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M - BORNOLOGICAL SPACES

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Introduction. In a recent paper (cf. [8]) the class of $\mathcal{M}$ -barrelled spaces which represents an extension of vector spaces of the second category, has been investigated. The purpose of this article is to give such extension of metrisable topological spaces to a wider class of $\mathcal{M}$ -bornological spaces.

It turns out that any $\mathcal{M}$ -bornological and locally convex space is at the same time a bornological space in the usual sense. Recall that a locally convex space E is termed bornological if any convex set B in E absorbing an arbitrary bounded subset in E is a neighborhood of the origin in E .

As to the denotation we preserve that one used in [8]. For the corresponding results and necessary notions in locally convex spaces as well as for the classical prototype of the theory of such spaces we refer to [2], [3],[4],[5] and to [6].

It should be pointed out that all vector spaces investigated in this paper are considered over the same and usual field of scalars.

1. M - BORNOLOGICAL SPACES

The purpose of this section is to generalize the properties of locally convex bornological spaces and metrisable vector spaces on the class of multibornological spaces.

<u>Definition 1.</u> Let E be a topological vector space. Then E will be called multibornological (abbreviated $\mathcal{M}$ -bornological) if for any sequence $(B_n ; n \in N)$ of balanced subsets in E, each B_n is absorbing any bounded subset in E, the set given by the formula

$$\Omega(B_n) = \bigcup_{k=1}^{\infty} \sum_{n=1}^{k} B_n \tag{1}$$

is a neighborhood of the origin in E.

<u>Remark 1</u>. Any locally convex multibornological space is bornological.

Proof. Take in a locally convex space E an absolutely convex subset B absorbing all bounded sets in E. Define a sequence $B_n = 2^{-n} \cdot B$, $n \in N$. But $\Omega(B_n)$ defined by Formula (1) is a neighborhood in E. From

$$\sum_{n=1}^{k} B_n = \sum_{n=1}^{k} 2^{-n} \cdot B \subseteq B$$

we obtain $\Omega(B_n) \subseteq B$. Hence B is a neighborhood in E.

<u>Theorem 1</u>. Suppose that E is a metrisable vector space. Then E is $\mathcal{M}$ -bornological.

The proof follows from the fact that a balanced subset B in E absorbing all sequences convergent to the origin is a neighborhood in E (cf.[2]).

<u>Corollary</u>. Any locally bounded space is M - bornological. Especially, if E is a p-normed space, $0 < p \leq 1$, then E is M-bornological.

<u>Theorem 2.</u> Let E be an M-bornological space, F an arbitrary topological vector space, H a subset of the space $\mathcal{L}(E;F)$ of all linear continuous transformations from E into F. If H is bounded in the strong topology in $\mathcal{L}(E;F)$ (i.e., in the topology of bounded convergence), then H is equicontinuous.

Proof. Take a closed neighborhood W in F and a sequence of closed and balanced neighborhoods V_n, $n \in N$ in E such that

(2) $$V_1 + V_2 + \dots + V_n + V_n \subseteq W$$

for each $n = 1, 2, \dots$. The subset

(3) $$B_n = \bigcap_{u \in H} u^{-1}(V_n)$$

is evidently closed, balanced and absorbing any bounded set in E. Since $\Omega(B_n) = \bigcup \sum B_n$ of the form (1) is a neighborhood in E and for each $k \in N$ it holds $u(\sum_{n=1}^{k} B_n) \subseteq W$, $u \in H$, we see that $u(\Omega(B_n)) \subseteq$ $\subseteq W$ for all $u \in H$. This implies the equicontinuity of H.

<u>Remark 2.</u> We would like to stress the fact that in the precedent proof we have established a more general statement. It was, namely, supposed that the

sequence $(B_n ; n \in N)$ consists of closed, balanced and any bounded set absorbing subsets in E. Further, it is to note that under the assumptions in the precedent proof it holds $u(\bar{\Omega}) \subseteq W$ for arbitrary $u \in \in H$.

We shall introduce for any topological vector space E the associated M-bornological topology T_B with E. For any sequence $(B_m ; m \in N)$ of balanced and any bounded set absorbing subsets B_m the set $\Omega(B_m)$ determined by Formula (1) is also balanced and absorbing in E. We prove that the system of all such sets $\Omega(B_m)$ satisfies the axiom of additivity of a vector topology. Indeed, take a sequence $(B'_k ; k \in N)$ defined as follows:

$$B'_k = B_{2k-1} \cap B_{2k} .$$

Obviously $(B'_k , k \in N)$ is a sequence of the required form and it is easy to observe that $\Omega(B'_k) + +\Omega(B'_k) \subseteq \Omega(B_m)$. The just defined vector topology on E will be called the associated M-bornological topology with E (or, equivalently, the T_B-modification of the initial topology in E).

Although $T_B = T_B(E)$ is finer than the initial topology of E it preserves, however, the same class of bounded sets in E. If T_o is now a vector topology on E preserving the same class of bounded sets as the initial topology in E, then for each neighborhood W we may define a sequence V_m of neighborhoods of the topology T_o such that

$\Omega(V_m) \subseteq W$ (see Formula (1)). Hence T_B is the finest vector topology with the stated property. Thus we have proved

Theorem 3. The topology $T_B = T_B(E)$ is the finest vector topology on E under which E has the same class of bounded subsets as the initial topology in E. The space E under the topology $T_B(E)$ is M-bornological.

Remark 3. The taking of the associated M-bornological topology may be considered as an isotonic operator on the complete lattice $\mathcal{T}(E)$ of all vector topologies on E. The M-bornological topologies are exactly the topologies invariant under the operator T_B. With respect to the relation $T_B(E) = T_B(T_B(E))$ T_B is a closure operator on $\mathcal{T}(E)$ in the sense of Moore (cf.[1],Chap.IV). Especially, the maximal vector topology on E being invariant under T_B is therefore M-bornological.

Recall that a mapping u of a topological vector space E into a topological vector space F is said to be bounded if $u(B)$ is bounded in F for any bounded subset B in E.

Theorem 4. Let u be a linear mapping from E into F. Then u is bounded if and only if u is continuous on E under the associated M-bornological topology T_B.

Proof. If u is continuous on (E, T_B), then obviously u is bounded. Conversely, suppose that u is bounded. Let $(V_n ; n \in N)$ be a sequence of neighborhoods in F satisfying (2) and let $(B_n ; n \in N)$ be a sequence of balanced sets defined by (3), where H consists of the element $\{u\}$ only. From the assumption that u is bounded it follows that any B_n is absorbing any bounded set in E . Similarly as in the proof of Theorem 2 we see that $u(\Omega(B_n)) \subseteq W$, that means the continuity of u on E under the topology $T_B(E)$.

Theorem 4 suggests a projective characterization of the associated M -bornological topology in the same way as in [10] is projectively characterized the Λ -structure $(E(X), t)$.

<u>Theorem 5.</u> Let E be a topological vector space. Then the associated M -bornological topology T_B is characterized as the uniquely defined vector topology t on E with the following properties:

(a) The canonical mapping $E \longrightarrow (E, t)$ is bounded.

(b) If u is a linear and bounded mapping from E into a topological vector space F , then u is continuous on E under the topology t .

The proof is based on the same ideas as that one of Theorem 5 in [10].

<u>Corollary 1.</u> A topological space E is M -bornological if and only if any bounded mapping $u : E \rightarrow F$, where F is an arbitrary topological space, is continuous on E .

<u>Remark 4.</u> If E is a locally convex space, then

there are associated with E two topologies of bornological type: the topology $\mathcal{T}_{\mathcal{B}}$ (the associated M-bornological topology) and the topology $\mathcal{T}'_{\mathcal{B}}$ (the associated usual locally convex bornological topology; cf. [2]). From the definitions of $\mathcal{T}_{\mathcal{B}}$ and $\mathcal{T}'_{\mathcal{B}}$ we may conclude that $\mathcal{T}'_{\mathcal{B}}$ is the maximal locally convex topology on E coarser than $\mathcal{T}_{\mathcal{B}}$.

<u>Remark 5.</u> Suppose that E is an M-bornological space, F a topological vector space. A linear mapping u from E into F is, according to Theorem 4, continuous if and only if u is a bounded mapping. Evidently we may modify this criterion of continuity taking into consideration various conditions for the boundedness of u. Under the stated conditions on E and on F, for example, it is easy to verify that u is continuous if and only if for any sequence $(x_n, n \in N)$ convergent in E to the origin the sequence $(u(x_n), n \in N)$ is bounded in F.

From Remark 5 we may now derive

<u>Corollary 2</u>. Suppose that u is a linear mapping of an M-bornological space E into a topological space F. Then the following properties are equivalent:

(a) u is continuous.

(b) The restriction of u is continuous on each precompact subset of E.

(c) The restriction of u is continuous on any compact subset of E.

(d) If $(x_n; n \in N)$ is an arbitrary sequence convergent to the origin in E , then the restriction of u on $(x_n ; n \in N)$ is continuous.

<u>Theorem 6</u>. Let E be an M -bornological space, F an arbitrary separated topological vector space. The space $\mathcal{L}(E;F)$ of all linear continuous mappings from E into F we consider under the topology of bounded (compact, precompact) convergence or under the topology of uniform convergence on the family of all sequences in E convergent to the origin, respectively. Then $\mathcal{L}(E;F)$ is complete (quasi-complete, sequentially complete) whenever F is complete (quasi-complete, sequentially complete).

Proof. Suppose that $\mathcal{F}$ is a Cauchy filter in $\mathcal{L}(E;F)$. In all cases we may now define the function u_0 on E by the formula

$$\lim \mathcal{F}(x) = u_0(x) .$$

According to Corollary 1 and 2 it is easy to verify that u_0 is continuous on E .

Note some permanence properties of M -bornological spaces. We refer for the concepts of the inductive limit and of the topological direct sum to [9].

<u>Theorem 7</u>. (a) A quotient space of an M -bornological space is M -bornological.

(b) The topological direct sum $\sum E_n$ of a sequence $(E_n ; n \in N)$ of topological vector spaces is M -bornological if and only if any E_n is M -bornological.

(c) The inductive limit of a spectrum $(E_n ; n \in N)$

of $\mathcal{M}$ -bornological spaces is $\mathcal{M}$ -bornological.

Proof. We sketch only the proof of the statement (c) which implies the statement (b).

Let $(B_n ; n \in N)$ be a sequence of balanced and any bounded set absorbing subsets in $E = \lim ind\, E_n$ and let $(B_{ik} ; i \in N, k \in N)$ be a double sequence obtained by rearrangement of $(B_n ; n \in N)$ as in [8]. Take the subsets $B'_{ik} = \varphi_n^{-1}(B_{ik})$, $i \in N$, $k \in N$, where φ_n is the projection of E_n into E. The subsets

$$V_n = \bigcup_{i=1}^{\infty} \sum_{k=1}^{i} B'_{nk}$$

are now neighborhoods in E_n for any $n \in N$. From

$$\sum_{i=1}^{m} \varphi_i \Big(\bigcup_{n=1}^{\infty} \sum_{k=1}^{n} B'_{ik} \Big) \subseteq \sum_{i=1}^{m} \bigcup_{n=1}^{\infty} \sum_{k=1}^{n} \varphi_i (B'_{ik}) \subseteq \bigcup_{k=1}^{\infty} \sum_{n=1}^{k} B_n$$

we obtain the assertion (c).

<u>Remark 6.</u> (a) The completion of an $\mathcal{M}$ -bornological space is quasi- $\mathcal{M}$ -barrelled (cf.[8]).
(b) As to the product $\prod_{n=1}^{\infty} E_n$ of a sequence of $\mathcal{M}$ -bornological spaces we could repeat the conjecture of Remark 6, (b) of [8].
(c) We presume that a closed subspace of an $\mathcal{M}$ -bornological space need not be $\mathcal{M}$ -bornological.

<u>Remark 7</u>. From the statement (b) and (c), respectively, of Theorem 7 it follows that an $\mathcal{M}$ -bornological space need not be, in general, metrisable. Thus the class of $\mathcal{M}$ -bornological spaces is actually wider than the class of all metrisable vector spaces.

<u>Remark 8.</u> Finally, we could also formally gene-

ralize the concept of an M-bornological space supposing in Definition 1 that any set $\Omega(B_n)$ is a neighborhood in E whenever each B_n absorbs any set of a prescribed covering γ consisting of bounded sets in E. Discuss only the case when γ is the family of all finite subsets in E. Thus any $\Omega(B_n)$ of the form (1) is a neighborhood in E, where each B_n is characterized by algebraical properties.

Evidently any vector space with the maximal topology compatible with the structure of a vector space is of this sort. If E is a topological vector space, then the modification of such a topology coincides with the maximal vector topology on E. Hence, the base of neighborhoods of the maximal vector topology on E is described by the subsets $\Omega(B_n)$, where each B_n is balanced and absorbent in E.

2. p-BORNOLOGICAL SPACES

In this section we deal with the case of locally p-convex spaces. Recall that a topological vector space is said to be locally p-convex if there exists a base of neighborhoods consisting of absolutely p-convex sets. It will be assumed that $0 < p \leq 1$.

Let us denote the class of all bounded subsets of a topological vector space E by $\mathcal{B}(E)$.

<u>Definition 2</u>. A locally p-convex space is said to be p-bornological if and only if any absolutely p-convex subset absorbing arbitrary $B \in \mathcal{B}(E)$ is a

neighborhood in E .

<u>Remark 9.</u> (a) Any locally p -convex and M - bornological space is p -bornological.
(b) If a locally convex space E is p -bornological, then E is bornological.

<u>Remark 10</u>. From Remark 9 and Theorem 1 we conclude that any metrisable locally p -convex space is p - bornological. Especially, a p -normable space is necessarily p -bornological. With regard to [7] we see that any locally bounded space is p -bornological for some $0 < p \leq 1$.

<u>Theorem 8</u>. Suppose that E and F are two locally p -convex spaces, E p -bornological. Then any subset $H \subseteq \mathcal{L}(E; F)$ bounded in the topology of bounded convergence is equicontinuous on E .

By $T_B^p(E)$ we mean the modification of the initial topology in E having as a base of neighborhoods the system of all absolutely p -convex subsets C in E , where each of such C is absorbing arbitrary $B' \in \mathcal{B}(E)$. With respect to the following theorem the topology $T_B^p(E)$ will be called the p -bornological modification of the initial topology in E .

<u>Theorem 9</u>. (a) The space E under the topology $T_B^p(E)$ is p-bornological. The p -bornological modification is the finest locally p -convex topology on E preserving the same class $\mathcal{B}(E)$.
(b) The p -bornological modification of the initial topology is characterized by the properties (a) and

(b) of Theorem 5, where F runs in the family of all locally p-convex spaces.

(c) A locally p-convex space E is p-bornological if and only if any bounded mapping u from E into an arbitrary locally p-convex space is continuous.

<u>Remark 11.</u> Similarly as before we could also formulate the statements of Theorem 6 for any p-bornological space E. The full conveyance of these assertions to the discussed case with modificated proofs may be obviously omitted.

Let B be a closed, bounded and absolutely p-convex set in a locally p-convex space E. Denote by E_B the set $\bigcup(n \cdot B;\, n \in N)$. With respect to

$$B + B \subseteq 2^{1/2} \cdot B$$

we may conclude that E_B is a linear subspace in E. The system of all $(\lambda B;\, \lambda > 0)$ defines on E_B a locally p-convex topology. The absolutely p-convex set B induces by usual rules a p-semi-norm on E_B. Obviously, if E is separated, then each of such E_B is separated, hence E_B is p-normable. If, moreover, E is complete, then E_B having a base of neighborhoods of closed and complete subsets in E is a complete space.

The system $\mathcal{B}'(E)$ of all closed, bounded and absolutely p-convex subsets in E is filtered, consequently $(E_B;\, B \in \mathcal{B}'(E))$ is filtered and it holds $E = \bigcup(E_B;\, B \in \mathcal{B}'(E))$. Further, for each $B \in \mathcal{B}'(E)$

the canonical embedding $E_B \longrightarrow E$ is evidently continuous. If $\mathcal{T}$ is the locally p-convex inductive topology on E, then the initial topology of E is coarser than $\mathcal{T}$. Conversely, it holds

Theorem 10. If E is a separated locally p-convex and p-bornological space, then E is the inductive limit (considered as the locally p-convex inductive limit) of p-normable spaces $(E_B; B \in \mathcal{B}'(E))$. If, in addition, E is complete, then E is the inductive limit of Day spaces.

Remark 12. The adequate results of locally convex spaces are included in that what has been stated in this section as a special case for $p = 1$.

References

[1] G. BIRKHOFF: Lattice theory, New York,1948.

[2] N. BOURBAKI: Espaces vectoriels topologiques, Hermann,Paris,1953-1955.

[3] N. BOURBAKI: Sur certains espaces vectoriels topologiques, Ann.Inst.Fourier(Grenoble) 2 (1951),5-16.

[4] J. DIEUDONNÉ, L. SCHWARTZ: La dualité dans les espaces $(\mathcal{F})$ et $(\mathcal{LF})$. Ann.Inst.Fourier(Grenoble) 1(1950),61-101.

[5] G. KÖTHE: Topologische lineare Räume I., Springer, 1960.

[6] A.P. ROBERTSON, W.J. ROBERTSON: Topological vector spaces, Cambridge,1964.

[7] S. ROLEWICZ: On a certain class of linear metric spaces, Bull.Acad.Polon.Sci.Cl.III 5 (1957),471-473.

[8] S. TOMÁŠEK: M-barrelled spaces, Comment.Math.Univ. Carolinae 11(1970),185-204.

[9] S. TOMÁŠEK: Spectra of topological vector spaces. I. Inductive and projective limits (to be submitted to CMUC).

[10] S. TOMÁŠEK: On a certain class of Λ -structures. I. (Submitted to Czech.Math.J.)

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