2} - q) ,$$

$$A(p,q) = A(p, \tbinom{p}{2} - q) ,$$

whenever the right hand side is meaningful (p, q finite),

(iv) $A(p,q) > 0 \Leftrightarrow A^+(p,q) > 0 \Leftrightarrow A^-(p,q) > 0$.

For proof of (i) - (iii) see [4] , Lemma 3.1,p.72, and [1], Lemmas 1,2, pp.295-6.

(iv) is obvious since all three statements express the same thing, namely that $\mathfrak{A}_{p,q} \neq \emptyset$.

Let us give the following simple sufficient condition for $A[X] \geq k$.

Lemma 2. Let X be a graph, P the set of all permutations on the set $V(X)$, U the set of all unordered pairs of elements of $V(X)$.
For $f \in P$ let $f^{\#}: U \to U$ be given by $f^{\#}([x,y]) = [f(x), f(y)]$, $[x,y] \in U$. If

$$|\Delta(f^{\#}E(X))| \geq 2k$$

for every $f \in P$, $f \neq id$, then $A[X] \geq k$.

Proof. Suppose $A[X] < k$. Then there are edges $\{e_1, \ldots, e_m\} \subset E(X)$ and $\{e'_1, \ldots, e'_{m'}\} \cap E(X) = \emptyset$ such that $m + m' < k$, and the graph Y with

$$E(Y) = (E(X) \cup \{e'_1, \ldots, e'_{m'}\}) - \{e_1, \ldots, e_m\}$$

is symmetric. Thus there is an $f \in P \cap G(Y)$, $f \neq id$. Now split $E(X) - f^{\#}E(X)$ into two disjoint sets $M = (E(X) - f^{\#}E(X)) \cap E(Y)$ and $N = (E(X) - f^{\#}E(X)) - E(Y)$. N is contained in $E(X) - E(Y)$, hence $|N| \leq m$. The map $e \to f^{\#-1}e$ is an injection $M \to E(Y) - E(X)$, hence $|M| \leq m'$. Thus $|E(X) - f^{\#}E(X)| \leq m + m'$. Similarly, one shows that $|f^{\#}E(X) - E(X)| \leq m + m'$, so that

$$|\Delta(f^{\#}E(X), E(X))| \leq 2(m + m') < 2k .$$

Let us solve now the case of infinite graphs.

Theorem 1. Let $\mathfrak{n}$ be an infinite cardinal. Then $A^0(\mathfrak{n},\mathfrak{q}) > 0 \Longleftrightarrow \mathfrak{n} = \mathfrak{q}$, moreover $A^0(\mathfrak{n},\mathfrak{n}) = \mathfrak{n}$.

Proof. It is obvious that if X is an asymmetric infinite graph then $|E(X)| = |V(X)|$ (since otherwise there would be in X two isolated vertices), hence $A(\mathfrak{n},\mathfrak{q}) = 0$ if $\mathfrak{q} \neq \mathfrak{n}$.

We now give a construction of a graph $X_{\mathfrak{n}}$ with $A[X_{\mathfrak{n}}] = \mathfrak{n}$.

Let $\mathfrak{n}$ mean the set of all ordinal numbers less than $\mathfrak{n}$.

Let $\mathfrak{n} = \bigcup_{\iota<\mathfrak{n}} M_\iota$, $|M_\iota| = \mathfrak{n}$, M_ι pairwise disjoint.

Define $E(X_{\mathfrak{n}}) = \{[i,j]; i<j, j \in \bigcup_{\iota>i} M_\iota\}$. We prove (using Lemma 2) that $A[X] = \mathfrak{n}$. Let $f \in P$, $f \neq id$. (We again denote by P the set of all permutations on the set $\mathfrak{n}$.)

Let $N = \{i; f(i) \neq i\}$, let i_0 be the first element of N (with respect to natural ordering of $\mathfrak{n}$). Since both $f(i_0)$ and $f^{-1}(i_0)$ belong to N it follows necessarily that $f(i_0) > i_0$ and $f^{-1}(i_0) > i_c$. Suppose first that $f(i_0) \leq f^{-1}(i_0)$. If $|\bigcup_{\iota>f(i_0)} M_\iota - f(\bigcup_{\iota>i_0} M_\iota)| = \mathfrak{n}$, then f satisfies the premise of Lemma 2, by definition $E(X_{\mathfrak{n}})$. Similarly if $|\bigcup_{\iota>i_0} M - f(\bigcup_{\iota>f^{-1}(i_0)} M_\iota)| = \mathfrak{n}$.

Assume f does not satisfy any of these equations. We prove that it is impossible. From the first inequality we have $|M_{f(i_0)} \cap f(\bigcup_{\iota>i_0} M_\iota)| < \mathfrak{n}$, from the second

$|M_{f(i_0)} \cap f(\bigcup_{\iota > f^{-1}(i_0)} M_\iota)| = p$. But $\bigcup_{\iota > i_0} M_\iota \supset \bigcup_{\iota > f^{-1}(i_0)} M_\iota$, this is a contradiction. Similarly if $f^{-1}(i_0) \leq f(i_0)$. Hence every $f \in P$ satisfies Lemma 2, consequently $A[X_p] = p$.

Let us finish this part with the following observation:

<u>Proposition 1</u>. Let $p < \aleph_0$, $a_p^0 = \max_q A^0(p, q)$. Then the function $A^0(p, \sigma)$ assumes all integral values in the interval $[0, a_p^0]$.

<u>Proof</u>. This fact follows from the property of the functions $A^0(p, \sigma)$ that $|A^0(p, q+1) - A^0(p, q)| \leq 1$. This is clear since every $(p, q+1)$-graph is obtained by adjoining some edge to some (p, q)-graph and vice versa.

<u>Corollary</u>. On every sufficiently large set, there is a graph with asymmetry k ($k \in Card$) .

<u>Proof</u>. If $k < \aleph_0$ then by [2] Theorem 2 there is a p_0 such that $p > p_0 \Longrightarrow a_p > k$. By the above proposition we have the existence of a graph with $A[X] = k$ on every finite set of cardinality $> p_0$. In the infinite case (and infinite k) it is enough to observe that the graph X_p constructed in the proof of Theorem 1 is p-connected if X is a finite graph then $A[X \cup X_p] = A[X]$ and $A[X_{p_1} \cup X_{p_2}] = \min(p_1, p_2)$.

Since the case of asymmetric infinite graphs is solved by Theorem 1, <u>from now on graph will mean finite graph</u>.

§ 3. Asymmetry equal 0 :

We are returning to our central problem of determining some values of the functions $A^\circ(p,q)$.
We will need the following simple lemma:

Lemma 3. For every p, q $A^\circ(p,q) > 0$ implies $A^\circ(p+1, q+1) > 0$.

Proof. It is enough to show that we can adjoin to every asymmetric graph X a single point and edge in such a way that the new graph is also asymmetric. Let $x \notin V(X)$, define the graph Y: $V(Y) = V(X) \cup \{x\}$, $E(Y) = E(X) \cup \cup [x, y]$, where y is an arbitrarily chosen point if X has no points of degree 1, and y is a vertex of the greatest from all points of degree > 2 otherwise. Y is obviously asymmetric, since $f \in G(X)$ implies $f(x) = x$ (degree and distance are invariants under automorphisms).

We will first characterize the support of A°.

Theorem 2. $A^\circ(p,q) = 0$ if and only if either $p < 6$ or $\aleph_0 > p \geq 6$, $q < m_p$, $q > M_p$ or p infinite $p \neq q$ (for m_p, M_p see [4], Theorem 1 or the proof below).

Proof. For the infinite case, see Theorem 2, Chapter 1.

The sufficiency of the condition was shown in [4], Theorem 1.

Let $\aleph_0 > p \geq 6$, $m_p \leq q \leq M_p$. We have to show that $\mathcal{O}_{pq} \neq \emptyset$.

Case 1: Let $m_p \leq q \leq p-1$. We need to write down the construction of the numbers m_p (see [4], pp.62-63): $m_6 = m_7 = 6$, and for $p \geq 8$, $m_p = p - m'_p$, where $m'_p = \sum_{n=1}^{N} a_n + w$, where a_n is the number of non-isomorphic asymmetric trees with n vertices (computed by Harary and Prins [2]) and the numbers N, w are defined as follows: $\sum_{n=1}^{N} a_n n \leq p < \sum_{n=1}^{n+1} a_n n$, $p = \sum_{n=1}^{N} a_n n + w(N+1) + r$ $(0 \leq w < a_{N+1}, 0 \leq r < N+1)$. From this we see that either $m'_{p+1} = m'_p$ (if $0 \leq r < N$) or $m'_{p+1} = m'_p$ (if $r = N$).

In [4] it is proved that $A(p, m_p) > 0$. From this also follows that if $m_p + 1 < p$, then $A(p, m_p + 1) > 0$ since we can take a forest $X \in \mathcal{A}_{p,m_p}$ (see [4],pp.62-63) and form the forest Y by omitting one component of X (say with n points) (since $m_p < p - 1$, X is dis connected) and enlarging the "greatest" of the remaining components by n points (for instance as in Lemma 3):

Now by Lemma 3 we have $A(p, q) > 0 \Longrightarrow$
$\Longrightarrow A(p+1, q+1) > 0$.

Thus, supposing $A(p,q) > 0$, $m_p \leq q < p$, we obtain $A(p+1, q) > 0$ for $m_p + 1 \leq q < p + 1$, but by the above observation $m_p + 1 \leq m_{p+1} + 2$ and we know already $A(p+1, m_{p+1}) > 0$. $A(p+1, m_{p+1} + 1) > 0$.

We have also $A(p,q) > 0$ for $\binom{p}{2} - p < q \leq \binom{p}{2} - M_p$, by Lemma 1 (iii - iv) and by $M_p = \binom{p}{2} - m_p$ (see [4]).

<u>Case 2</u>: $p \leq q \leq \binom{p}{2} - p$.

We use induction again. In [1], Chapter 1, it is shown that $A(6,6) = A(6,7) = A(6,8) = A(6,9) = 1$. Suppose that $A(p,q) > 0$ for $p \leq q \leq \binom{p}{2} - p$. Then we have again by Lemma 3 that $A(p+1, q) > 0$ for $p + 1 \leq q \leq \binom{p}{2} - p + 1$. But

$$\binom{p}{2} - p + 1 = \frac{p}{2}(p-3) + 1 > \frac{p}{2}\,\frac{p+1}{2} = \binom{p+1}{2}/2$$

for every $p \geq 7$, and if $p = 6$, then $\binom{6}{2} - 6 + 1 = 10 = \left[\frac{\binom{7}{2}}{2}\right]$. Hence, using Lemma 1 (iii - iv) we have $A(p+1, q) > 0$ for $p + 1 \leq q \leq \binom{p}{2} - p - 1$.

By Lemma 1 (iv) the support of the functions A^+, A^- coincides with that of the function A.

§ 4. Asymmetry equal 1.

We give a particular result concerning the set $\{(p,q) \mid A^\circ(p,q) = 1\}$, the full characterization of this set is in [3]. In connection with it we have to distinguish more carefully between the functions A, A^+, A^- because we do not have the analogy of (iv) Lemma 1, § 2 for $A^\circ(p,q) \geq 1$.

In the sufficiency part of our theorem we will use the following lemma:

For i is natural, X graph let $D_i(X) = \{x ; d(x, X) = i\}$.

Lemma 4. $A^\circ[X] = k$, $i + j \leq k$ then

(i) $V(D_i, X) \cap D_j = 0$,

$V(D_i, X) \cap V(D_j, X) = 0$,

(ii) if $i+j<k$, then $\min\{|D_i|, |D_j|\} \leq 1$

Proof. (i) By [4], Lemma 3.2, $A^-[X] \leq \Delta_{xy}$, where Δ_{xy} is the cardinality of symmetric difference of the neighbourhoods of the points x, y.

In fact $A^+[X] \leq \min \Delta_{x,y}$ also holds (if $\Delta_{x,y} < k$, then the addition $\Delta_{x,y}$ edges $[x,z]$ $z \in V(y,X) - V(x,X)$ and $[y,z]$ $z \in V(x,X) - V(y,X)$ will produce the symmetric graph).

It is easy to show that if one of the conditions in (i) is not satisfied, then also there is x, y $\Delta_{x,y} < k$.

(ii) is obvious, since $\Delta_{x,y} < k$, for every two vertices.

Lemma 5. (i) T Tree $\Longrightarrow A^\circ[T] \leq 1$,

(ii) X unicyclic $\Longrightarrow A^\circ[X] \leq 1$,

(iii) X bicyclic $\Longrightarrow A^\circ[X] \leq 1$,

(iv) If $A[X] > 1$ and C is the union of all cycles in X, then $x \in V(X) \Longrightarrow \rho(x, C) \leq 1$.

Proof. (i) $A^-[T] \leq 1$ is part of [1], Theorem 5. By Lemma 4 if $A^+[T] > 1$ then the tree does not contain two endpoints which are connected with the same point and every endpoint is connected with the point of degree > 2. It is easy to show that this is impossible.

(ii) and (iii) is [1], Theorem 7.

(iv) is the essential part of the proofs of (ii) and (iii).

Let $x \notin C$ and $x_0 \in C$ such that $\rho(x, x_0) =$

$= \varphi(x, C)$. It is clear that x_0 is determined uniquely. It is also clear that x_0 is a cut point of X and that the component T of $X - x_0$ which contains x is a tree. If $\varphi(x, x_0) > 1$, we can apply (i) to T and get a contradiction.

To characterize $A^0(p, q) = 1$ we begin with

Proposition 2. $A^0(p, q) = 1$ for $m_p \leq q \leq p$, $\binom{p}{2} - p \leq q \leq M_p$.

Proof. We use a well known connection between the cyclomatic number $N(X)$, the number of components $c(X)$ and the number of edges and points of a graph X.

Let X be a p, q-graph, $q \leq p$. Then
$N(X) = q - p + c(X)$.

If all components X_i of X are $p_i q_i$-graphs, we have

$$N(X_i) = q_i - p_i + 1 ,$$
$$N(X) = \Sigma N(X_i) = \Sigma(q_i - p_i) + c(X) \leq c(X) .$$

Hence there is a component for which $N(X_i) \leq 1$ (otherwise $N(X) > c(X)$).

By Lemma 5 (i) - (iii) and Lemma 1 (ii) $A^0[X] \leq 1$, therefore $A^0(p, q) \leq 1$. But by Theorem 2 $A^0(p, q) = 1$, $q \geq m_p$. The statement for $\binom{p}{2} - p \leq q \leq M_p$ we obtain by Lemma 1 (iii).

Remark. We show that this bound is best possible for a large enough p ($p \geq 18$) by exhibiting a bicyclic graph X with no point of degree 1 (which is of course a necessary condition for a graph with $A^0[X] \geq 2$

to have an isolated point (see Lemma 4)). For smaller values of p this can be improved (see [3]). Investigating the p, q -graphs for small $q - p$ we use the concept of <u>subdivision</u> of a graph with the following notation.

Let the graph X be a subdivision of the graph $\tilde{X}$, $D_2(X) = 0$. (At this point we admit $\tilde{X}$ to be a multigraph.) For every $[a, b]_i$ (the i-th edge connecting a, b in $\tilde{X}$) we denote by $W_i(a, b)$ the path in X which arises by subdividing $[a, b]_i$, and $p_i(a, b)$ the number of points of degree 2 belonging to $W_i(a, b)$ (i.e. $p_i(a, b) + 1$ is the length of $W_i(a, b)$).

<u>Lemma 6.</u> Let X be a connected $p, p+2$ -graph, $D_1(X) = 0$, $A^{\circ}[X] = 2$. Then X is a subdivision of K_4 and $[i, j] \neq [j, k] \Rightarrow p(i, j) \neq p(j, k)$ for $i, j, k \in V(K_4)$.

<u>Proof</u>. Let X be a graph satisfying the hypothesis. Then the cycle base of X consists of the three cycles C_1, C_2, C_3 by the formula $N(X) = q - p + c(X)$ used in the proof of Proposition 2.

Since X is an asymmetric $p, p+2$ -graph with $D_1(X) = 0$, we have that for every $i = 1, 2, 3$, there is a $j \neq i$ such that $|C_i \cap C_j| > 1$.
Now there are only four possible multigraphs for $\tilde{X}$ (we write only edges):

$X_1 = [a, b]_i,\ i = 1, 2, 3, 4$,

$X_2 = [a, b]_1, [a, b]_2, [b, c]_1, [b, c]_2, [a, c]$,

$X_3 = [a, b]_1, [a, b]_2, [c, d]_1, [c, d]_2, [a, c], [b, d]$,

$X_4 = K_4$, the complete graph on the four vertices a, b, c, d .

Every subdivision of X_1 is obviously symmetric. It can be shown that every subdivision of X_2 can be made symmetric by deleting or adjoining one edge. The same holds for the graph X_3 and thus the graph X is necessarily a subdivision of K_4 .

Let us suppose by way of contradiction that $p(a,b) = p(a,c)$ for $b \neq c$. If $d' \in W(a,d)$ is such that $[d',d] \in E(X)$ then $X - [d,d']$ is obviously symmetric.

We show that we can get a symmetric graph also by adjoining one edge. Let $p = \min\{p(b,d), p(c,d)\}$ say $p < p(b,d)$ (since by asymmetry necessarily $p(b,d) \neq p(c,d)$); let $d'' \in W(b,d)$ be such that $|W(d'',b)| = p+1$. Then $X \cup [d,d'']$ is again symmetric.

<u>Proposition 3</u>. Let X be a connected $p, p+2$-graph, $D_1(X) = 0$. Then $A^\circ[X] = 2 \Rightarrow p \geq 18$ and for every $p \geq 18$ such a graph exists.

<u>Proof</u>. By Lemma 6, X is a subdivision of K_4 . We use the following observation:

$A^\circ[X] = 2$, X a subdivision of K_4 implies $p(a,b) > 0$ for every $a \neq b \in K_4$. This is obvious, since if $[a,b] \in E(X)$, $a, b \in D_3(X)$, then $X - [a,b]$ is a subdivision of $[c,d]_1, [c,d]_2, [c,d]_3$ and thus a symmetric graph. One can find also points $x, y \in X$ such that $X \cup [x,y]$ is a symmetric graph.

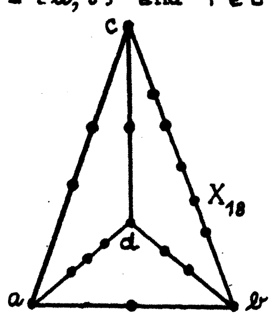
By this and by Lemma 6, we have $p \geq 16$. There is exactly one such graph with $p = 16$ and this is symmetric.

Up to isomorphism, there is also exactly one graph for $p=17$ ($p(a,b)=p(c,d)=1$, $p(a,c)=p(b,d)=2$, $p(a,d)=3$, $p(b,c)=4$). This graph can be made by deleting the edge $[d,d']$, where $d'\in W(a,d)$, and by adjoining the edge $[d',c']$, where $c'\in W(b,c)$, $\rho(c,c')=2$.

Define the graph $X_{18}=X$ as the subdivision of K_4 with $p(a,b)=p(c,d)=1$, $p(a,c)=p(b,d)=2$, $p(a,d)=3$, $p(c,b)=5$. Then $A^-[X]=2$, because

1. $p(x,y)\neq p(x,z)+p(z,y)$ for every $x\neq y\neq z$ $D_3(X)$ and hence $X-[u,t]$, where $u\in D_3(X)$ is asymmetric;

2. $\{u,t\}\cap D_3(X)=0 \Rightarrow (f\in G(X-[u,t]) \Rightarrow f\{u,t\}=\{u,t\}$ and $f\in G(X)$.



By the same method we can prove that the graph X_n a subdivision of K_4 defined by $p(a,b)=p(c,d)=1$, $p(a,c)=p(b,d)=2$, $p(a,d)=3$, $p(b,c)=n=13$ satisfies $A^-[X_n]=2$. $A^+[X_n]=2$ can be easily proved in view of the fact that $(D_3(X_n)\times D_3(X_n))\cap E(X_n)=\emptyset$ and thus every graph $X\cup[x,y]$ has at most 3 points of degree 3 which are in relation.

Let us add one remark. We were led in previous considerations roughly by the connection between $\Delta(q,p)=q-p$ and $A^\circ(p,q)$. Now we show that, limitwise, constant difference Δ characterizes only the values

1 and 2 of the functions A° .

<u>Proposition 4</u>. $k \leq \frac{4}{3}$ then $\lim\limits_{\substack{p \to \infty \\ \frac{q}{p} \to k}} A^{\circ}(p, q) \leq 2$.

The proof is essentially the proof of Erdös-Rényi [1], Theorem 6 where it is shown $A(p, q) \leq 2$ for $q < \frac{4}{3} p - \frac{2}{3}$. One is essentially using the fact that if $A^{-}[X] > 2$, then $m_0(X) \leq 1$, $m_1(X) \leq 2$ (see [3]). Since the same thing holds if $A^{+}[X] > 2$, we can use the proof in [1] in "limit modification". The limit has to exist by [3] Theorems 3,4.

<u>Corollary</u>. $\lim\limits_{p \to \infty} A^{\circ}(p, p+\Delta) = 2 \Longleftrightarrow \Delta > 0$,

$$\lim_{p \to \infty} A^{\circ}(p, p+\Delta) = 1 \Longleftrightarrow \Delta \leq 0 \ .$$

<u>Proof</u>. If $\Delta > 0$ then by Proposition 4, $\lim\limits_{p \to \infty} A^{\circ}(p, p+\Delta) \leq 2$, but by [3] Theorem 4, $A^{\circ}(p, p+\Delta) > 1$ for every sufficiently large p.

If $\Delta \leq 0$ then there is p_0 such that $p > p_0 \Rightarrow m_p < p + \Delta$ (since $m_p \to -\infty$).

§ 5. <u>Asymmetric bounds.</u>

In [1], Chapter 4, Erdös and Rényi have defined the following numbers:

$F(k, p)$ is the smallest q such that $A(p, q) \geq k$.(The numbers $F^{+}(k, p)$, $F^{-}(k, p)$ are defined in the obvious analogy.)

As an immediate consequence of finding the best possible lower bound for $A(p, q) = 1$, the values $F(p, 1)$,

$F^+(p,1)$, $F^-(p,1)$ were determined in [4], Theorem 7 (of course by Lemma 1 (iv) - they coincide).

Having here characterized when $A^\circ(p,q)=1$, we have

Corollary. Let $p \geq 10$. Then $F^\circ(p,2)=p+2$ for $p \leq 17$, $F^\circ(p,2)=p+1$ for $p > 17$.
The proof is an immediate consequence of Proposition 2, 3, examples for $A^\circ(p,p+2)=2$, $p \leq 17$ may be found in [3].

Let $C^\circ(p,k)$ be the smallest q such that there is connected p,q-graph X such that $A^\circ[X] \geq k$. It is conjectured in [1] that $C(n,k)=F(n,k)$ for all $k \geq 2$.

By the above corollary, we see that for $k=2$ this is false for all $n \geq 17$ since $C(p,2)=p+2$ (see Lemma 6) but $F(p,2)=C(p,2)-1$. (We have an analogous observation for the values of F^+, F^-, C^+, C^-, too.) We see that $F^\circ(p,2)=C^\circ(p,2)$ for the first few values of p, where $F^\circ(p,2)$ is defined. This holds generally.

Proposition 5. Let p_0 be the first p such that $F^\circ(p,k)$ is defined. Then $C^\circ(p,k)$ is defined and $F^\circ(p,k)=C^\circ(p,k)$.

Proof. We have the graph X $|V(X)|=p_0$ $|E(X)| = F^\circ(p_0,k)$. Since there is no graph with asymmetry k on $< p_0$ vertices, the X must obviously be connected by Lemma 1 (ii).

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