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ON EXISTENCE OF THE WEAK SOLUTION FOR NON-LINEAR PARTIAL DIFFERENTIAL EQUATIONS OF ELLIPTIC TYPE

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Introduction. In this paper we shall be concerned with existence and uniqueness of the weak solution for a boundary value problem of equations of the form

$$\sum_{|i| \le k} (-1)^{|i|} D^i a_i(x, D^j u) = f \quad ,$$

where the growth of $a_i(x, \xi_j)$ in ξ_j is considered in a wide span.

We use well-known methods in reflexive spaces, namely the calculus of variations and the method of monotone operators. These methods are discussed and developed in the works of Browder [5],[6]; Nečas [1],[2]; Vajnberg [8]; Leray-Lions [7] etc.

The mentioned authors consider the growth from below and from above, given by polynomials, having the same degree, e.g.,

$$-c + c_1 |\xi|^m \le \sum_{|i| \le k} \xi_j a_i(x, \xi_j) \le c_2 (1 + |\xi|^m) \quad ,$$

$m > 1$ is a real number.

This condition can be weakened for the derivatives $D^j u$ with $|j| < k$, because of theorems of imbedding.

We shall use the same notations as in [1],[2], as there are here many references to those works. We shall denote

$$\mathcal{D}^{i} \equiv \frac{\partial^{|i|}}{\partial x_1^{i_1} \dots \partial x_N^{i_N}}\,,$$ where i is a multiindex, i.e.,

$i \equiv (i_1, \dots, i_N)$ is a vector, i_ℓ for $\ell = 1, \dots, N$ are non-negative integers and $|i| = \sum_{\ell=1}^{N} i_\ell \le k$.

In the present paper, the growth $a_i(x, \xi_j)$ in ξ_j is described by functions of certain classes.

Let us consider real functions $g(u)$, for which there exists a positive number u_0 such that

I $g(u) \in C(-\infty, \infty) \cap C^1(u_0, \infty)$; for $u \ge u_0$, $g(u) + u\,g'(u)$ is non-decreasing and $\lim\limits_{u \to \infty}(g(u) + u\,g'(u)) = \infty$; $u \cdot g(u)$ is even for $|u| \ge u_0$.

II For each $\ell > 1$ there exists a constant $c(\ell)$ such that $g(\ell u) \le c(\ell) \cdot g(u)$ for each $u \ge u_0$.

III There exists $\ell > 1$ such that

$$g(u) \le \tfrac{1}{2}\, g(\ell u) \text{ for each } u \ge u_0 .$$

Now, we shall denote $\mathcal{M}_1$; $\mathcal{M}_2$; $\mathcal{M}_3$ the classes of the functions $g(u)$ satisfying I; I and II; I,II and III. Let us have $g_i(u) \in \mathcal{M}_1$ for all $|i| \le k$ and suppose $g_i(u) \ge g_j(u)$ (resp. $g_i(u) \le g_j(u)$) for each i, j with $|i|, |j| \le k$ and $u \ge u_0$.

Then the condition for the growth possesses the form

$$|a_i(x, \xi_j)| \le C\,(1 + \sum_{|j| \le k} g_{ij}(\xi_j)) \quad \text{for } |i| \le k$$

and $g_{ij}(u) \le \min(|g_i(u)|, |g_j(u)|)$; $(|u| \ge u_0)$.

Thus, the growth $a_i(x, D^j u)$ in $D^i u$ is limited only by the properties of $g_i(u)$ and growth $a_i(x, D^j u)$ in $D^j u$ for $i \neq j$ is limited by the functions $g_i(u)$ and $g_j(u)$ in a very simple form.

In this work we find the weak solution even in such cases, when the degrees of polynomials differ by estimating from above and from below at the same member.

We construct Orlicz spaces $L^*_{G_i}(\Omega)$ by means of functions $G_i(u) = u\, g_i(u)$ - see Krasnosel'skij - Rutickij [4]. Then we construct a space $W^k_{\vec{G}}$ of Sobolev's type in the following way: $W^k_{\vec{G}}(\Omega) \equiv \{u \in L^*_{G_0}(\Omega)$, for which the distribution derivatives $D^i u \in L^*_{G_i}(\Omega)$, $|i| \leq k\}$.

Ω is a bounded domain of R^N (N-dimensional Euclidean space).

To the given equation we choose $g_i(u)$ so closely as to obtain even a coerciveness. In special cases, the algebraic condition for coerciveness is of the form

$$\sum_{|i| \leq k} \xi_i a_i(x, \xi_j) \geq c_1 \sum_{|i| \leq k} \xi_i g_i(\xi_i) - c \quad .$$

When the growth is described by $g_i(u) \in \mathcal{M}_3$ for $|i| \leq k$, then $W^k_{\vec{G}}$ is reflexive.

By the class $\mathcal{M}_2$ we can describe even a very small growth, e.g.,

$a_i(x, \xi_j) = \frac{\xi_i}{|\xi_i|} \ln(|\xi_i| + 1)$. Euler's equation of Example a) in §3, being of this type, stands very near to the equation for minimal surfaces.

By means of the class $\mathcal{M}_1$ we can describe a very wide span of growths even very fast, e.g. $a_i(x, \xi_j) = \xi_i e^{(\xi_i)^2}$. In both the cases $\mathcal{M}_1$ and $\mathcal{M}_2$

the space $W^{k}_{\vec{G}}$ need not be reflexive.

In § 1, a preliminary material on Orlicz spaces will be found.

§ 2 deals with existence and uniqueness of the weak solution, if the growth is given by the class $\mathcal{M}_3$.

§ 3 involves solving of existence and uniqueness of the minimum of functional constructed to an equation, when the growth is given by the class $\mathcal{M}_1$. Generally, we work with a non-reflexive space. In the space $W^{k}_{\vec{G}}$ we define a convergence which is weaker than the weak one but with respect to which $W^{k}_{\vec{G}}$ is sequentially compact.

Using Serrin's result [9], we prove lower semicontinuity of the functional with respect to the convergence just defined.

In § 4, existence and uniqueness of the weak solution is studied, when the growth is given by the class $\mathcal{M}_2$. In this case, too, $W^{k}_{\vec{G}}$ need not be reflexive.

§ 1.

We begin by presenting some fundamental notions from the theory of Orlicz spaces (see [4]). $G(u)$ is called to be an N-function if it is of the form $G(u) = \int_0^{|u|} p(t)\,dt$, where $p(t) > 0$ for $t > 0$ is a continuous on the right, non-decreasing function satisfying $p(0) = 0$ and $\lim_{t \to \infty} p(t) = +\infty$. When $p(t)$ is a continuous increasing function, let us denote by $r(t)$ its inverse function and define $P(v) =$

$= \int_0^{|v|} \kappa(t)\,dt$. $P(v)$ is an $\mathcal{N}$-function, too, and is called to be conjugate to $G(u)$. In the general case $\kappa(t)$ is inverse in some sense to $s(t)$ (see [4]). Further, we shall understand $G(u)$, $P(u)$ - maybe with indices - to stand for the $\mathcal{N}$-functions. There holds the Young's inequality $u \cdot v \le G(u) + P(v)$ for u, $v \ge 0$. If $Q(u)$ is a contiuous, convex and even function defined for $|u| \ge u_0$ and satisfying $\lim_{u \to \infty} \frac{Q(u)}{u} = \infty$, then there exists an $\mathcal{N}$-function $G(u)$ such that

$$G(u) = \begin{cases} c \cdot |u|^{\alpha} & \text{for } |u| \le u_1 \\ Q(u) & \text{for } |u| \ge u_1 \end{cases}$$

, where c, α and u_1 are suitable constants (they exist) and $\alpha > 1$. $Q(u)$ is called the principal part of $G(u)$ and it is denoted p.p. $G(u) = Q(u)$. $G(u)$ satisfies Δ_2 -condition, if for arbitrary $k > 1$ there exist constants $c(k)$ and u_0 such that $G(ku) \le c(k) G(u)$ for each $u \ge u_0$.

Suppose Ω is a bounded domain of R^N .

The Orlicz class $L_G(\Omega)$ is the set of all real functions $u(x)$ defined on Ω and satisfying $\varphi(u, G) = \int_\Omega G(u(x))\,dx < \infty$.

Orlicz space $L^*_G(\Omega)$ is the set of all $u(x)$ on Ω , for which $(u, v) = \int_\Omega u(x) v(x)\,dx < \infty$ holds for all functions $v(x) \in L_P(\Omega)$, where $P(u)$ is conjugate to $G(u)$, with the norm $\|u\|_G = \sup_{\varphi(v,P) \le 1} |(u, v)|$.

$L_G^*(\Omega)$ is a Banach space. $E_G(\Omega)$ is the closure of bounded functions in the norm of $L_G^*(\Omega)$. If $G(u)$ satisfies Δ_2-condition, then $E_G(\Omega) \equiv L_G(\Omega) \equiv L_G^*(\Omega)$. In the other case $E_G(\Omega)$ is a nowhere dense set in $L_G^*(\Omega)$ and $E_G(\Omega) \subset L_G(\Omega) \subset L_G^*(\Omega)$. For $u \in L_G^*$ and $v \in L_P^*$ ($G(u)$, $P(v)$ being conjugate) there holds the Hölder inequality $|(u,v)| \leq \leq \|u\|_G \cdot \|v\|_P$.

Assertion 1. If $g(u) \in \mathcal{M}_3$, then there exist $p, q > 1$ and constants c_1, c_2, u_o such that

(1.1) $c_1 |u|^p \leq u g(u) \leq c_2 |u|^q$ for $u \geq u_o$.

Proof. There exists $G(u)$ satisfying Δ_2-condition with p.p. $G(u) = u g(u)$. The existence of $q > 1$ and c_2 is a consequence of [4] (Theorem 4.1).

Iterating the inequality in III, we obtain

$$2^n g(u) \leq g(\ell^n u), \quad (u \geq u_o).$$

If $0 < \alpha \leq \log_\ell 2$, then the preceding inequality implies

$$(1.2) \qquad \frac{g(u)}{u^\alpha} \leq \frac{g(\ell^n u)}{(\ell^n u)^\alpha}, \quad u \geq u_o .$$

We prove the existence of $p > 1$ and c_1 by contradiction. Thus, there exists $\{u_m\}$ such that $u_m \to \infty$ and

$$\lim_{m\to\infty} \frac{g(u_m)}{u_m^\alpha} = 0 .$$

Let us denote $K = \inf_{u \in (u_o, \ell u_o)} \frac{g(u)}{u^\alpha}$. $K > 0$ (see II,I). Every u_m is of the form $u_m = \ell^m u_m'$, where

$u_n' \in (u_0, l u_0\rangle$ and m is a positive integer. According to (1.2) we have a contradiction:

$$0 < K \leq \frac{g(u_n')}{(u_n')^{\alpha}} \leq \frac{g(l^m u_n')}{(l^m u_n')^{\alpha}} = \frac{g(u_n)}{u_n^{\alpha}} .$$

Example. $g(u) = u^{\alpha} \cdot \ln^{\gamma_1} u \cdot (\ln \ln u)^{\gamma_2} \ldots (\ln \ldots \ln u)^{\gamma_n}$ where $u \geq u_0$, $\alpha \geq 0$ and $\gamma_1, \ldots, \gamma_n$ are real numbers. Let us extend $g(u)$ continuously on $(-\infty, +\infty)$ to obtain an odd function for $|u| \geq u_0$. If $\alpha > 0$, then $g(u) \in \mathcal{M}_3$. If $\alpha = 0, \gamma_1 > 0$, then $g(u) \in \mathcal{M}_2$.

Let us have $g_i(u) \in \mathcal{M}_1$ for $|i| \leq k$. There exist $G_i(u)$ with p.p. $G_i(u) = u\, g_i(u)$. Now, we construct $W^i_{G_i}(\Omega) \equiv \{u \in L_1(\Omega)$, for which $D^i u \in L_{G_i}(\Omega)\}$, where $D^i u$ is the distribution derivative and i is multi-index with $|i| \leq k$. We define $W^k_{\vec{G}}(\Omega) \equiv W^k_{\vec{G}} = \bigcap_{|i| \leq k} W^i_{G_i}$ (intersection) with the norm $\|u\|_{W^k_{\vec{G}}} = \sum_{|i| \leq k} \|D^i u\|_{G_i}$.

Let $\mathcal{E}(\overline{\Omega})$ be the set of all functions defined on Ω having derivatives of all orders extendable continuously on $\overline{\Omega}$. Let $\mathcal{D}(\Omega)$ be a subset of all functions from $\mathcal{E}(\overline{\Omega})$ which have support in Ω.

We define $\overset{\circ}{W}{}^k_{\vec{G}} = \overline{\mathcal{D}(\Omega)}$, where the closure is taken in the norm of $W^k_{\vec{G}}$.

Lemma 1. $W^k_{\vec{G}}$ is a Banach space. If $g_i(u) \in \mathcal{M}_3$ for $|i| \leq k$, then it is reflexive and separable.

Proof. $W^k_{\vec{G}}$ is a closed subspace of $\prod_{|i| \leq k} L^*_{G_i}(\Omega)$ (topological product of spaces $L^*_{G_i}(\Omega)$).

If $g_i(u) \in \mathcal{M}_3$, then $L^*_{G_i} \equiv E_{G_i}$ is a reflexive and separable space (see [4],Theorems 14.2 ; 8.2 and 10.1).

Let $\mathcal{E}_{\bar{\Omega}}(R^N)$ be the set of all functions from $\mathcal{E}(R^N)$ restricted on $\bar{\Omega}$.

Lemma 2. Suppose $g_i(u) \in \mathcal{M}_1$ for $|i| \leq k$. There holds $\overline{\mathcal{E}_{\bar{\Omega}}(R^N)} = W^k_{\vec{G}}(\Omega)$, where the closure is taken in the norm of $W^k_{\vec{G}}$.

Proof is very similar to that made in [3] (Theorem 3.1) and thus we verify only the basic points of this proof.

$u \in E_G(\Omega)$ possesses the following property: $\|u \cdot \chi(x, F)\| < \varepsilon$, if $\operatorname{mes} F < \sigma(\varepsilon)$, where $\chi(x,F)$ is the characteristic function of the set $F \subset \bar{\Omega}$. Using the Lusin's theorem, we conclude that $\|f(x+z)-f(x)\|_G < \varepsilon$, if $|z| < \sigma(\varepsilon)$.

Let us denote $u_h(x)$ the mollified function of u .

$$(*)\ u_h(x)=\frac{1}{\varkappa h^N}\int_{|\xi-x|\leq h} \exp\frac{|\xi-x|^2}{|\xi-x|^2-h^2}\, u(\xi)d\xi ,$$ where

$h>0,\ \varkappa = \int_{|\xi|\leq 1} \exp\frac{|\xi|^2}{|\xi|^2-1}\, d\xi$ and $u(\xi) = 0$ for $\xi \notin \Omega$. Suppose $v(x) \in E_p(\Omega)$ and $\rho(v, P) \leq 1$. We have

$$\int_\Omega v(x)(u_h(x)-u(x))dx = \frac{1}{\varkappa}\int_{|z|\leq 1}\exp\frac{|z|^2}{|z|^2-1}\int_\Omega v(x)\cdot \tag{1.3}$$

$$\cdot(u(x+hz)-u(x)dx\cdot dz \leq$$

$$\leq \frac{1}{\varkappa}\int_{|z|\leq 1}\exp\frac{|z|^2}{|z|^2-1}\|v\|_p\cdot\|u(x+hz)-u(x)\|_G\, dz .$$

By reason of the previous fact, (1.3) and $\|v\|_p \leq 2$, we obtain $\|u_h - u\|_G \to 0$ with $h \to 0$.

The rest of the proof is the same as that in [3].

Lemma 3. Suppose $g(u) \in \mathfrak{M}_2$. Then $g(u(x))$ is a bounded mapping from $L^*_G(\Omega)$ into $L^*_P(\Omega)$, where p.p. $G(u) = u\,g(u)$ and $P(u)$ is its conjugate. This mapping is continuous if and only if $g(u) \in \mathfrak{M}_3$.

Proof. For conjugate functions $G(u)$, $P(u)$ there holds $P\left(\frac{G(u)}{u}\right) < G(u)$, $(u > 0)$ (see [4],p.25). (This is easy to see in a geometrical sketch.)

From this inequality we conclude

(1.4) $\frac{G(u)}{u} < P^{-1}(G(u))$, or $g(u) < P^{-1}(G(u))$

$(u \geq u_0)$.

$P^{-1}(u)$, $G^{-1}(u)$ are inverse functions to $P(u)$, $G(u)$ for $u > 0$. As a consequence of the definition of the norm, (1.4) and the Jensen's inequality we have

(1.5) $\|g(u(x))\|_P \leq \int_\Omega P[g(u(x))]\,dx + 1 \leq$

$$\leq c + \int_\Omega G[u(x)]\,dx ,$$

where c is a constant.

$G(u)$ satisfies Δ_2 -condition and the first part of the lemma is a consequence of (1.5) (see [4] p.95).

If $g(u) \in \mathfrak{M}_3$, then (1.4),(1.5) imply continuity by reason of [4] (Theorem 17.3).

In the case $g(u) \notin \mathfrak{M}_3$ we prove discontinuity of the mapping $g(u(x))$. At first, from the Young's inequality we have

(1.6) $G^{-1}(u)P^{-1}(v) \leq u + v$ and hence

$P^{-1}(G(u)) \leq 2 \dfrac{G(u)}{u}$.

The fact that $q(u)$ does not possess III, implies $P(u)$ does not satisfy Δ_2-condition. From this we conclude that there exists $v(x) \in L_P(\Omega)$ such that $\| v - v_n \|_P \geq d > 0$, where

$$v_n(x) \begin{cases} v(x) & \text{if } |v(x)| \leq n, \\ 0 & \text{if } |v(x)| > n. \end{cases}$$

We can suppose $v(x) \geq u_1$, where $u_1 = P^{-1}(G(u_0))$ and $G(u) = u\,q(u)$, $u \geq u_0$.

For $u_n(x) = G^{-1}[P(v(x) - v_n(x))]$ we have $\lim\limits_{u \to \infty} \int_\Omega G(u_n(x))\,dx = 0$ and because of Δ_2-condition for $G(u)$, $\| u_n \|_G \longrightarrow 0$ (see [4], Theorem 9.4).

With respect to (1.6) we have

(1.7) $\| q(u(x)) \|_P \geq \frac{1}{2} \| v_n - v \|_P > \frac{d}{2} > 0$.

If $q(0) = 0$, the proof is finished; otherwise we prove (1.7) with $q^*(u) = q(u) - q(0)$, $G^*(u)$, $P^*(u)$. However, $L^*_G \equiv L^*_{G^*}$, $L^*_{P^*} \equiv L^*_P$ and, in addition,, they have equivalent norms.

<u>Theorem 1.</u> If $q(u) \in \mathcal{M}_3$, then

(1.8) $\lim\limits_{\|u\|_G \to \infty} \int_\Omega \dfrac{G[u(x)]}{\|u\|_G}\,dx = \infty$ (p.p. $G(u) = u\,q(u)$).

If $q(u) \in \mathcal{M}_3$ satisfies

there exist $r, s > 0$ such that $r - s < 1$

(1.9) and $c_1 \lambda^s \cdot q(u) \leq q(\lambda u) \leq c_2 \cdot \lambda^r \cdot q(u)$

for $\lambda \geq \lambda_0$, $u \geq u_0$, then

$$\lim_{\|u\|_G \to \infty} \int_\Omega \frac{G[u(x)]}{\left\| \frac{G(u)}{u} \right\|_P}\,dx = \infty .$$

Proof. From (1.2) we conclude

$$\ell^{n\alpha} g(u) \leq g(\ell^n u).$$

For arbitrary $\lambda > \ell$ we find an integer $n > 0$ such that $\ell^{n-1} \leq \lambda < \ell^n$. There holds

$$(1.10)\quad g(\lambda u) = g\left(\ell^{n-1} u \cdot \frac{\lambda}{\ell^{n-1}}\right) \geq \ell^{(n-1)\alpha} g(u) = \lambda^\alpha g(u) \frac{\ell^{(n-1)\alpha}}{\lambda^\alpha} \geq \frac{1}{\ell^\alpha} \cdot \lambda^\alpha g(u),$$

where $0 < \alpha \leq \log_\ell 2$, $\ell > 1$ being fixed. Suppose $G(u) = u\, g(u)$ for $|u| \geq u_1 > u_0$ and let us denote

$$g^*(u) = \begin{cases} g(u), & \text{for } |u| \geq u_1 \\ 0, & \text{for } |u| < u_1 \end{cases}.$$

$g^*(u)$ is an odd, non-decreasing function satisfying (1.10) for all $\lambda \geq \ell > 1$ and $u \geq 0$.

Now, we prove (1.8) by contradiction.

Thus, there exists $\{u_n(x)\}$ satisfying $\|u_n\|_G \to \infty$ and $\int_\Omega \frac{G[u_n(x)]}{\|u_n\|_G} dx \leq A$ for all n; A is a constant. Let us consider $u_n(x) = \lambda_n v_n(x)$, where $\|v_n\|_G = R > 2$ and hence $\lambda_n \to \infty$. Evidently, $-c + G(u) \leq u g^*(u) \leq G(u)$ holds for each u, where c is a suitable constant. There holds

$$AR \geq \int_\Omega \frac{G[u_n(x)]\,dx}{\lambda_n} \geq \int_\Omega |v_n(x)|\,|g^*(\lambda_n v_n(x))|\,dx \geq$$

$$\geq c_1 \lambda_n^\alpha \int_\Omega |v_n(x)| \cdot |g^*(v_n(x))|\,dx \geq$$

$$\geq c_1 \lambda_n^\alpha \left(\int_\Omega G[v_n(x)]\,dx - c\right).$$

In regard to the known inequality $\rho(u, G) \geq \frac{1}{2}\|u\|_G$ for $\|u\|_G \geq 2$, it suffices to take $R = 2(c+1)$ and hence $\lambda_n^\alpha \leq \frac{A}{c_1} 2(c+1)$, which gives us a contradiction. The remaining part of the theorem will be proved analogously. We set again $u_n = \lambda_n v_n$,

$$\int_\Omega \frac{G[u_n(x)]}{\left\| \frac{G(u_n)}{u_n} \right\|_p} dx \le A \quad \text{and} \quad \| v_n \|_G = R > 2 .$$

From the first inequality (1.9) we obtain, as in the previous part of the theorem, the following estimate:

$$(1.11)\ c_1 \lambda_n^{1+s} \left(\int_\Omega G[v_n(x)]\, dx - c \right) \le \int_\Omega G[u_n(x)]\, dx .$$

From the second inequality in (1.9) we deduce $|g(u)| \le c_2' \cdot |u|^r + c$ for all u and taking

$$g^{**}(u) = \begin{cases} g(u), \text{for } |u| \ge u_1 \\ \operatorname{sgn} u \cdot g(u_1), \text{ for } |u| < u_1 , \end{cases}$$

in account of (1.9) there holds

$|g^{**}(\lambda u)| \le c_3 \lambda^r |g^{**}(u)| + c$ for all u and $\lambda \ge$ $\ge \lambda_0$, where c_3 is a suitable constant.

Thus, considering the inequality $G(u) \le u\, g^{**}(u) \le$ $\le G(u) + c$ for all u, we obtain

$$\left\| \frac{G[u_n(x)]}{u_n(x)} \right\|_p \le \left\| \frac{\lambda_n |v_n(x)|\, |g^{**}(v_n(x))|}{\lambda_n v_n(x)} \right\|_p \le$$

$$\le \| g^{**}(\lambda_n v_n(x)) \|_p \le c_3 \lambda_n^r \| g^{**}(v_n(x)) \|_p + c \le$$

$$\le c_3 \lambda_n^r \left(\int_\Omega G[v_n(x)]\, dx - c \right) + c .$$

Now, considering R sufficiently large but fixed, we conclude from (1.11) and the last inequality $c \cdot \lambda_n^{1+s-r} \le A$ which gives a contradiction and the theorem is proved.

<u>Corollary</u>. If $g(u) \in \mathcal{M}_3$ satisfies (1.9), then $\lim\limits_{\|u\|_G \to \infty} \int_\Omega \frac{G[u(x)]}{\|g(u(x))\|} dx = \infty$. (p.p. $G(u) = u\, g(u)$).

Proof. There exists a c such that $|g(u)| \leq$ $\leq |\frac{G(u)}{u}| + c$ for each u. We have

$$\int_\Omega \frac{G[u(x)]}{\|\frac{G(u)}{u}\|_p + c}\,dx \leq \int_\Omega \frac{G[u(x)]}{\|g(u)\|_p}\,dx .$$

It suffices to know $\lim_{\|u\|\to\infty} \|\frac{G(u)}{u}\|_p = \infty$. If $\|\frac{G(u)}{u}\|_p \leq c$, the Hölder inequality implies

$$\int_\Omega \frac{G[u(x)]}{\|u\|_G}\,dx = \int_\Omega \frac{G[u(x)]}{u(x)} \cdot \frac{u(x)}{\|u\|_G}\,dx \leq c$$ and hence we have a contradiction with (1.8).

Lemma 4. Suppose $g(u) \in \mathcal{M}_3$ and p.p. $G(u) = u\,g(u)$. Then there exist constants c_1, c_2 such that

(1.12) $\int_\Omega G(D^i u(x))\,dx \leq c_1 \sum_{|j|=k} \int_\Omega G(D^j u(x))\,dx + c_2$

for $u \in \overset{\circ}{W}{}^k_{\vec{G}}$, where $G_i(u) \equiv G(u)$ and $|i|, |j| \leq k$.

Proof. Firstly, we prove (1.12) for $u \in \mathcal{D}(\Omega)$ and to this purpose it suffices to prove

(1.13) $\int_\Omega G[u(x)]\,dx \leq c_1 \sum_{|i|=1} \int_\Omega G[\frac{\partial u}{\partial x_i}]\,dx + c'$.

We imbed Ω into the cube $\sigma \subset R^N$ with the length a of the edge and with a center in origin. Putting $u(x) \equiv 0$ for $x \in R^N - \Omega$ we have

(1.14) $u(x_1,\dots,x_N) = \int_{-a}^{x_1} \frac{\partial u}{\partial x_1}(\xi_1, x_2,\dots,x_N)\,d\xi_1 = (x_1+a) \cdot \frac{\int_{-a}^{x_1} \frac{\partial u}{\partial x_1}\,d\xi_1}{x_1+a}$.

There hold

(1.15) $G[u(x)] \leq (x_1+a)\, G[\frac{\int_{-a}^{x_1} \frac{\partial u}{\partial x_1} \cdot d\xi_1}{x_1+a}]$, if $0 < x_1+a < 1$

from convexity for $G(u)$ $(G(\alpha u) \leq \alpha G(u), \alpha < 1)$;

$$G[u(x)] \leq G[2a \frac{\int_{-a}^{x_1} \frac{\partial u}{\partial x_1} d\xi_1}{x_1 + a}] \leq c_1 G[\frac{\int_{-a}^{x_1} \frac{\partial u}{\partial x_1} d\xi_1}{x_1+a}] + c ,$$

if $1 \leq x_1 + a \leq 2a$ from Δ_2-condition for $G(u)$ ($G(2au) \leq c_1 G(u) + c$ for each u).

Applying the Jensen's inequality in (1.15), we have $G[u(x)] \leq c_1 \int_{-a}^{x_1} G[\frac{\partial u}{\partial x_1}] d\xi_1 + c$ and hence (1.13).

The functional $\int_\Omega G[u(x)]dx = \int_\Omega u(x) \cdot \frac{G[u(x)]}{u(x)} dx$ is continuous from $L_G(\Omega)$ into $L_1(\Omega)$ as a consequence of Lemma 3. If $u \in \mathring{W}_G^k(\Omega)$, we choose $u_m \in \mathcal{D}(\Omega)$ satisfying $\| u_m - u \|_{W_G^k} \to 0$.
Clearly, we may allow $m \to \infty$ in (1.12) for $u_m \in \mathcal{D}(\Omega)$, and thus we obtain the required assertion.

§ 2.

In this section, we establish two general theorems for existence of a weak solution, where the coerciveness is assumed and then we state algebraic conditions to assure the coerciveness in special cases. We work entirely with the reflexive spaces, except Theorem 4, concerning the compactness of the imbedding.

The boundary $\partial\Omega$ of the bounded domain $\Omega \subset R^N$ is supposedto be Lipschitzian (see [3]).

We shall denote positive constants by c with or without subscripts and in the same discussion it may denote different constants.

Suppose $a_i(x, \xi_j)$ for $|i| \leq k$ real functions defined for $x \in \Omega$ and $-\infty < \xi_j < \infty$ with $|j| \le k$ (i, j are multiindices). They are continuous in ξ_j for almost every $x \in \Omega$ and measurable in x by fixed ξ_j. (By this designation we understand $a_i(x, \xi_j)$ to be a function of x and a vector $\xi \equiv (\xi_1, \dots, \xi_d)$, where the integer $d \le \mathrm{card}\,\{i, |i| \le k\}$.)

Let us denote $K \equiv \{i, |i| \le k\}$, $L \equiv \{i; |i| = k\}$ and M some subset of K with $K \supset M \supset L$.

We assume $q_i(u) \in \mathcal{M}_3$, $i \in M$ being chosen with respect to an equation given in such a way that

(2.1) $\quad |a_i(x, \xi_j)| \le c\,(1 + \sum_{j \in M} q_{ij}(\xi_j))$

for all $i \in M$, where $q_{ij}(u) \in C(-\infty, \infty)$ with $0 \le q_{ij}(u) \le q_i[G_i^{-1}(G_j(u))]$, $(|u| \ge u_0)$, p.p. $G_i(u) = u\,q_i(u)$ and G_i^{-1} its inverse function for $u > 0$.

If every pair of $q_i(u)$, $q_j(u)$ for $i, j \in M$ satisfies one of the inequalities $q_i(u) \underset{(\le)}{\ge} q_j(u)$ for $u \ge u_0$, then the condition (2.1) can be rewritten in a slightly stronger but synoptical form:

(2.2) $\quad |a_i(x, \xi_j)| \le c\,(1 + \sum_{j \in M} q_{ij}(\xi_j)) \quad i \in M,$

where $q_{ij}(u) \in C(-\infty, \infty)$ and $0 < q_{ij}(u) \le$ $\le \min(|q_i(u)|, |q_j(u)|)$ for $|u| \ge u_0$.

Condition (2.1) or (2.2) involves $a_i(x, \xi_j) \equiv 0$ for $i \notin M$ and $a_l(x, \xi_j)$ are independent on ξ_i for all $l \in M$ and $i \notin M$.

To the equation given with (2.1) or (2.2) we construct a space $W_{\vec{G}}^{k} \equiv \bigcap_{i \in M} W_{G_i}^{i}$ with the norm $\|u\|_{W_{\vec{G}}^{k}} = \sum_{i \in M} \|D^i u\|_{G_i}$ to which we add $\|u\|_{L_1(\Omega)}$ in the case $(0,\ldots,0) \notin M$.

From (1.1) (Assertion 1) there exist $p, q > 1$ such that $W_q^k(\Omega) \subset W_{\vec{G}}^{k} \subset W_p^k(\Omega)$ (algebraically and topologically). Condition (2.1) or (2.2) can be weakened by some information of imbeddings of $W_{\vec{G}}^{k}$.

Lemma 1. Suppose (2.1) or (2.2). Then $a_i(x, D^j u)$, $i \in M$ is a bounded, continuous mapping from $W_{\vec{G}}^{k}$ into $L_{P_i}(\Omega)$ (P_i being conjugate to $G_i(u)$).

Proof. From Lemma 3, § 1 and (2.1) we conclude
$g_i[G_i^{-1}(G_j(u))] < P_i^{-1}(G_i[G_i^{-1}(G_j(u))]) = P_i^{-1}(G_j(u))$
for each $|u| \geq u_0$.
Similarly as in Lemma 3, § 1 we obtain from this inequality that $a_i(x, D^j u)$ is a bounded mapping from $W_{\vec{G}}^{k}$ into $L_P(\Omega)$. The continuity follows from the results [4] (Lemma 17.2, Theorem 17.3).

Condition (2.1) is stronger than (2.2). Indeed, we prove $\min(|g_i(u)|, |g_j(u)|) \leq 2 g_i[G_i^{-1}(G_j(u))]$ for $|u| \geq u_1$. If $G_i(u) \leq G_j(u)$, then $|u| \leq$ $\leq G_i^{-1}(G_j(u))$ and hence $|g_i(u)| \leq g_i[G_i^{-1}(G_j(u))]$ in regard to I. If $G_j(u) \leq G_i(u)$ ($|u| \geq u_0$), then $P_i(v) \leq P_j(v)$ for $|v| \geq v_1(u_0)$ (see [4], Theorem 2.1) and hence $P_j^{-1}(v) \leq P_i^{-1}(v)$ ($|v| \geq v_1$).

Using Lemma 3, § 1, we have
$g_i[G_i^{-1}(G_j(u))] \geq \frac{1}{2} P_i^{-1}(G_i[G_i^{-1}(G_j(u))]) =$

$= \frac{1}{2} P_i^{-1}[G_j(u)] \geq \frac{1}{2} P_j^{-1}(G_j(u)) > \frac{1}{2} |g_j(u)|$

for $|u| \geq u_1$ and the proof is complete.

Having Lemma 1, we are able to present the definition of the weak solution of a boundary value problem (see [2],[3]). Let $\mathcal{V}$ be a linear subset of $\mathcal{E}(\overline{\Omega})$ with $\mathcal{D}(\Omega) \subset \mathcal{V} \subset \mathcal{E}(\overline{\Omega})$. Let us denote $V_{\vec{G}} \equiv \overline{\mathcal{V}}$, where the closure is in the norm of $W^{k}_{\vec{G}}$. Let $u_0(x) \in W^{k}_{\vec{G}}$ represent a stable boundary value condition and $g \in (V_{\vec{G}})'$ (dual space), $g(v) = 0$ for $v \in \overset{\circ}{W}{}^{k}_{\vec{G}}$, the non stable one. (For the Dirichlet's problem, i.e. $V_{\vec{G}} \equiv \overset{\circ}{W}{}^{k}_{\vec{G}}$, the functional g is not given.)

$u \in W^{k}_{\vec{G}}$ is called to be a weak solution of the boundary value problem, if $u - u_0 \in V_{\vec{G}}$ and for all $v \in V_{\vec{G}}$

$$(2.3)\quad \int_\Omega \sum_{i \in M} D^i v\, a_i(x, D^j u)\, dx = (v, f)_\Omega + (v, g)_{\partial\Omega}$$

holds, where $f \in (V_{\vec{G}})'$ and $(v, f)_\Omega$, $(v, g)_{\partial\Omega}$ are the values of the functionals at the point v .

Using a variational method we shall suppose the symmetry:

$$(2.4)\quad \frac{\partial a_i(x, \xi_j)}{\partial \xi_l} = \frac{\partial a_l(x, \xi_j)}{\partial \xi_i} \quad \text{in the sense}$$

of distribution for all i, $l \in M$.

<u>Lemma 2</u>. Suppose (2.1),(2.4) and $f, g \in (V_{\vec{G}})$. Then the functional

(2.5) $$\phi(v) = \int_0^1 dt \int_\Omega \sum_{i\in M} D^i v a_i(x, D^j(u_0 + tv))dx - (v, f)_\Omega - (v, g)_{\partial\Omega}$$

is continuous on $V_{\vec{G}}$ and has a Gâteaux differential at every point equal to

(2.6) $$D\phi(v, \tilde{v}) = \int_\Omega \sum_{i\in M} D^i \tilde{v} a_i(x, D^j(u_0 + v))\, dx - (\tilde{v}, f)_\Omega - (\tilde{v}, g)_{\partial\Omega} .$$

Proof of this lemma is the same as that in [2] Theorem 2.1). We use Lemma 1 and Lemma 2,§ 1, only.

Now, monotonicity conditions and a general condition for coerciveness will be written:

(2.7) $$\lim_{\|v\|_{W_{\vec{G}}^{m}} \to +\infty} \frac{1}{\|v\|_{W_{\vec{G}}^{m}}} \int_\Omega \sum_{i\in M} D^i v a_i(x, D^j(u_0+v))\,dx = \infty$$

(coerciveness).

(2.8) $$\sum_{i\in M} (\xi_i - \eta_i)[a_i(x, \xi_j) - a_i(x, \eta_j)] \geq 0$$

(monotonicity).

(2.8 a) $$\sum_{i\in M} (\xi_i - \eta_i)[a_i(x, \xi_j) - a_i(x, \eta_j)] > 0$$

for $\xi \neq \eta$.

$\xi \equiv (\xi_1, \ldots, \xi_d), \eta \equiv (\eta_1, \ldots, \eta_d)$ are real vectors with $d = \operatorname{card} M$.

A functional $\phi(u, v)$ defined on $V_{\vec{G}} \times V_{\vec{G}}$ is called to be semi-convex (see Browder [6]), if it is convex and continuous at u by each v fixed and if $v_n \rightharpoonup v$ (weak convergence), then $\phi(u, v_n) \to \phi(u, v)$ uniformly for u belonging to a bounded set.

Theorem 2. Suppose (2.1),(2.4),(2.7) and one of the following conditions: i) (2.8); ii) there exists a semi-convex $\phi(u,v)$ where $\phi(u,u) = \phi(u)$ is from (2.5). Then there exists the solution of (2.3). If (2.8a) is satisfied then the solution of (2.3) is unique.

Proof. Let us define

$$\inf_{\|v\|_{W^k_{\vec{G}}}=R} \frac{1}{\|v\|_{W^k_{\vec{G}}}} \int_\Omega \sum_{i\in M} D^i v\, a_i(x, D^j(u_0+v))\,dx = \\ = \lambda(R).$$

$\lambda(R)$ is measurable and $\lim_{R\to\infty} \lambda(R) = \infty$ on the ground of (2.7). There holds

$\phi(v) \geq \int_0^1 R\cdot\lambda(tR)dt - CR = R\left(\frac{1}{R}\int_0^R \lambda(s)ds - c\right)$, where $\|v\|_{W^k_{\vec{G}}} = R$. But $\lim_{R\to\infty} \frac{1}{R}\int_0^R \lambda(s)ds = \infty$ and hence $\lim_{\|v\|_{W^k_{\vec{G}}}\to\infty} \phi(v) = \infty$. $V_{\vec{G}}$ is reflexive and i) or ii) imply the lower semi-continuity for $\phi(u)$ and hence there exists a point $v \in V_{\vec{G}}$ at which $\phi(u)$ attains its minimum.

If we construct a Gâteaux differential at the point v, we find - with respect to (2.6) - that $v + u_0$ is the solution of (2.3). Uniqueness is clear from (2.8a).

Now we shall apply the theory of monotone operators - see e.g. Browder [5],[6], Leray-Lions [7]. Let us assume $M = M_1 \cup M_2$ with $M_1 \supseteq L$.

(2.9) The imbedding $W^k_{\vec{G}} \longrightarrow \bigcap_{i\in M_2} W^i_{G_i}$ is compact.

$$(2.10)\ \sum_{i\in M_1}(\xi_i-\eta_i)[a_i(x,\xi_\alpha,\gamma_\beta)-a_i(x,\eta_\alpha,\gamma_\beta)]>0\ ,$$

if $\xi \neq \eta$; almost everywhere in Ω , where $\alpha \in M_1$ and $\beta \in M_2$.

$$(2.11)\quad \sum_{i\in M_1}\xi_i a_i(x,\xi_j)/\Big(\sum_{i\in L}|\xi_i|+\sum_{i,j\in L} q_{ij}(\xi_j)\Big)\to\infty\ ,$$

if $\sum_{i\in L}|\xi_i|\to\infty$, uniformly for ξ_ℓ , $\ell\in M-L$ from a bounded set and $x\in\Omega$, where $q_{ij}(u)$ are from (2.1).

$$(2.11a)\ \sum_{i\in M_1}\xi_i a_i(x,\xi_j)/\Big(\sum_{i\in L}|\xi_i|+\sum_{i\in L}|q_i(\xi_i)|\Big)\to\infty\ ,$$

if $\sum_{i\in L}|\xi_i|\to\infty$ uniformly for ξ_ℓ, $\ell\in M-L$ from a bounded set and $x\in\Omega$.

Let us denote $(w, A(v,u))=\sum_{i\in M_1}\int_\Omega D^i w\, a_i(x,$

$$D^i w a_i(x,D^\alpha(u_0+v),D^\beta(u_0+u))dx+\sum_{i\in M_2}\int_\Omega D^i w a_i(x,D^j(u_0+u))dx,$$

where $\alpha\in M_1$ and $\beta\in M_2$, $|j|\le k$.

Theorem 3. Suppose (2.7) and (2.1) (resp.(2.2)). If one of the following two conditions i) (2.8), ii) (2.9),(2,10),(2.11) (resp. (2.11a)) is satisfied, then there exists the solution of (2.3). If (2.8a) is satisfied then the solution of (2.3) is unique.

Proof. It is sufficient to verify the hypotheses of Leray-Lions Theorem [7].

The operator $A(u,v)$ is continuous and bounded from $V_{\vec G}\times V_{\vec G}$ into $(V_{\vec G})'$ because of Lemma 1. If (2.8) holds, then $M_2=\emptyset$ and the mentioned hypotheses are verified.

In the other case we must verify:

1) if $u_n \rightharpoonup u$ in $V_{\vec{G}}$ and $(u_n - u, A(u_n, u_n) - A(u, u_n)) \to 0$ then $A(v, u_n) \rightharpoonup A(v, u)$ in $(V_{\vec{G}})'$ for all $v \in V_{\vec{G}}$;

2) if $u_n \rightharpoonup u$ and $A(v, u_n) \rightharpoonup v'$ in $(V_{\vec{G}})'$, then $(u_n, A(v, u_n)) \longrightarrow (u, v')$, for all $v \in V_{\vec{G}}$.

In the case 1) we can prove from (2.1),(2.10) and (2.11), resp. (2.2),(2.10) and (2.11a) similarly as in [7](see [2] Lemma 3.2) that it is possible to select a subsequence still called $\{u_n\}$, satisfying $D^i u_n(x) \to D^i u(x)$ for $|i| \le k$ almost everywhere in Ω. For $i \in M$ we have

(2.12) $a_i(x, D^j(u_0 + u_n)) \to a_i(x, D^j(u_0 + u))$

almost everywhere in Ω. From Lemma 1 we have

(2.13) $\| a_i(x, D^j(u_0 + u_n)) \|_{P_i} \le c$ for all n.

If $\varphi \in \mathcal{D}$, then we have (2.13) for $D^i \varphi a_i(x, D^j(u_0 + u_n))$. From the Young's inequality and (2.13) we conclude

$$\sum_{i \in M_2} \int_\Omega P_i [D^i \varphi a_i(x, D^j(u_0 + u_n))] \, dx \le C ,$$

for all n.

As a consequence of the Valleé-Pousin's theorem (see [4],p.113) $D^i \varphi a_i(x, D^j(u_0 + u_n))$ have uniformly absolutely continuous integrals and thus from (2.12),(2.9) and Lemma 1 we conclude $(\varphi, A(v, u_n)) \to (\varphi, A(v, u))$. On account of (2.13),(2.9) and Lemma 1, $\| A(v, u_n) \|_{(V_{\vec{G}})'} \le c$ holds. $\overline{\mathcal{D}} = V_{\vec{G}}$

implies $A(v, u_n) \rightharpoonup A(v, u)$ in the space $(V_{\vec{G}})'$ for all $v \in V_{\vec{G}}$.

In the case 2) we obtain

$$(2.14) \quad \int_\Omega \sum_{i \in M_2} D^i(u_n - u)\, a_i(x, D^j(u_0 + u_n))\, dx \xrightarrow[n\to\infty]{} 0,$$

because $\| D^i(u_n - u \|_{G_i} \xrightarrow[n\to\infty]{} 0$ and $\| a_i(x, D^j(u_0 + u_n)) \|_{P_i} \le C$ for $i \in M_2$.

$$\int_\Omega \sum_{i \in M_2} D^i u\, a_i(x, D^j(u_0 + u_n))\, dx = (u, A(v, u_n)) -$$

$$- \int_\Omega \sum_{i \in M_1} D^i u\, a_i(x, D^\alpha(u_0 + v), D^\beta(u_0 + u_n))\, dx \xrightarrow[n\to\infty]{} (u, v') -$$

$$- \int_\Omega \sum_{i \in M} D^i u\, a_i(x, D^\alpha(u_0 + v), D^\beta(u_0 + u))\, dx .$$

From this and (2.14) we conclude

$$(2.15) \quad \int_\Omega \sum_{i \in M_2} D^i u_n\, a_i(x, D^\alpha(u_0 + v), D^\beta(u_0 + u_n))\, dx \to (u, v') -$$

$$- \int_\Omega \sum_{i \in M_1} D^i u\, a_i(x, D^\alpha(u_0 + v), D^\beta(u_0 + u))\, dx .$$

But

$$\int_\Omega \sum_{i \in M_1} D^i u_n\, a_i(x, D^\alpha(u_0 + v), D^\beta(u_0 + u_n))\, dx \longrightarrow$$
$$\longrightarrow \int_\Omega \sum_{i \in M_1} D^i u\, a_i(x, D^\alpha(u_0 + v), D^\beta(u_0 + u))\, dx$$

holds as a consequence of $u_n \rightharpoonup u$ in $V_{\vec{G}}$ and $a_i(x, D^\alpha(u_0 + v), D^\beta(u_0 + u_n)) \to a_i(x, D^\alpha(u_0 + v), D^\beta(u_0 + u)$ in the norm of the space L_{P_i}, because of (2.9) and Lemma 1. Thus, from (2.15) we have

$$(u_n, A(v, u_n)) \longrightarrow (u, v') .$$

In the next we shall establish some sufficient conditions for coerciveness, compactness of imbedding and equivalence of norms.

We shall use the following condition for coerciveness:

(2.16) $\sum_{i \in M} \xi_i a_i(x, \xi_j) \geq c_1 \sum_{i \in M} \xi_i g_i(\xi_i) - c$.

(2.17) $\sum_{i \in M} \xi_i a_i(x, \eta_j) \geq c_1 \sum_{i \in M} \xi_i g_i(\eta_i) - c$.

In the case of non-Dirichlet problem we suppose $(0 \ldots 0) \in M$.

Lemma 3. Suppose (2.2),(2.16) and let $g_i(u) \in \mathcal{M}_3$ satisfy (1.9) for $i \in M$. Then (2.7) holds.

Proof. From every sequence $\| v_n \|_{W_\sigma^k} \longrightarrow \infty$ it suffices to select a subsequence v_{n_k} satisfying (2.7).

According to (2.16) we have

$$\int_\Omega \sum_{i \in M} D^i v_n a_i(x, D^j(u_0 + v_n))\,dx = \int_\Omega \sum_{i \in M} D^i(v_n + u_0) a_i(x, D^j(u_0 + v_n))\,dx - \int_\Omega \sum_{i \in M} D^i u_0 a_i(x, D^j(u_0 + v_n))\,dx \geq$$

$$\geq c_1 \sum_{i \in M} \int_\Omega G_i(D^i(u_0 + v_n))\,dx -$$

$$- c_2 \sum_{i \in M} \| g_i(D^i(u_0 + v_n)) \|_{p_i} - c \, .$$

Let us divide this inequality by $\| v_n + u_0 \|_{W_\sigma^k}$. If

(2.8) $$\sum_{i \in M} \frac{\| g_i(D^i(u_0 + v_n)) \|_{p_i}}{\| v_n + u_0 \|_{W_\sigma^k}}$$

is bounded, then the assertion for v_n is true as a consequence of Theorem 1. Otherwise the fraction in (2.18) converges to infinity for a suitable v_{n_k} . Then, with regard to (1.9), the corollary of Theorem 1 gives us

$$\frac{\sum_{i \in M} \int_\Omega G(D^i(u_0 + v_{n_k}))}{\sum_{i \in M} \| g_i(D^i(u_0 + v_{n_k})) \|_p} \xrightarrow[k \to \infty]{} \infty \, .$$

Lemma 4. If we substitute (2.7) by Condition (2.17), Theorem 2 remains true.

Proof. From every sequence $\|v_n\|_{W_{\mathcal{G}}^k} \to \infty$ it suffices to select a subsequence v_{n_k} satisfying $\phi(v_{n_k}) \to \infty$. Let us define $F(s) = \int_0^s g(t)\,dt$, $g(u) \in \mathcal{M}_1$. $g(u)$ is increasing to infinity because of I.

The following estimations hold:

$$F(s) = F(s_0) + \int_{s_0}^{s} g(t)\,dt \leq F(s_0) + s\,g(s)$$

for $s \geq s_0$, where s_0 is a suitable positive number and hence $F(s) \leq 2s\,g(s)$ for $s \geq s_1$. $g(u)$ is an odd function and thus $F(-s) \leq 2s\,g(s)$ for $s \geq s_2$. On the other hand,

$$F(s) = F(s_0) + \int_{s_0}^{s} g(t)\,dt \geq F(s_0) + \int_{\frac{s}{2}}^{s} g(t)\,dt \geq \frac{1}{2}\cdot\frac{s}{2}\,g\left(\frac{s}{2}\right)$$

for $s \geq s_3$ and $F(-s) \geq \frac{1}{2}\frac{s}{2}\,g\left(\frac{s}{2}\right)$ for $s \geq s_4$. (s_1, s_2, s_3, s_4 are suitable positive numbers.) Thus, there exists a constant c such that

$$-c + \frac{1}{2}\frac{s}{2}\,g\left(\frac{s}{2}\right) \leq F(s) \leq 2s\,g(s) + c,\ s \in (-\infty, \infty).$$

From this estimate and (2.17) we obtain

$$\begin{aligned}\phi(v) &\geq c_1 \sum_{i\in M} \int_\Omega dx \int_0^1 D^i v\, g_i(D^i(u_0 + tv))\,dt - \\ &- c_2\|v\|_{W_{\mathcal{G}}^k} - c = c_1 \sum_{i\in M} \int_\Omega dx \int_{D^i u_0(x)}^{D^i(u_0(x)+v(x))} g_i(s)\,ds - \\ &- c_2\|v\|_{W_{\mathcal{G}}^k} - c \geq c_1 \sum_{i\in M} \int_\Omega G_i(D^i(u_0+v))\,dx - \\ &- c_2\|v\|_{W_{\mathcal{G}}^k} - c,\end{aligned}$$

where the inner integral has a definite sense for almost all $x \in \Omega$. By reason of this inequality and

Theorem 1, Lemma 4 follows.

σ, Ω will denote the bounded domains of R^N. Let T be a mapping from σ onto Ω. We shall call T regular, if it is of the class C^1 and a 1-1-mapping σ onto Ω. Let us denote D_T the Jacobi's determinant of T.

Lemma 5. Let $G(u)$ be an N-function and $u \in L^*_G(\Omega)$. If T is a regular mapping from σ onto Ω with $c_1 \leq |D_T(y)| \leq c_2$ in σ, then $v(y) = u(Ty)$ belongs to $L^*_G(\sigma)$ and $\|v\|_{L_G(\sigma)} \leq c\,\|u\|_{L_G(\Omega)}$, where c is independent on u.

Proof. Let us assume $\varphi(y) \in E_P(\sigma)$, $\int_\sigma P[\varphi(y)]dy \leq 1$. We have

$$\int_\sigma \varphi(y)v(y)dy = \int_{T(\sigma)} \varphi(T^{-1}(x))v(T^{-1}(x))|D_{T^{-1}}(x)|dx \leq$$
$$\leq \frac{1}{c_1}\int_\Omega \varphi(T^{-1}(x))u(x)dx .$$

On the other hand, we have

$$\int_\Omega P[\varphi[T^{-1}(x)]]dx = \int_{T^{-1}(\Omega)} P[\varphi(y)]|D_T(y)|dy \leq c_2\int_\sigma P[\varphi(y)]dy .$$

From both inequalities and [4] (Lemma 9.1) we conclude

$$\|v\|_{L_G(\sigma)} \leq c\,\|u\|_{L_G(\Omega)} \quad \text{for each } u \in L^*_G(\Omega) .$$

Lemma 6. Let T be a regular mapping from σ onto Ω with $c_1 \leq |D_T(y)| \leq c_2$ in σ and $G(u)$ N-function. Suppose $u \in W^1_G(\Omega)$. If $v(y) = u(Ty)$, then $v \in W^1_G(\sigma)$ and $\|v\|_{W^1_G(\sigma)} \leq c\,\|u\|_{W^1_G(\Omega)}$.

We recall $W^1_G(\Omega) \equiv W^1_{\vec{G}}$, where $G_i(u) = G(u)$, $|i| \leq 1$.

Proof is the same as that in [3] (Lemma 3.2). We use only Lemma 5 and the fact that

$\|u_h - u\|_{W^1_G(\Omega^*)} \xrightarrow[h \to 0]{} 0$, for each $\bar{\Omega}^* \subset \Omega$,
where $u_h(x)$ is the mollified function of $u(x)$ (see the proof of Lemma 2, § 1).

<u>Lemma 7</u>. Suppose $\partial\Omega \in C^1$. Then for each bounded domain $\Omega^* \supset \bar{\Omega}$ there exists an extension for functions from $W^1_G(\Omega)$ to functions belonging to $W^1_G(\Omega^*)$ and

$$\|u\|_{W^1_G(\Omega^*)} \leq c\,\|u\|_{W^1_G(\Omega)} ,$$

where c is independent on u.

Having Lemmas 5 and 6, the proof of Lemma 7 is the same as that in [3] (Theorem 3.9).

<u>Theorem 4</u>. If $\partial\Omega \in C^1$ and G is an N-function, then the imbedding $W^1_G(\Omega) \longrightarrow E_G(\Omega)$ is compact.

<u>Proof</u>. Let $\{u_m\}$ be a bounded sequence from $W^1_G(\Omega)$, i.e. $\|u_m\|_{W^1_G} \leq c$. Let us take an arbitrary $\Omega^* \supset \bar{\Omega}$. We extend every u_m to a function belonging to $W^1_G(\Omega^*)$, still called u_m with $\|u_m\|_{W^1_G(\Omega^*)} \leq c$, because of Lemma 7. We can suppose $\partial\Omega^*$ Lipschitzian. For a smooth function $u \in W^1_G(\Omega^*)$ and $|h| \leq h_0 = dist\,(\partial\Omega, \partial\Omega^*)$ we have

$$u(x+h) - u(x) = \int_0^1 h_i \sum_{i=1}^{N} \frac{\partial u}{\partial x_i}(x+th)\,dt ,$$

where $x \in \Omega$. Supposing $v(x) \in E_P(\Omega)$ and $\int_\Omega P[v(x)]\,dx \leq 1$ we have

$$\int_\Omega v(x)\,[u(x+h) - u(x)]\,dx = \tag{2.19}$$
$$= \sum_{i=1}^{N} h_i \int_0^1 dt \int_\Omega v(x) \frac{\partial u}{\partial x_i}(x+th)\,dx .$$

(2.19) holds also for $u_m(x)$ by Lemma 2,§ 1. Using the Hölder's inequality in (2.19), we have

$$(2.20)\int_\Omega v(x)[u_m(x+h)-u(x)]\,dx \le |h|\sum_{i=1}^{N}\int_0^1 \|v\|_p \cdot$$

$$\cdot\left\|\frac{\partial u}{\partial x_i}(x+th)\right\|_G dt \le c\cdot|h|\,\|u_m\|_{W^1_G(\Omega^*)} \le c\cdot|h| .$$

Taking supremum in (2.20) with respect to $v(x)$ we obtain $\|u_m(x+h)-u_m(x)\|_G \le c\cdot|h|$. From [4] (Theorem 11.4 and Lemma 11.1) the compactness in $E(\Omega)$ follows.

<u>Corollary.</u> Let us have N-functions $G_i(u)$ for every $i \in M$ satisfying $G_i(u) \ge G_j(u)$ for $u \ge u_0$ and $|i| > |j|$. Suppose $\partial\Omega \in C^1$. Then the imbedding $W^k_{\vec{G}} \longrightarrow \bigcap_{i\in M-L} W^i_{G_i}$ is compact.

<u>Assertion 2</u>. Let $G_i(u)$ for each $i \in K$ satisfy $\frac{G_i(u)}{u} \in \mathcal{M}_3$ and

(2.21) $G_i(u) \ge G_0(u) = G_j(u)$ for $(u \ge u_0)$,

where $|i| = k$, $|j| < k$.

If there exist numbers p_0, q_0 from (1.1) corresponding to $G_0(u)$ and satisfying

$$(2.22)\qquad q_0 < p_0\cdot\frac{N}{N-p_0}$$

then $\|u\|_{W^k_{\vec{G}}} \le c\left(\sum_{|i|=k}\|D^i u\|_{G_i} + \|u\|_{G_0}\right)$ and the imbedding $W^k_{\vec{G}}(\Omega) \longrightarrow W^{k-1}_{G_0}$ is compact.

<u>Proof.</u> $W^k_{\vec{G}} \subset W^k_{p_0}$ (algebraically and topologically).

Using the known imbeddings we have

$\| D^i u \|_{G_0} \le c \| D^i u \|_{L_{q_0}} \le c \cdot \| u \|_{W^k_{p_0}}$ for $|i| < k$.
There holds (see [3] § 7)

$\| u \|_{W^k_{p_0}} \le c (\sum_{|i|=k} \| D^i u \|_{L_{p_0}} + \| u \|_{L_{p_0}})$. From both inequalities we obtain the required inequality.

Finally, $W^1_{G_0} \subset W^1_{p_0}$ and the imbedding $W^1_{p_0} \to \longrightarrow L_{q_0}(\Omega)$ is compact.

Assertion 3. Suppose (2.1),(2.7),(2.9) and (2.10), where the equality is admitted. If $a_i(x, \xi_j)$ is independent on ξ_l for all $i \in M_2$, $l \in M_1$, then there exists the solution of (2.3).
Indeed, the hypotheses of Leray-Lions Theorem are evidently satisfied.

Assertion 4. If (2.1),(2.4),(2.9),(2.10) (the equality admitted in (2.10)) hold and $a_i(x, \xi_j)$ is independent on ξ_l for all $i \in M_2$, $l \in M_1$, then the functional from (2.5) is semi-convex.

Proof. Let us define

$$\phi(u, v) = \sum_{i \in M_1} \int_0^1 dt \int_\Omega D^i u \, a_i(x, D^j(u_0 + tu)) dx +$$

$$+ \sum_{i \in M_2} \int_0^1 dt \int_\Omega D^i v \, a_i(x, D^j(u_0 + tv)) dx + (f, v) +$$

$$+ (g, v)_{\partial\Omega} = \phi_1(u) + \phi_2(v) .$$

$\phi_1(u)$ and $\phi_2(v)$ are continuous and bounded over the space $V_{\vec{G}}$ as a consequence of (2.1), Lemma 3§1 and Lemma 1. Regarding the properties of $a_i(x, \xi_j)$

and (2.10), the functional $\phi_1(u)$ is convex (see [6]). By reason of (2.9) nad Lemma 1, $\phi_2(v)$ is continuous with respect to the weak convergence in $V_{\vec{G}}$.

Assertion 5. If the stable boundary value condition $u_0(x) \equiv 0$ in the problem (2.3), then (2.7) follows from Conditions (2.16) and (2.1).

Indeed, we use Theorem 1 in the estimate

$$\frac{1}{\|v\|_{W_{\vec{G}}}} \int_\Omega \sum_{i \in M} D^i v\, a_i(x, D^j v)\, dx \geq$$

$$\geq c_1 \sum_{i \in M} \int_\Omega \frac{G_i[D^i v(x)]}{\|v\|_{W_{\vec{G}}}}\, dx - c \,.$$

In many cases the weaker conditions than (2.16) and (2.17) will be sufficient.

$$(2.16a) \quad \sum_{i \in M} \xi_i a_i(x, \xi_j) \geq c_1 \sum_{i \in L} \xi_i q_i(\xi_i) - c$$

for almost all $x \in \Omega$.

$$(2.17a) \quad \sum_{i \in M} \xi_i a_i(x, \eta_j) \geq c_1 \sum_{i \in L} \xi_i q_i(\eta_i) + c_2\, \xi_0 q_0(\eta_0) - c$$

for almost all $x \in \Omega$. For the Dirichlet's problem there is $c_2 \geq 0$.

One can see easily that one of the conditions (2.16),(2.17),(2.16a),(2.17a) implies Condition (2.11a).

Assertion 6. Let us have $q_i(u) \in \mathcal{M}_3$ for all $i \in K$ and suppose $q_l(u) \geq q_j(u) = q_0(u)$ for $u \geq u_0$, where $l \in L$, $j \in K - L$. Suppose $q_i(u)$ satisfies (1.9) for all $i \in K$. If (2.2),(2.16a),(2.10) and (2.22) hold, where $M_1 \equiv L$, $M_2 \equiv K - L$, then there exists the solution of the Dirichlet's problem (2.3).

Proof. We define $A(v, u)$ as in Theorem 3, putting only $M_1 \equiv L$, $M_2 \equiv K - L$.

Condition (2.9) is a consequence of Assertion 2. If we have $\partial\Omega \in C^1$, we do not need Condition (2.22), because (2.9) is a consequence of Theorem 4.) One can see easily that Condition (2.16a) implies (2.11a). Thus, it suffices to prove coerciveness of the operator From (2.16a) and Lemma 4, §1 we conclude

$$(v, A(v)) \geq c_1 \sum_{i \in L} \int_\Omega G_i [D^i(u_0+v)]dx -$$

$$- c_2 \sum_{i \in K} \| q_i(D^i(u_0+v)) \|_{p_i} - c \geq c_1' \sum_{i \in K} \int_\Omega G_i [D^i(u_0+v)]dx -$$

$$- c_2 \sum_{i \in K} \| q_i(D^i(u_0+v)) \|_{p_i} - c' .$$

Similarly as in Lemma 3, we deduce from the last inequality $\dfrac{(v, A(v))}{\|v\|_{W^k_{\vec{G}}}} \longrightarrow \infty$, if $\|v\|_{W^k_{\vec{G}}} \longrightarrow \infty$.
The rest of the proof is the same as that in Theorem 3.

Remark. If the stable boundary value condition $u_0(x) \equiv 0$, then in Assertion 6 we do not need Condition (1.9) for $q_i(u)$, $i \in K$, by the same argument as in Assertion 5.

Assertion 7. Let us have $q_i(u) \in \mathcal{M}_3$ for all $i \in K$ and suppose $q_l(u) \geq q_j(u) = q_0(u)$ for $u \geq u_0$, where $l \in L, j \in K - L$. Suppose (2.1), (2.4), (2.17a) and (2.8). For non-Dirichlet problem suppose, in addition, (2.22). Then there exists the solution of (2.3).

Proof. It suffices to prove $\phi(v) \longrightarrow \infty$, if $\|v\|_{W^k_{\vec{G}}} \to \infty$, where $\phi(v)$ comes from (2.5). Similarly as in Lemma 4 we have

$$\phi(v) \geq c_1 \sum_{i \in L} \int_\Omega G_i[D^i(u_0+v)]dx +$$
$$+ c_2 \int_\Omega G_0[u_0+v]dx - c \cdot \|v\|_{W^k_{\vec{G}}} - c .$$

If (2.22) holds, then the required result will be obtained from Assertion 2 and Theorem 1. We use Lemma 4,§1 for the Dirichlet's problem, and

$$\phi(v) \geq c_1' \sum_{i \in K} \int_\Omega G_i[D^i(u_0+v)]\,dx - c\,\|v\|_{W^k_{\vec{G}}} - c'$$

holds. The assertion follows from Theorem 1.

<u>Examples</u>. Suppose $g_i(u) \in \mathcal{M}_3$ for all $i \in M$ and let $f(x)$, $g(s)$ be measurable bounded functions defined on Ω, $\partial\Omega$. Let us consider an equation of the form

(2.23) $\sum_{i \in M} (-1)^{|i|} D^i[li(x)\, g_i(D^i u)] = f$,

where $li(x) \geq c > 0$ are measurable bounded functions. $u_0 \in W_{\vec{G}}$ and $g(s)$ give the stable and non-stable boundary value conditions.

a) If $g_i'(u) \geq 0$ for $u \in (-\infty, \infty)$ and for all $i \in M_1$ and if the imbedding $W^k_{\vec{G}} \rightarrow \bigcap_{i \in M_2} W^i_{G_i}$ is compact, then there exists a weak solution of the equation (2.23).

b) Let $g_i'(u) \geq 0$ for $u \in (-\infty, \infty)$ and $|i| = k$.
For $|i| < k$ we assume $g_i(u) \geq 0$ for $u > 0$ and $g_i(u) \leq 0$ for $u < 0$. Then there exists a weak solution of (2.23).

c) If $g_i'(u) > 0$ for $u \in (-\infty, \infty)$ and for all $i \in M$, then there exists the unique weak solution of (2.23).

The cases a), c) are evident from Theorem 2 and Lemma 4. In the case b), if $v_n \rightharpoonup v$ in $W^{k}_{\vec{G}}$, then for

$$\phi_2(v) = \sum_{i \in M-L} \int_0^1 dt \int_\Omega li(x) D^i v\, q_i\,(D^i(u_0 + tv))\, dx$$

holds.

$\phi_2(v) \leq \liminf_{k\to\infty} \phi_2(v_{n_k})$, where v_{n_k} is a suitable subsequence of $\{v_n\}$. Indeed, it is possible to select $\{v_{n_k}\}$ from $\{v_n\}$ satisfying $D^i v_{n_k}(x) \to$ $\to D^i v(x)$ for all $i \in M-L$, almost everywhere in Ω and $0 \leq F(s) = \int_0^s q(t)\,dt$ for $s \in (-\infty, \infty)$.

Thus, Assertion b) is a consequence of the Fatou's lemma.

The concrete examples of this type are

$$-\Delta u + q(u) = f ,$$

$$\sum_{i=1}^{N} -\frac{\partial}{\partial x_i}\left[li(x)\frac{\partial u}{\partial x_i}\cdot\left|\frac{\partial u}{\partial x_i}\right|^{n_i} \ln^{m_i}\left(\left|\frac{\partial u}{\partial x_i}\right|+1\right)\right] = f ,$$

where $n_i > -1$ and $m_i \geq 0$; n_i, m_i real numbers.

§ 3.

Now, let us consider a wide span of the growths (2.1) given by the class $\mathcal{M}_1$. If $q(u)$ does not possess Condition II, then $q(u(x))$ is not a mapping from $L^*_G(\Omega)$ into the dual space $L^*_P(\Omega)$, where p.p. $G(u) = u\,q(u)$. Indeed, in such a case there exists $v(x) \in L^*_G(\Omega)$ such that $\infty = \int_\Omega G[v(x)]dx \leq c +$ $+ \int_\Omega v(x)\, q(v(x))\, dx$, c being a finite constant.

$g(v(x)) \notin L_p^*(\Omega)$ because of the Hölder inequality. Thus, the method of monotone operators is not directly applicable as in § 2. In addition, we must admit the values $+\infty$ for a functional from (2.5) if we intend to use the calculus of variations. Finally, if the functional from (2.5) is finite at the point v, it need not be finite at the point $v + v'$ and thus there are difficulties with the Gateaux differential.

The weak solution for special cases of this direction was obtained by M.I. Višik [10], by means of the Galerkin's method.

We shall solve the Dirichlet's boundary value problem for the minimum of the functional

$$(3.1)\quad \phi(v) = \int_\Omega f(x, D^i v)\,dx + \int_\Omega g(x, D^j v)\,dx + F(v) \;,$$

$$\frac{\partial^\ell v}{\partial \nu^\ell} = \frac{\partial^\ell u_0}{\partial \nu^\ell} \text{ on } \partial\Omega \,, \text{ for } \ell = 0, 1, \dots, k-1 \,,$$

where ν is an exterior normal, i, j are multi-indices with $|i| \le k$, $|j| < k$. $u_0 \in W_1^k(\Omega)$ satisfying $\phi(u_0) < \infty$ gives us the boundary values. $F(v)$ is some linear functional.

Let M, M_1, M_2, K and L be from § 2.

1) $f(x, \xi_i) \ge 0$ is continuous in all variables

$$(3.2)\quad x \in \bar\Omega, |\xi_i| < \infty \text{ for } i \in M_1 \text{ and } (0, \dots, 0) \notin M \,.$$

2) $f(x, \xi_i)$ is convex in ξ ,

3) $|f(x, \xi_i) - f(y, \xi_i)| \le \lambda(|x-y|)[1 + f(x, \xi_i)]$,

where $\lambda(\sigma)$ is a positive function with $\lim\limits_{\sigma \to 0} \lambda(\sigma) = 0$.

(3.3) $g(x, \xi_j)$ is a real-valued function for $x \in \Omega$, $|\xi_j| < \infty$ with $j \in M_2$. It is continuous in ξ_j for almost every $x \in \Omega$ and measurable in x by ξ_j fixed.

(3.4) $|g(x, \xi_j)| \leq c\,(1 + \sum_{j \in M_2} G_j(\xi_j))$.

(3.5) $f(x, \xi_i) + g(x, \xi_j) \geq c_1 \sum_{i \in M} G_i(\frac{\xi_i}{\kappa}) - c$,

where $\kappa > 1$ is constant.

(3.6) $\dfrac{G_i(u)}{u} \in \mathcal{M}_1$ for all $i \in M_1$ and $\dfrac{G_j(u)}{u} \in \mathcal{M}_3$

for all $j \in M_2$.

Let us construct a space $W^{i}_{G_i^*}(\Omega) \equiv \{u \in L_1(\Omega);$ $D^i u \in L^*_{G_i}(\Omega)\}$, where $D^i u$ is the distribution derivative, $i \in M$. Let us denote $W^{k}_{\vec{G}^*} = \bigcap_{i \in M} W^{i}_{G_i^*}(\Omega)$ with the norm $\|u\|_{W^{k}_{\vec{G}^*}} = \sum_{i \in M} \|D^i u\|_{G_i}$, to which we add $\|u\|_{L_1(\Omega)}$ in the case $(0, \dots, 0) \notin M$.

(3.7) Let the imbedding $W^{k}_{\vec{G}^*} \longrightarrow \bigcap_{i \in M_2} W^{i}_{G_i}(\Omega)$ be compact.

Let us choose $1 \leq p_i$ for all $i \in L$ such that $|u|^{p_i} \leq G_i(u)$ for $u \geq u_0$ (the case " p_i are larger" is of more interest) and denote $p = \min\{p_i, i \in L\}$.

(3.8) $F(v) \in (W^{k-1}_p)'$ (dual space) ;

(3.9) $f(x, \xi_i)$ is strictly convex and $g(x, \xi_j)$

is convex in ξ.

Theorem 5. Suppose $\partial\Omega$ Lipschitzian for $p > 1$ and $\partial\Omega \in C^1$ for $p = 1$. If (3.2) to (3.8) are satisfied, then there exists a minimum of (3.1) in $W^k_{\vec{G}*}$. If, in addition, (3.9) holds, the minimum is unique.

At first we prove two lemmas.

Let us define $\overset{\circ}{W}{}^k_{\vec{G}} = \{u \in W^k_{\vec{G}*}$, for which $\frac{\partial^l u}{\partial \nu^l} = 0$ on $\partial\Omega$ for $l = 0, 1, \ldots, k-1\}$. In this case $\overline{\mathcal{D}(\Omega)} = \overset{\circ}{W}{}^k_{\vec{G}*}$ need not be true. $\overset{\circ}{W}{}^k_{\vec{G}*}$ is evidently a closed subspace of $W^k_{\vec{G}*}$.

We introduce $^*X$ convergence in the space $W^k_{\vec{G}*}$ by the following way: $u_n \xrightarrow[*X]{} u$, $u_n, u \in W^k_{\vec{G}*}$, if $\int_\Omega D^i u_n(x) v^{(i)}(x)\,dx \longrightarrow \int_\Omega D^i u(x) v^{(i)}(x)\,dx$, for all $v^{(i)}(x) \in E_{P_i}(\Omega)$ and for each $i \in M$; P_i being conjugate to G_i.

In general, $W^k_{\vec{G}*}$ need not be reflexive and $^*X$ convergence can be weaker than the weak convergence.

Lemma 1. $W^k_{\vec{G}*}$ is compact with respect to $^*X$ convergence; more exactly, from any bounded subset $B \subset W^k_{\vec{G}*}$ it is possible to select $\{u_n\} \subset B$ and $u \in W^k_{\vec{G}*}$ such that $u_n \xrightarrow[*X]{} u$. If $\partial\Omega \in C^1$ for $p = 1$, then $\overset{\circ}{W}{}^k_{\vec{G}*}$ is closed with respect to $^*X$ convergence.

Proof. The space $L^*_G(\Omega)$, G being an N-function, possesses the properties (see [4], Theorems 14.3 and 14.4):

1) From any bounded subset $A \subset L_G^*(\Omega)$ it is possible to select $\{u_n\} \in A$, $u \in L_G^*(\Omega)$ such that

(3.10) $\int_\Omega u_n(x)v(x)dx \longrightarrow \int_\Omega u(x)v(x)dx$

for all $v(x) \in E_p(\Omega)$;

2) whenever (3.10) holds, then there exists c such that $\|u_n\|_G \leq c$.

$W_{\vec{G}}^k$ is a closed linear subset of $\prod_{i \in M} L_{G_i}^*(\Omega)$ (cartesian product). By a successive selection we find $u_n \in B$ and $u^{(i)} \in L_{G_i}^*(\Omega)$ for all $i \in M$ satisfying $\int_\Omega D^i u_n(x) v^{(i)}(x)dx \longrightarrow \int_\Omega u^{(i)}(x)v^{(i)}(x)dx$ for all $v^{(i)} \in E_{P_i}(\Omega)$ and for each $i \in M$. There exists $u(x) \in L_1(\Omega)$ such that $u_n \xrightarrow[L_1(\Omega)]{} u$. We find easily that $D^i u(x) = u^{(i)}(x)$ for all $i \in M$. and thus the first part of the lemma is proved.

Now, suppose $u_n \xrightarrow[*X]{} u$ for $u_n \in \overset{\circ}{W}{}_{\vec{G}^*}^k$, $u \in W_{\vec{G}^*}^k$. In accordance with (3.10) there exists c such that $\|u_n\|_{W_{\vec{G}^*}^k} \leq c$ and hence $u_n \to u$ in the norm of the space $W_1^{k-1}(\Omega)$. Thus, we have $\frac{\partial^\ell u}{\partial \nu^\ell} = 0$ on $\partial\Omega$ for $\ell = 0, 1, \dots, k-2$. Now, let us suppose $p = 1$ and $\partial\Omega \in C^1$. Using the Green's theorem, we obtain for each $\varphi \in \mathcal{E}(\bar{\Omega})$

$$\int_{\partial\Omega} D^i u_n \nu_j \varphi \, ds = \int_\Omega D^{i+1} u_n \varphi \, dx + \int_\Omega D^i u_n \frac{\partial\varphi}{\partial x_j} dx ,$$

where ν_j is j-th component of the exterior normal ν and $i+1 \equiv i + (0, \dots 1, \dots 0)$, $|i| = k-1$.

From $^*X$ convergence we conclude

$$(3.11) \qquad \int_{\partial\Omega} D^i u_n \nu_j \varphi \, ds \longrightarrow \int_{\partial\Omega} D^i u \nu_j \varphi \, ds .$$

Let us denote $f_n(\varphi) = \int_{\partial\Omega} D^i u_n \nu_j \varphi \, ds$ and similarly $f(\varphi)$. There holds

$$(3.12) \quad |f_n(\varphi)| \le c \, \|u_n\|_{W^k_{\vec{G}^*}} \cdot \|\varphi\|_{C(\partial\Omega)} \le c \|\varphi\|_{C(\partial\Omega)}, \quad \varphi \in \mathcal{E}(\bar{\Omega}).$$

Restrictions of functions from $\mathcal{E}(\bar{\Omega})$ on $\partial\Omega$ form a dense subset in $C(\partial\Omega)$. (3.12) holds for $f(\varphi)$, too. We can uniquely extend f_n, f on $C(\partial\Omega)$ and thus $f_n(\varphi) \longrightarrow f(\varphi)$ for each $\varphi \in C(\partial\Omega)$. In (3.11) we substitute $\varphi = \nu_j \psi$, $\psi \in C(\partial\Omega)$ and then we sum up (3.11) through $j = 1, 2, \dots, N$. And hence

$$(3.13) \qquad \int_{\partial\Omega} D^i u_n \psi \, ds \longrightarrow \int_{\partial\Omega} D^i u \psi \, ds$$

for $|i| = k - 1$.

From (3.13) we deduce

$\int_{\Omega} \dfrac{\partial^{k-1} u}{\partial \nu^{k-1}} \psi \, ds = 0$ for all $\psi \in C(\partial\Omega)$ and thus $u \in \overset{\circ}{W}{}^k_{\vec{G}^*}$.

In the case $p > 1$, $\partial\Omega$ is Lipschitzian. Suppose $1 < s < \dfrac{N}{N-1}$, $s \le p$. For $u \in W^k_{\vec{G}^*}$ we have at least $D^i u \, \nu_j \in L_q(\partial\Omega)$, where $\dfrac{1}{q} = \dfrac{1}{s} - \dfrac{p-1}{(N-1)s}$ and $|i| = k-1$. For $\varphi \in \mathcal{E}(\bar{\Omega})$ there holds (3.13) and

$$\int_{\partial\Omega} D^i u_n \nu_j \varphi \, ds \le c \|u_n\|_{W^k_{\vec{G}^*}} \cdot \|\varphi\|_{L_{q'}(\partial\Omega)} \le c \cdot \|\varphi\|_{L_{q'}(\partial\Omega)} ,$$

where $q^{-1} + q'^{-1} = 1$. Restrictions of functions from $\mathcal{E}(\overline{\Omega})$ are dense in $L_{q'}(\partial\Omega)$. We can uniquely extend f_n, f on $L_{q'}(\partial\Omega)$ and $f_n(\varphi) \rightarrow f(\varphi)$ for each $\varphi \in L_{q'}(\partial\Omega)$. From this we deduce

$\dfrac{\partial^{k-1} u}{\partial \nu^{k-1}} = 0$ on $\partial\Omega$ again.

Lemma 2. Let us assume (3.2) to (3.4) and (3.6) to (3.8). If $u_n \xrightarrow[*X]{} u$; u_n, $u \in \mathring{W}^{k}_{\vec{G}*}$, then

$$\phi(u_0 + u) \leq \liminf_{n\to\infty} \phi(u_0 + u_n) .$$

Proof. $u_n \xrightarrow[*X]{} u$, where u_n, $u \in \mathring{W}^{k}_{\vec{G}*}$ implies $u_n \longrightarrow u$ in the norm of the space W^{k-1}_{p}. The results of J. Serrin [9] can be extended to the higher derivatives and hence

$$\int_\Omega f(x, D^j(u_0+u))\,dx \leq \liminf_{n\to\infty} \int_\Omega f(x, D^j(u_0+u_n))\,dx .$$

The functional $\int_\Omega g(x, D^j u)\,dx$ is continuous from $\bigcap_{i \in M_2} W^{i}_{G_i}$ into $L_1(\Omega)$ as a consequence of (3.3), (3.4), (3.6) and (3.7) and with respect to [4] (Lemma 17.2 and Theorem 17.3 where we set $M_2(u) = u$). The functional $F(v)$ is continuous because of (3.8). Thus, we have

$$(3.14)\quad \phi(u_0+u) \leq \liminf_{n\to\infty} \int_\Omega f(x, D^i(u_0 + u_n))\,dx \; +$$
$$+ \lim_{n\to\infty} \int_\Omega g(x, D^j(u_0+u_n))\,dx + \lim_{n\to\infty} F(u_0 + u_0) \leq$$
$$\leq \liminf_{n\to\infty} \phi(u_0 + u_n) .$$

Proof of Theorem 5. Let us consider $\phi(u_0 + u)$ over the space $\mathring{W}^{k}_{\vec{G}*}$. We admit the value $+\infty$ for $\phi(u_0 + u)$. From (3.5) and (3.8) we conclude

$$\phi(u_0+u)\geq c_1\sum_{i\in M}\int_\Omega G_i\left(\frac{D^i(u_0+u)}{\kappa}\right)-c_2\|u_0+u\|_{W_p^{k-1}}-c\ .$$

On the ground of the property of p we have

$$\sum_{i\in M}\int_\Omega G_i\left(\frac{D^i(u_0+u)}{\kappa}\right)dx\geq c_3\|u_0+u\|^p_{W_p^k}-c\ .$$

On the other hand, $\|u_0+u\|_{W_p^{k-1}}\leq c\|u_0+u\|_{W^k_{\vec{G}*}}$ holds and $\|u\|_G\leq 2\int_G G(u(x))\,dx$ for $\|u\|_G\geq 2$. (See [4],Theorem 9.5.) Thus, we conclude that from any sequence $\|u_m\|_{W^k_{\vec{G}*}}\to\infty$ it is possible to choose a subsequence u_{m_k} for which $\phi(u_0+u_{m_k})\xrightarrow[k\to\infty]{}\infty$ and hence $\phi(u_0+u)\to\infty$ if $\|u\|_{W^k_{\vec{G}*}}\to\infty$.

The last statement is true in the case $p=1$, too, by reason of the inequality $G_i(\frac{u}{\kappa})-c_1|u|\geq c_1'\,G_i(\frac{u}{\kappa})-c$, for each $|i|=k$; for suitable constants c_1' and c.

Let $\{u_m\}$ be a minimizing sequence for the functional $\phi(u_0+u)$. By reason of the previous fact there exists c such that $\|u_m\|_{W^k_{\vec{G}*}}\leq c$. Using Lemma 1 we find $u\in \overset{\circ}{W}{}^k_{\vec{G}*}$ and a suitable subsequence still called u_m such that $u_m\xrightarrow[*X]{}u$. With regard to Lemma 2 we have

$$\inf_{v\in \overset{\circ}{W}{}^k_{\vec{G}*}}\phi(u_0+v)=\phi(u_0+u)\leq\liminf_{m\to\infty}\phi(u_0+u_m)\ .$$

If $v\in W^k_{\vec{G}*}$ and $\dfrac{\partial^\ell v}{\partial\nu^\ell}=\dfrac{\partial^\ell u_0}{\partial\nu^\ell}$ on $\partial\Omega$ for $\ell=0,1,\dots,k-1$,then $v=v-u_0+u_0$ and $v-u_0\in\overset{\circ}{W}{}^k_{\vec{G}*}$ and hence $\phi(u_0+u)\leq\phi(v)$.

If (3.9) holds and u_1,u_2 are two points of minimum, then we have for $u_t=tu_1+(1-t)u_2$, $t\in(0,1)$

$$\int_\Omega[tf(x,D^iu_1)+(1-t)f(x,D^iu_2)-f(x,D^iu_t)]\,dx+$$
$$+\int_\Omega[tg(x,D^iu_1)+(1-t)g(x,D^iu_2)-g(x,D^iu_t)]\,dx=0$$

and thus $D^i u_1 = D^i u_2$ for $|i| = k$ almost everywhere in Ω. Considering the Dirichlet problem $u_1(x) \equiv \equiv u_2(x)$ almost everywhere in Ω.

Remark 1. Theorem 5 remains true if we substitute (3.7) and (3.6) by

(3.15) $g(x, \xi_j) \geq 0$ almost everywhere in Ω for all $|\xi_j| < \infty$, $j \in M_2$ and $\frac{G_i(u)}{u} \in m_1$ for all $i \in M$.

Indeed, if $u_m \xrightarrow[*X]{} u$, u_m, $u \in \overset{\circ}{W}{}^{k}_{\vec{G}*}$, then a suitable subsequence still called u_m $D^i u_m(x) \longrightarrow \longrightarrow D^i u(x)$ holds for all $i \in M - L$, almost everywhere in Ω. Using Fatou's Lemma, we obtain

$$\int_\Omega g(x, D^j(u_0+u))dx \leq \liminf_{m\to\infty} \int_\Omega g(x, D^j(u_0+u_m))dx$$

and hence Lemma 2.

Remark 2. In the case $k = 1$ Condition (3.2) can be weakened to (3.2') with respect to the results of J. Serrin [9] (Theorem 12).

(3.2') 1) $f(x, u, \xi_i) \geq 0$ is continuous in all variables, $|i| = 1$.

2) $f(x, u, \xi_i)$ is convex in ξ for each $x \in \Omega$, $|u| < \infty$.

Without loss of generality it is possible to suppose in (3.2) or (3.2') $f(x, \xi_i) \geq -c$ only.

Examples. Theorem 5 is applicable in the following types of examples:

a) $\phi(u) = \int_\Omega \sqrt{(\frac{\partial u}{\partial x} \ln^2(|\frac{\partial u}{\partial x}|+1) + (\frac{\partial u}{\partial y})^2 \ln^2(|\frac{\partial u}{\partial y}|+1)+1}\, dx\, dy -$

$-\int_\Omega u \cdot f\,dx\,dy$.

b) $\phi(u)=\int_\Omega \sqrt{(\frac{\partial u}{\partial x})^2 \ln^2(|\frac{\partial u}{\partial x}|+1)+e^{(\frac{\partial u}{\partial y})^2}}\,dx\,dy-\int_\Omega u\cdot f\,dx\,dy$.

c) $\phi(u)=\int_\Omega \sqrt{e^{(\frac{\partial u}{\partial x})^2}+e^{(\frac{\partial u}{\partial y})^2}}\,dx\,dy - \int_\Omega u f\,dx\,dy$.

d) $\phi(u)=\int_\Omega [(\frac{\partial u}{\partial x})^2+(\frac{\partial u}{\partial y})^2+h(u(x,y))-uf]\,dx\,dy$,

where $0 \le h(t) \in \mathcal{M}_1$.

§ 4.

In this section we establish a weak solution of those equations when the growth (2.1) or (2.2) is given by the class $\mathcal{M}_2$. We shall consider $\partial\Omega \in C^1$. Let $u_0 \in W^{k}_{\vec{G}}$ give us a boundary value.

<u>Theorem 6.</u> Suppose (2.1),(2.4) and let the functional from (2.5) have the form (3.1). Suppose (3.2) to (3.5),(3.7),(3.8) and $g_i(u) \in \mathcal{M}_3$ for all $i \in M_2$. Then there exists the weak solution of the Dirichlet's problem (2.3). In the case (2.8a) the solution is unique.

By reason of Lemma 3,§1 we must prove at first that (2.5) defines a functional over $W^{k}_{\vec{G}}$.

$a_i(x, D^{i}(u_0+tv))$ are measurable functions on $\Omega\times\langle 0,1\rangle$. Using (2.1), we have

(4.1) $\sum_{i\in M} |D^i v|\,|a_i(x, D^{i}(u_0+tv))| \le c(\sum_{i\in M} |D^i v| +$

$$+\sum_{i,j\in M}|D^i v||q_{ij}(D^j(u_0+tv))| .$$

Let us consider some member from the right side of (4.1) by t fixed. Using Lemma 3,§1, we obtain successively

$$(4.2)\quad \int_\Omega |D^i v||q_{ij}(D^j(u_0+tv))|\,dx \le$$

$$\le \|D^i v\|_{G_i}\cdot\|q_{ij}(D^j(u_0+tv))\|_{P_i} \le \|D^i v\|_{G_i}(1+$$

$$+\int_\Omega P_i[q_{ij}(D^j(u_0+tv))]dx \le \|D^i v\|_{G_i}(c+$$

$$+\int_\Omega G_j(D^j(u_0+tv))dx \le \|D^i v\|_{G_i}(c+\int_\Omega G_j(2D^j u_0)dx+$$

$$+\int_\Omega G_j(2D^j v)dx) .$$

Thus, on account of (4.1),(4.2) and (2.1), the functional (2.5) is well defined over $W^k_{\vec{G}}$. In addition, it is bounded on the bounded sets, because of Lemma 3,§ 1.

By Theorem 5, the functional (2.5) attains its minimum at a point $v\in \mathring{W}^k_{\vec{G}}$.

We shall construct a Gâteaux differential v only in some directions; precisely, we shall prove

$$(4.3)\quad \lim_{\tau\to 0}\frac{\phi(v+\tau\varphi)-\phi(v)}{\tau}=0 \text{ for each } \varphi\in\mathcal{D}(\Omega).$$

We use the idea of [8] (Theorem 5.1) and [2] (Theorem 2.1). Let us denote $a_{ih}(x,\xi_j)$ the mollified function of $a_i(x,\xi_j)$ in ξ_j by $x\in\Omega$ fixed (see (*) of Lemma 2,§ 1). Let $h\le h_0$ be fixed. There holds

$$(4.4)\ |a_{ih}(x,\xi_j)|\le c(1+\sum_{j\in M}q_i[G_i^{-1}(G_j(2\xi_j))])$$

for all $i \in M$.

(4.5) $|a_{ijh}(x, \xi_j)| \leq c(h)(1 + \sum_{j \in M} q_i [G_i^{-1}(G_j(2\xi_j))])$,

where $a_{ijh}(x, \xi_\ell) = \dfrac{\partial a_{ih}(x, \xi_\ell)}{\partial \xi_j}$ for all $i, j,$ $\ell \in M$.

By means of $a_{ih}(x, \xi_j)$ let us define the functional $\phi_h(v)$ from (2.5). Similarly as in [5], we obtain, with respect to (4.2),(4.4) and (4.5),

$$(4.6)\quad \phi_h(v+\varphi) - \phi_h(v) = \int_0^1 ds \int_\Omega \sum_{i \in M} D^i \varphi a_{ih}(x, D^j(u_0 + v + s\varphi))\, dx - F(\varphi) .$$

The inner integral is a continuous function in s, because of

$$a_{ih}(x, D^j(u_0+v+s\varphi)) \xrightarrow[s \to s_0]{} a_i(x, D^j(u_0+v+s_0\varphi))$$

for almost all $x \in \Omega$ and

$$(4.7)\qquad \| a_{ih}(x, D^j(u_0+v+s\varphi)) \|_{P_i} \leq c$$

for all $s \in (0,1)$.

Using the Valeé-Poussin´s theorem analogically as in the proof of Theorem 3, § 2. $D^i\varphi a_{ih}(x, D^j(u_0+v+s\varphi))$ have the uniformly absolutely continuous integrals. (h , $\varphi(x)$ being fixed.)

Thus, for suitable $s_0 \in (0,1)$

$$\phi_h(v+\varphi) - \phi(v) = \sum_{i \in M} \int_\Omega D^i \varphi a_{ih}(x, D^j(u_0 + v + s_0\varphi))\, dx - F(\varphi)$$

holds and hence there exists a derivative in the direction φ , from which

$$(4.8)\qquad \frac{\phi_h(v+\tau\varphi) - \phi_h(\varphi)}{\tau} =$$

$$= \frac{1}{\tau}\int_0^{\tau} dt \int_{\Omega} \sum_{i\in M} D^i \varphi\, a_{ih}(x, D^j(u_0 + v + t\varphi))\,dx - F(\varphi).$$

$$a_{ih}(x, D^j(u_0 + v + t\varphi) \xrightarrow[h\to 0]{} a_{ih}(x, D^j(u_0 + v + t\varphi))$$

holds almost everywhere in Ω.

By the reason of (4.1),(4.2) and (4.4) the Lebesgue's theorem gives $\phi_h(v) \xrightarrow[h\to 0]{} \phi(v)$. Now, we are allowed to let h become to infinity in (4.8). The inner integral in (4.8) is again continuous at $t = 0$. Thus, in the point v of the minimum we obtain

$$\int_{\Omega} \sum_{i\in M} D^i \varphi\, a_i(x, D^j(u_0 + v))dx - F(\varphi) = 0$$

for each $\varphi \in \mathcal{D}(\Omega)$. But, $\overline{\mathcal{D}(\Omega)} = \overset{\circ}{W}{}^{k}_{\vec{G}}$ and the theorem is proved.

Examples. a) Let us construct the Euler's equation to $\phi(v)$ from Example a), § 3. This equation possesses a weak solution.

b) Let us consider the example from § 2, where $q_i(u) \in \mathfrak{M}_2$ and $\partial\Omega \in C^1$. There exists a weak solution of (2.23) in Case b.

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