Pavel Čihák

On an exposed element of a set of doubly stochastic rectangular matrices

Commentationes Mathematicae Universitatis Carolinae, Vol. 11 (1970), No. 1, 99--113

Persistent URL: http://dml.cz/dmlcz/105268

Commentationes Mathematicae Universitatis Carolinae
11,1 (1970)

ON AN EXPOSED ELEMENT OF A SET OF DOUBLY STOCHASTIC RECTANGULAR MATRICES

Pavel ČIHÁK, Praha

In the present paper the notion of doubly stochastic matrix of the type (m, n), the notion of $\mathcal{U}$-exposed element of any subset of a linear space and the notion of a doubly stochastic unit matrix E of the type (m, n) are introduced. The main result of this paper is to obtain some analogous properties to those of the square unit matrix. It will be proved that the matrix E is $\mathcal{U}$-exposed, V-monotonic, V^o-monotonic and middle-symmetric. Moreover, these results are used to obtain a necessary and sufficient conditions for a m-vector to be a doubly stochastic image of a n-vector.

Contents:

1. Notations

The euclidean space of dimension n will be denoted by R_n. Define $b \cdot x = \sum_{k=1}^{n} b_k x_k$ for $b, x \in R_n$,

$$e = (1, 1, \dots, 1), \quad e^{(r)} = (0, 0, \dots, 1_r, 0, \dots, 0),$$

$$\mathcal{P}_n = \{x \in R_n ;\ x = (x_k)_{k=1}^{n},\ x_k \geq 0,\ \sum_{k=1}^{n} x_k = 1\} .$$

If $Q = (q_{jk})_{j,k}$ is a matrix of the type (m, n) then the matrix $Q^* = (q^*_{kj})_{k,j}$ of the type (n, m) fulfils the inequality $Qb \cdot x = b \cdot Q^* x$ for all $b \in R_n$, $x \in R_m$, if and only if $q^*_{kj} = q_{jk}$ for all j, k (i.e. Q^* is the adjoint matrix to Q).

Throughout this paper the term map Q will be used to mean a map from the euclidean space R_n to R_m such that

$$a_j = \sum_{k=1}^{n} q_{jk} b_k \quad \text{for } b = (b_k)_{k=1}^{n} \in R_n,\ a = (a_j)_{j=1}^{m} \in R_m,$$
$$a = Qb .$$

2. Doubly stochastic rectangular matrices

(2.1) A matrix $Q = (q_{jk})_{j,k}$ of the type (m, n) is called doubly stochastic iff

$$q_{rs} \geq 0,\ \sum_{k=1}^{n} q_{rk} = 1,\ \frac{1}{m} \sum_{j=1}^{m} q_{js} = \frac{1}{n}$$
for $r = 1, 2, \dots, m$ and $s = 1, 2, \dots, n$.

A set of all doubly stochastic matrices of the type (m, n) is denoted by $\mathcal{D}_{m,n}$.

(2.2) If a matrix Q of the type (m, n) and a matrix S of the type (n, p) are both doubly stochastic then the product QS is a doubly stochastic matrix of the type (m, p).

(2.3) If $Q \in D_{m,n}$ then $\frac{n}{m} Q^* \in D_{n,m}$.

(2.4) $D_{m,n}$ is a convex subset of $R_{m \cdot n}$.

3. Orderings

Define the following sets:

$$V_n = \{b \in R_n ;\ b_1 \geq b_2 \geq \ldots \geq b_n\} ,$$

$$U_n = \{b \in R_n ;\ b_1 > b_2 > \ldots > b_n\} ,$$

$$V_n^o = \{c \in R_n ;\ b \cdot c \geq 0 \text{ for all } b \in V_n\} .$$

Then V_n is a convex wedge, $V_n \cap -V_n = [e]$ and

U_n is a convex cone, $U_n \cap -U_n = \emptyset$ [3] .

(3.1) Lemma. The convex wedge V_n is generated by the following elements:

$$v^s = (\tfrac{1}{s}, \tfrac{1}{s}, \ldots, \tfrac{1}{s}, 0, 0, \ldots, 0) \in \mathcal{P}_n \quad \text{for } s = 1, 2, \ldots, n$$

and $v^{-n} = -v^n$, i.e. $v^s \in V_n$ for $s = 1, 2, \ldots, n, -n$

and if $b \in V_n$ then there are nonnegative numbers γ_s such that

$$b = \sum_{s=1}^{n} \gamma_s v^s + \gamma_{-n} v^{-n} .$$

Proof. If $b \in V_n$ then $b = (b_1 - b_2) v^1 + (b_2 - b_3) 2 \cdot v^2 + \ldots + (b_{n-1} - b_n) \cdot (n-1) \cdot v^{n-1} + |b_n| \cdot n \cdot v^{n \cdot \mathrm{sign}\, b_n}$.

(3.2) Lemma. V_n^o is a convex cone generated by the following elements:

$$c^1 = (1, -1, 0, 0, \ldots, 0), c^2 = (0, 1, -1, 0, \ldots, 0), \ldots, c^{n-1} = (0, 0, \ldots, 0, 1, -1).$$

Proof. Clearly $V_n^\circ = \{c \in R_n ;\ v^p \cdot c \geq 0$ for $p = 1, 2, \dots, n, -n\}$, i.e. $c \in V_n^\circ$ iff $c \in R_n$ and $\sum_{k=1}^{p} c_k \geq 0$ for $p = 1, 2, \dots, n$, $-\sum_{k=1}^{n} c_k \geq 0$. It follows that $c^1, c^2, \dots \dots, c^{n-1}$ are elements of V_n°, $V_n^\circ \cap - V_n^\circ = [0]$. If we put $c^n = (0, 0, \dots, 0, 1) \in R_n \dot{-} V_n^\circ$ then $\{c^p\}_{p=1}^{n}$ is a basis for the space R_n. If $c \in V_n^\circ$ then $c = \sum_{p=1}^{n} \gamma_p c^p$, where $\gamma_p = \sum_{k=1}^{p} c_k \geq 0$, $\gamma_n = \sum_{k=1}^{n} c_k = 0$. The proof is complete.

(3.3) Define a relation $\geq$ on the set $D_{m,n}$:

$Q_1 \geq Q_2$ iff $Q_1, Q_2 \in D_{m,n}$ and $Q_1 b \cdot x \geq Q_2 b \cdot x$ for all $b \in V_n$, $x \in V_m$, i.e. $Q_1 b - Q_2 b \in V_m^\circ$ for all $b \in V_n$. If moreover $Q_1 b \cdot x > Q_2 b \cdot x$ for all $b \in U_n$, $x \in U_m$ then this property will be denoted by $Q_1 > Q_2$.

Clearly $\geq$ is a quasiordering. Moreover, this quasiordering is an ordering. Suppose $Q_1 \geq Q_2$, $Q_2 \geq Q_1$. If $b \in V_n$, then $Q_1 b - Q_2 b \in V_m^\circ \cap - V_m^\circ = [0]$. Put $C = (c_{jk})_{j,k} = Q_1 - Q_2$. Then $Cb = 0$ for all $b \in V_n$. Hence $(c_{jk})_{k=1}^{m} \in V_n^\circ \cap (-V_n^\circ) = [0]$ for $j = 1, 2, \dots, m$ and we have $C = 0$, $Q_1 = Q_2$.

4. Permutations

The group G_m of all permutations of m elements can be considered as a subset of the set of doubly stochastic matrices $\mathcal{D}_{n,n}$. Define the identical permutation $E_n : E_n x = x$, the converse permutation E'_n : $: E'_n x = (x_n, x_{n-1}, \ldots, x_1)$ and the transposition $P_{i,j} : P_{i,j} x = (x_1, x_2, \ldots, x_{i-1}, x_j, x_{i+1}, \ldots, x_{j-1}, x_i, x_{j+1}, \ldots, x_n)$ for all $x = (x_1, x_2, \ldots, x_n) \in R_n$.

(4.1) Theorem. Let P be a permutation in the group G_n. Then there is a finite sequence $P_k \in G_n$, $k = 0, 1, \ldots, q$, having the following properties:

1° $P \in \{P_k\}_{k=0}^{q}$.

2° For each $k \in \{1, 2, \ldots, q\}$, $P_{k-1} P_k^{-1}$ is a transposition of the form $P_{i,i+1}$, $i \in \{1, 2, \ldots, n-1\}$.

3° $E_n = P_0 > P_1 > \ldots > P_q = E'_n$.

Hence $E_n > P$ for $P \neq E_n$ and $P > E'_n$ for $P \neq E'_n$. For each sequence fulfilling the properties 1°, 2° and 3° the inequality $q \leq \frac{(n-1)n(n+1)}{6}$ holds.

Proof. Let $I(P)$ ($I'(P)$ resp.) be the number of inversions (noninversions resp.) of the permutation P. Hence $I(P) + I'(P) = \binom{n}{2}$. If $P \neq E_n$ ($P \neq E'_n$ resp.) then $I(P) > 0$ ($I'(P) > 0$ resp.). Hence there is a number $i \in \{1, 2, \ldots, n-1\}$ such that

$$a_i < a_{i+1} \quad (a_i > a_{i+1} \text{ resp.})$$

for all $a = Pb$, $b \in U_n$, $a = (a_k)_{k=1}^{n}$. Hence $P_{i,i+1} Pb \cdot x - Pb \cdot x = a_{i+1} x_i + a_i x_{i+1} - a_i x_i - a_{i+1} x_{i+1} = (a_{i+1} - a_i)(x_i - x_{i+1}) \geq 0$ (≤ 0 resp.) for all $b \in V_n$, $x \in V_n$, moreover the strong inequality is fulfilled for all $b \in U_n$ and

$x \in \mathcal{U}_m$, i.e.

$$P_{i,i+1} P > P \qquad (P > P_{i,i+1} P \quad \text{resp.}).$$

There is a maximal finite decreasing sequence $\{P_r\}_{r=0}^{q}$ of permutations satisfying conditions 1^o and 2^o. If $P_0 \neq E_m$ or $P_q \neq E'_m$ then this sequence (by the first part of this proof) is not maximal. Hence $P_0 = E_m$, $P_q = E'_m$.

If $x = b = (n, n-1, \dots, 1)$ then $P_r b \cdot x - P_{r-1} b \cdot x \geq 1$. Hence

$$q \leq E_m b \cdot x - E'_m b \cdot x = \sum_{k=1}^{m} k(2k-n-1).$$

5. <u>Doubly stochastic unit matrix E</u>

Now, let us try to define some matrix $E \in D_{m,n}$ which would have analogous properties to those of the unit matrix and coincide with it in the case $m = n$. This matrix E will be called <u>doubly stochastic unit matrix</u>.

Consider elements w^r, $r = 1, 2, \dots, m$ of the set $\mathcal{P}_n \cap V_n$ which are defined in the following way: Put $s' = \frac{r \cdot n}{m}$. If $s \leq s' < s+1$ we define $w^r_k = \frac{1}{s'}$ for $k \leq s$, $w^r_{s+1} = 1 - \frac{s}{s'}$ and $w^r_k = 0$ for $k > s+1$. Clearly $0 \leq w^r_{s+1} < \frac{1}{s'}$.

Put $v^r = (v^r_k)_{k=1}^{n}$ for $r = 1, 2, \dots, n$, where $v^r_k = \frac{1}{r}$ for $k \leq r$, $v^r_k = 0$ for $k > r$. Since the elements $\{v^r\}_{r=1}^{n}$ are independent in the linear space R_m, there exists one and only one linear map $E^*: R_m \to R_n$ such that $E^* v^\kappa = w^\kappa$ for $\kappa = 1, 2, \ldots, m$.

(5.1) The matrix $E = (E^*)^*$ is doubly stochastic, $E \in D_{m,n}$.

Proof. If $E = (e_{jk}; j = 1, 2, \ldots, m, k = 1, 2, \ldots, n)$, $e^{(\kappa)} = (0, 0, \ldots, 0, 1_\kappa, 0, \ldots, 0)$ then $e^{(\kappa)} = \kappa v^\kappa - (\kappa - 1) v^{\kappa-1}$,

$E^* e^{(\kappa)} = \kappa w^\kappa - (\kappa - 1) w^{\kappa-1}$ (put $v^0 = v^1$, $w^0 = w^1$),

$\sum_{k=1}^{n} e_{\kappa k} = E^* e^{(\kappa)} \cdot e = \kappa w^\kappa \cdot e - (\kappa - 1) w^{\kappa-1} \cdot e = 1$ for $\kappa = 1, 2, \ldots, m$.

On the other hand $E^* v^m = w^m = (\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n})$, i.e. $\frac{1}{m} \sum_{j=1}^{m} e_{js} = \frac{1}{n}$ for $s = 1, 2, \ldots, n$.

It remains to show that all elements e_{jk} of the matrix E are nonnegative. This property is equivalent to $E^* e^{(\kappa)} = (e_{\kappa k})_{k=1}^{n}$ are nonnegative vectors for $\kappa = 1, 2, \ldots, m$.

If $\kappa = 1$ then $E^* e^{(1)} = w^1$ is a nonnegative vector. If

$\kappa \in \{2, 3, \ldots, m\}$, $s' = \frac{\kappa \cdot n}{m}$, $t' = \frac{(\kappa-1) \cdot n}{m} = s' - \frac{n}{m}$, $s, t \in \{0, 1, \ldots, n\}$, $s \leq s' < s + 1$, $t \leq t' < t + 1$ then $t \leq s$ and $e_{\kappa k} = 0$ for $k \leq t$.

1° If $t = s$ then $e_{\kappa, t+1} = \kappa (1 - \frac{s}{s'}) - (\kappa - 1)(1 - \frac{t}{t'}) = 1 - \frac{m}{n} s + \frac{m}{n} t = 1$ and $e_{\kappa k} = 0$ for $k > t + 1$.

2° If $t < s$ then

$e_{\kappa,t+1} = \frac{m}{n} - (\kappa-1)(1-\frac{t}{t'}) = \frac{\kappa-1}{t'}(1+t-t') \ge 0$ and $e_{\kappa k} = \kappa w^{\kappa}_{k} \ge 0$ for $k > t+1$. The proof is complete.

(5.2) Note. 1° $E^{*}\mathcal{P}_{m} \subset \mathcal{P}_{n}$, $E^{*}V_{m} \subset V_{n}$.

2° $E^{*}e^{(1)} = w^{1} = (1,0,0,\dots,0)$ for $m \ge n$.

3° If $m = n$ then $v^{\kappa} = w^{\kappa}$ for all κ and E is the unit matrix.

(5.3) Theorem. $E \ge Q \ge E'_{m}E$ for all $Q \in D_{m,n}$.

Proof. 1° Suppose $b \in V_{n}$, $a = Qb$, $\kappa \in \{1,2,\dots,m\}$. Then $Qb\cdot v^{\kappa} = a\cdot v^{\kappa} = \frac{1}{\kappa}\sum_{j=1}^{\kappa} a_{j} = \sum_{k=1}^{n}(\frac{1}{\kappa}\sum_{j=1}^{\kappa} q_{jk})b_{k} = \sum_{k=1}^{n} r_{k}b_{k}$, where

$$Q = (q_{jk})_{j,k}, \quad r_{k} = \frac{1}{\kappa}\sum_{j=1}^{\kappa} q_{jk} .$$

Choosing a number $s \in \{0,1,2,\dots,n\}$ such that $s \le s' = \frac{\kappa\cdot n}{m} < s+1$ we obtain the following inequalities:

$$0 \le r_{k} = \frac{1}{\kappa}\sum_{j=1}^{\kappa} q_{jk} \le \frac{1}{s'}, \quad \sum_{k=1}^{n} r_{k} = \frac{1}{\kappa}\sum_{j=1}^{\kappa}\sum_{k=1}^{n} q_{jk} = 1, \quad s r_{k} \le \frac{s}{s'} .$$

There is a number $b^{*} \le b_{s+1}$ such that

$$\sum_{k=1}^{n} r_{k}b_{k} = \sum_{k=1}^{s} r_{k}b_{k} + (1-\sum_{k=1}^{s} r_{k})b^{*} \le$$
$$\le \frac{1}{s}\sum_{k=1}^{n}(s r_{k}b_{k} + (1-s r_{k})b_{s+1}) \le \frac{1}{s}\sum_{k=1}^{s}(\frac{s}{s'}b_{k} + (1-\frac{s}{s'})\cdot$$
$$\cdot b_{s+1}) = \frac{1}{s'}\sum_{k=1}^{s} b_{k} + (1-\frac{s}{s'})b_{s+1} = b\cdot w^{\kappa} ,$$

i.e.

$$Qb\cdot v^{\kappa} \le b\cdot w^{\kappa} .$$

Since $E^{*}v^{\kappa} = w^{\kappa}$ we can write $Qb\cdot v^{\kappa} \le b\cdot E^{*}v^{\kappa} = Eb\cdot v^{\kappa}$. Moreover $Qb\cdot v^{m} = \sum_{k=1}^{n}\frac{1}{m}\sum_{j=1}^{m} q_{jk})b_{k} = \frac{1}{n}\sum_{k=1}^{n} b_{k} = b\cdot w^{m}$. Using

(3.1) we obtain $Q b \cdot x \leq E b \cdot x$ for all $x \in V_m$ and $b \in V_n$ i.e. $E \geq Q$.

2° If $b \in V_n$, $y \in V_m$ then $x = -E'_m y \in V_m$. Clearly $E'_m Q \in D_{m,n}$ and $E \geq E'_m Q$ by 1°. Hence

$$E'_m E b \cdot y = E b \cdot E'_m y = -E b \cdot x \leq -E'_m Q b \cdot x = Q b \cdot (-E'_m x) = Q b \cdot y$$

and $E'_m E \leq Q$.

(5.4) Definition. Let D be a subset of a linear space R. Let U be a nonvoid subset of the space of all linear functionals on the space R. Then an element $Q \in D$ will be called U-exposed iff

$$Z(Q_0) > Z(Q) \text{ for all } Q \in D,\ Q \neq Q_0 \text{ and all } Z \in U.$$

If $Z \in R_{m \cdot n}$, $Z = (z_{jk})_{j,k}$ then the symbol $Z \cdot Q$ will denote the functional $Z(Q) = \sum_{j=1}^{m} \sum_{k=1}^{n} z_{jk} q_{jk}$.

(5.5) Theorem. The matrix $E \in D_{m,n}$ is an $U_m \otimes U_n$-exposed element of the set $D_{m,n}$ in the linear space $R_{m \cdot n}$.

Proof. If $x \in R_m$ and $b \in R_n$ then the symbol $x \otimes b$ will denote the matrix $Z = (x_j b_k)_{j,k}$ of the type (m, n). Hence

$x \otimes b \cdot Q = Q b \cdot x$ and $x \otimes b \cdot Q \leq x \otimes b \cdot E$ for all $x \in V_m$, $b \in V_n$, $Q \in D_{m,n}$ (by (5.3)).

Now, suppose $x \in U_m$, $b \in U_n$. Then there are real numbers ξ_k and η_k such that $x = \sum_{k=1}^{m} \xi_k v^k$,

$y = \sum_{s=1}^{m} \eta_s v^s$. We obtain

$\xi_k = k(x_k - x_{k+1}) > 0$ for $k = 1, 2, \ldots, m-1$, $\xi_m = m x_m$,

$\eta_s = s(y_s - y_{s+1}) > 0$ for $s = 1, 2, \ldots, n-1$, $\eta_n = n y_n$

and

$$x \otimes y = \sum_{k=1}^{m} \sum_{s=1}^{n} \xi^k \eta^s v^k \otimes v^s .$$

If $x \otimes y \cdot Q_o = x \otimes y \cdot E$ for some matrix $Q_o \in$ $\in D_{m,n}$ then

$$\sum_{k=1}^{m} \sum_{s=1}^{n} \xi_k \eta_s v^k \otimes v^s \cdot Q_o = \sum_{k=1}^{m} \sum_{s=1}^{n} \xi_k \eta_s v^k \otimes v^s \cdot E .$$

Since $v^k \otimes v^s \cdot Q_o \leq v^k \otimes v^s \cdot E$, $v^k \otimes v^n \cdot Q_o = v^k \otimes v^n \cdot E$ and $v^m \otimes v^s \cdot Q_o = v^m \otimes v^s \cdot E$ for all k and s, we obtain mn independent equalities:

$v^k \otimes v^s \cdot Q_o = v^k \otimes v^s \cdot E$ for $k = 1, 2, \ldots, m$ and $s = 1, 2, \ldots, n$. Hence $Q_o = E$. The proof is complete.

(5.6) <u>Note.</u> Using the symbol $>$ defined in (2.3) which denotes a transitive and nonreflexive relation on the set $D_{m,n}$, and using theorem (5.5), we obtain the following statement:

The matrix E is the unique element of the set $D_{m,n}$ having the following property: $E > Q$ for all $Q \in D_{m,n}$ such that $Q \neq E$.

(5.7) <u>Definition</u>. A matrix $S = (s_{jk})_{j,k}$ of the type (m,n) will be called <u>middle-symmetric</u> iff $E'_m S E'_n = S$, i.e. $s_{jk} = s_{m-j+1,\, n-k+1}$ for all j, k.

(5.7) Remark. If a matrix is a product of two middle-symmetric matrices then this matrix is also middle-symmetric.

(5.8) Theorem. Let E be the doubly stochastic unit matrix of the set $D_{m,n}$. Then the matrix E is middle-symmetric and the matrix $\frac{n}{m} E^*$ is the doubly-stochastic unit matrix of the set $D_{n,m}$.

Proof. If $E \in D_{m,n}$ and $E'_m E E'_n \neq E$ then by the note (5.6) $E'_n E E'_n < E$. Take an element $x \in U_m$ and $b \in U_n$. Then $-E'_m x \in U_m$ and $-E'_n b \in U_n$. Using theorem (5.5), we obtain the following inequalities:

$$E b \cdot x > E'_m E E'_n b \cdot x = E(-E'_n b) \cdot (-E'_m x) > E'_m E E'_n (-E'_n b) \cdot (-E'_m x) = E b \cdot x,$$

which is a contradiction.

Let E_1 be the doubly stochastic unit matrix of the set $D_{n,m}$, $x \in U_m$, $b \in U_n$. If $E_1 \neq \frac{n}{m} E^*$ then by (5.6)

$$E_1 > \frac{n}{m} E^*, \; E_1 x \cdot b > \frac{n}{m} E^* x \cdot b, \; \frac{m}{n} E_1^* b \cdot x > E b \cdot x .$$

Hence $\frac{m}{n} E_1^* > E$. Using (5.6), we obtain a contradiction.

(5.9) Examples of doubly stochastic unit matrices of various types (m, n):

$(m,n) = (2,1),\ (3,2),\ (4,3),\ (5,3),\ (5,4),\ (6,5)$

$$E = \begin{bmatrix}11\\11\end{bmatrix} \begin{bmatrix}1&0\\ \frac12&\frac12\\ 0&1\end{bmatrix} \begin{bmatrix}1&0&0\\ \frac13&\frac23&0\\ 0&\frac23&\frac13\\ 0&0&1\end{bmatrix} \begin{bmatrix}1&0&0\\ \frac23&\frac13&0\\ 0&1&0\\ 0&\frac13&\frac23\\ 0&0&1\end{bmatrix} \begin{bmatrix}1&0&0&0\\ \frac14&\frac34&0&0\\ 0&\frac12&\frac12&0\\ 0&0&\frac34&\frac14\\ 0&0&0&1\end{bmatrix} \begin{bmatrix}1&0&0&0&0\\ \frac15&\frac45&0&0&0\\ 0&\frac25&\frac35&0&0\\ 0&0&\frac35&\frac25&0\\ 0&0&0&\frac45&\frac15\\ 0&0&0&0&1\end{bmatrix}$$

6. Doubly stochastic maps of the euclidean space

(6.1) Definition. A matrix $Q \in D_{m,n}$ is said to be V-monotone (V^o-monotone resp.) iff

$$Q V_n \subset V_m \quad (Q V_n^o \subset V_m^o \quad \text{resp.}).$$

(6.2) Lemma. The doubly stochastic unit matrix $E \in D_{m,n}$ is V^o-monotone and V-monotone.

Proof. 1^o By (5.2) $E^* V_m \subset V_n$. If $c \in V_n^o$ then $0 \le c \cdot E^* x$ for all $x \in V_m$. But $c \cdot E^* x = Ec \cdot x$. Hence $Ec \in V_m^o$.

2^o Define a matrix $E_1 = \frac{n}{m} E^*$. Then by (5.8) E_1 is the doubly stochastic unit matrix of $D_{m,n}$. Hence

$$E V_n = \frac{m}{n} E_1^* V_n = E_1^* V_n \subset V_m .$$

(6.3) Theorem. Let a be an element of the set V_m, let b be an element of the set V_n and let E be the doubly stochastic unit matrix in $D_{m,n}$. Then the following conditions are equivalent:

1° There is a doubly stochastic matrix $Q \in \mathcal{D}_{m,n}$ such that $a = Qb$.

2° $Eb - a \in V_m^o$.

3° $a \cdot v^k \leq b \cdot w^k$ for $k = 1, 2, \ldots, m, -m$.

4° If $k \in \{1, 2, \ldots, m-1\}$, $s' = \frac{k \cdot n}{m}$, $s \in \{0, 1, \ldots, n\}$, $s \leq s' < s+1$

then

$$\frac{1}{k}\sum_{j=1}^{k} a_j \leq \frac{1}{s'}\sum_{k=1}^{s} b_k + \left(1 - \frac{s}{s'}\right) b_{s+1} \quad \text{and} \quad \frac{1}{m}\sum_{j=1}^{m} a_j = \frac{1}{n}\sum_{k=1}^{n} b_k .$$

5° There is a doubly stochastic matrix $S \in \mathcal{D}_{m,m}$ such that $a = SEb$.

<u>Proof</u>. $1^\circ \Longrightarrow 2^\circ$, $2^\circ \Longleftrightarrow 3^\circ$, $3^\circ \Longleftrightarrow 4^\circ$:
If the condition 1° holds then $a \cdot x = Qb \cdot x \leq Eb \cdot x$ for all $x \in V_m$ by (5.3). Hence $Eb - a \in V_m^o$. Using (3.1) we obtain:

$$\frac{1}{k}\sum_{j=1}^{k} a_j = a \cdot v^k \leq Eb \cdot v^k = b \cdot E^* v^k = b \cdot w^k = \frac{1}{s'}\sum_{k=1}^{s} b_k + \left(1 - \frac{s}{s'}\right) b_{s+1}$$

for $k = 1, 2, \ldots, m$ and

$$-\frac{1}{m}\sum_{j=1}^{m} a_j = a \cdot v^{-m} \leq b \cdot w^{-m} = -\frac{1}{n}\sum_{k=1}^{n} b_k ,$$

and the equivalence of conditions 2°, 3° and 4°.

$2^\circ \longrightarrow 5^\circ$: If $a \notin \{SEb \,;\, S \in \mathcal{D}_{m,m}\}$ then by the separation theorem [6] there is an element $y \in R_m$ such that

$$a \cdot y > SEb \cdot y \quad \text{for all } S \in \mathcal{D}_{m,m} .$$

Let P be such permutation in G_m that $x = P^{-1} y \in V_m$.

Then, using theorem (4.1), we obtain:

$$a \cdot x \geq a \cdot Px = a \cdot y > PEb \cdot y = Eb \cdot P^{-1} y = Eb \cdot x \,.$$

Hence $(Eb - a) \cdot x < 0$ which is a contradiction.

$5^{\circ} \Longrightarrow 1^{\circ}$: This follows from (2.2).

(6.4) <u>Corollary.</u> Let T be a doubly stochastic matrix of the type (n, m), let E be the doubly stochastic unit matrix of the type (m, n) and let b be an element of the set V_m. Then there is a doubly stochastic matrix S of the type (m, m) such that

$$SEb = ETb.$$

(6.5) <u>Note.</u> The equivalence of the conditions 1° and 4° in theorem (6.3) is well-known for a special case $m = n$ (see [4]), the condition 4° in the following form:

$$\sum_{k=1}^{\kappa} a_k \leq \sum_{k=1}^{\kappa} b_k \quad \text{for } \kappa = 1, 2, \ldots, m-1 \quad \text{and} \sum_{k=1}^{m} a_k = \sum_{k=1}^{m} b_k \,.$$

But the proof is given by another way.

Acknowledgment. The author wishes to thank Professor Vl. Pták for his advice to consider square doubly stochastic matrices and dilatation of measures and for his kind encouragement.

References

[1] CARTIER P., FELL J.M.G., MEYER P.A.: Comparaison des measures portées par un ensemble convex compact. Bull.Soc.Math.France 92 (1964),435-445.

[2] CHOQUET G., CORSON H., KLEE V.: Exposed points of convex sets. Pacific J.Math.17(1966),33-43.

[3] DAY M.: Normed linear spaces (Berlin 1958).

[4] HARDY G.H.; LITTLEWOOD J.E.,PÓLYA G.: Inequalities

[5] SCHWARZ Št.: A note on the structure of the semigroups of doubly stochastic matrices. Matem. čas.SAV 17(1967),308-316.

[6] VALENTINE F.A.: Convex sets (New York 1952).

Matematicko-fyzikální fakulta
Karlova universita
Sokolovská 83
Praha 8 Karlín
Československo

(Oblatum 7.7.1969)