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*Commentationes Mathematicae Universitatis Carolinae*, Vol. 11 (1970), No. 1, 91--98

Persistent URL: http://dml.cz/dmlcz/105267

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11,1 (1970)

# AN EMBEDDING OF GROUPOIDS AND MONOMORPHISMS INTO SIMPLE GROUPOIDS

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1. Introduction. By a type we mean a family $(n_i)_{i\in I}$ of natural numbers $n_i \geq 0$. By an algebra of type $(n_i)_{i\in I}$ we mean an ordered pair $\mathbf{A} = \langle A, (f_i)_{i\in I}\rangle$ where $A$ is a set (called the underlying set of $\mathbf{A}$) and $f_i$ (for every $i \in I$) is an $n_i$-ary operation in $A$. The underlying set of $\mathbf{A}$ (of $\mathbf{B}, \dots,$ resp.) is denoted by $A$ (by $B, \dots,$ resp.). If I consists of a single element $i_0$ and $n_{i_0} = 2$, then algebras of this type $(n_i)_{i\in I}$ are called groupoids. A groupoid $\mathbf{A}$ is thus a set $A$ together with a binary operation in $A$; this operation is denoted by $x \cdot y$.

An algebra is called simple if it has no non-trivial congruence relations. We shall be concerned with the following question: given a type $(n_i)_{i\in I}$, how large is the class of all simple algebras of this type?

Let firstly $n_i \leq 1$ for all $i \in I$. If $\mathbf{A}$ is a simple algebra of type $(n_i)_{i\in I}$, then $\mathbf{A}$ is generated by each at least two-element subset of $A$ and thus $\operatorname{Card} A \leq \max(\aleph_0, \operatorname{Card} I)$; if $\mathbf{B}$ is a subalgebra of $\mathbf{A}$, then the relation $\theta$ defined by

$\langle x, y\rangle \in \theta$ if and only if either $\langle x, y\rangle \in B \times B$ or $x = y \in A$ is evidently a congruence relation of $\mathbf{A}$. This shows that the class of all simple algebras of such a type $(n_i)_{i\in I}$ is not too large.

Let secondly $n_i \geq 2$ for at least one $i \in I$. The class of all simple algebras of type $(n_i)_{i\in I}$ is sufficiently large in the following sense: the category of all monomorphisms of algebras of type $(n_i)_{i\in I}$ is isomorphic to a full subcategory of the category of all homomorphisms of simple algebras of type $(n_i)_{i\in I}$. This statement will be proved only in the case of groupoids (see the Theorem below); the general case is an easy generalization.

For some theorems and methods concerning full embeddings of categories of algebras see [1] and [2].

2. <u>An auxiliary construction</u>. If $\mathcal{O}\!\mathcal{L}$ is a class of groupoids, then we assign to $\mathcal{O}\!\mathcal{L}$ two categories:

(i) $h(\mathcal{O}\!\mathcal{L})$ is the category of all groupoids from $\mathcal{O}\!\mathcal{L}$, morphisms being the homomorphisms;

(ii) $\mu(\mathcal{O}\!\mathcal{L})$ is the category of all groupoids from $\mathcal{O}\!\mathcal{L}$, morphisms being the monomorphisms, i.e. injective homomorphisms, i.e. isomorphisms into.

The following three classes of groupoids will be spoken of:

(i) $\mathcal{G}$ is the class of all groupoids;

(ii) $\mathcal{G}'$ is the class of all groupoids $\mathbf{A}$ satisfying $x\cdot((x\cdot x)\cdot x)\neq x$ for all $x\in A$. Notice that every $\mathbb{A}\in\mathcal{G}'$ is a groupoid without idempotents. (An element $x$ is called idempotent if $x\cdot x = x$.)

(iii) $\mathcal{G}_s$ is the class of all simple groupoids without idempotents. Notice that $\mathfrak{h}(\mathcal{G}_s) = \mu(\mathcal{G}_s)$.

<u>Lemma 1.</u> $\mu(\mathcal{G})$ is isomorphic to a full subcategory of $\mu(\mathcal{G}')$.

<u>Proof</u>. Let us assign to every groupoid $\mathbb{A}$ a groupoid $\Psi(\mathbb{A})$ with the underlying set $(A\times\{0\})\cup(A\times\{1\})$ in this way:

$$\langle a_1, 0\rangle\cdot\langle a_2, 0\rangle = \langle a_1\cdot a_2, 1\rangle ;$$

$$\langle a_1, 1\rangle\cdot\langle a_2, 1\rangle = \langle a_1\cdot a_2, 0\rangle ;$$

$$\langle a_1, 0\rangle\cdot\langle a_2, 1\rangle = \langle a_2, 1\rangle\cdot\langle a_1, 0\rangle = \langle a_2, 0\rangle .$$

This $\Psi(\mathbb{A})$ belongs to $\mathcal{G}'$, as $x\cdot((x\cdot x)\cdot x)\in A\times\{1\}$ for all $x\in A\times\{0\}$ and $x\cdot((x\cdot x)\cdot x)\in A\times\{0\}$ for all $x\in A\times\{1\}$.

Let us assign to every monomorphism $\varphi$ of a groupoid $\mathbb{A}$ into a groupoid $\mathbb{B}$ a monomorphism $\Psi(\varphi)$ of $\Psi(\mathbb{A})$ into $\Psi(\mathbb{B})$ in this way:

$$\Psi(\varphi)(\langle a, 0\rangle) = \langle\varphi(a), 0\rangle ;$$

$$\Psi(\varphi)(\langle a, 1\rangle) = \langle\varphi(a), 1\rangle .$$

It is easy to prove that $\Psi(\varphi)$ is really a monomorphism. $\Psi$ is evidently a functor and it is an isomorphism of $\mu(\mathcal{G})$ onto a subcategory of $\mu(\mathcal{G}')$; it is sufficient to prove that this subcategory is full. Let $\mathbb{A}$ and $\mathbb{B}$ be two groupoids and $\psi$ a monomorphism of $\Psi(\mathbb{A})$ into $\Psi(\mathbb{B})$; we are to prove that there exists a monomorphism $\varphi$ of $\mathbb{A}$ into $\mathbb{B}$ such that $\psi = \Psi(\varphi)$.

As $\psi(\langle a, 0\rangle)\cdot\psi(\langle a, 1\rangle) = \psi(\langle a, 0\rangle\cdot\langle a, 1\rangle) = \psi(\langle a, 0\rangle)$, we have evidently $\psi(\langle a, 0\rangle) \in B \times \{0\}$ for all $a \in A$. Define a mapping $\varphi$ of $A$ into $B$ by $\psi(\langle a, 0\rangle) = \langle\varphi(a), 0\rangle$.

From $\psi(\langle a, 0\rangle)\cdot\psi(\langle a, 1\rangle) = \psi(\langle a, 0\rangle)$ we get $\psi(\langle a, 1\rangle) \in B \times \{1\}$. Put $\psi(\langle a, 1\rangle) = \langle b, 1\rangle$. We have $\langle b, 0\rangle = \langle\varphi(a), 0\rangle\cdot\langle b, 1\rangle = \psi(\langle a, 0\rangle)\cdot$ $\cdot\psi(\langle a, 1\rangle) = \psi(\langle a, 0\rangle\cdot\langle a, 1\rangle) = \psi(\langle a, 0\rangle) = \langle\varphi(a), 0\rangle$, so that $b = \varphi(a)$, so that $\psi(\langle a, 1\rangle) = \langle\varphi(a), 1\rangle$ for all $a \in A$.

It remains to show that $\varphi$ is a monomorphism. We have $\langle\varphi(a_1\cdot a_2), 0\rangle = \psi(\langle a_1\cdot a_2, 0\rangle) = \psi(\langle a_1, 1\rangle\cdot\langle a_2, 1\rangle) =$ $= \psi(\langle a_1, 1\rangle)\cdot\psi(\langle a_2, 1\rangle) = \langle\varphi(a_1), 1\rangle\cdot\langle\varphi(a_2), 1\rangle =$ $= \langle\varphi(a_1)\cdot\varphi(a_2), 0\rangle$, so that $\varphi(a_1\cdot a_2) = \varphi(a_1)\cdot$ $\cdot\varphi(a_2)$ for all $a_1, a_2 \in A$. The injectivity of $\varphi$ is evident.

3. <u>A full embedding of the category of monomorphisms of groupoids into the category of homomorphisms of simple groupoids without idempotents</u>. If $A$ is a set, then put $F(A) = A_0 \cup A_1 \cup A_2 \cup \ldots$ where $A_0 = A \times \{0\}$ and for every $n > 0$, $A_n$ is the set of all ordered triples $\langle x, y, n \rangle$ such that $x$ and $y$ are different elements of $A_0 \cup \ldots \cup A_{n-1}$, at least one of them belonging to $A_{n-1}$.

If $\varphi$ is an injective mapping of a set $A$ into a set $B$, then define a mapping $F(\varphi)$ of $F(A)$ into $F(B)$ by

$$F(\varphi)(\langle a, 0 \rangle) = \langle \varphi(a), 0 \rangle ;$$

$$F(\varphi)(\langle x, y, n \rangle) = \langle F(\varphi)(x), F(\varphi)(y), n \rangle .$$

It is evidently again injective.

If $\mathbb{A}$ is a groupoid, then define a groupoid $\Phi(\mathbb{A})$ with the underlying set $F(A)$ by

$$\langle a_1, 0 \rangle \cdot \langle a_2, 0 \rangle = \langle a_1 \cdot a_2, 0 \rangle ;$$
$$\langle x, y, n \rangle \cdot \langle a, 0 \rangle = \langle x, y, n \rangle ;$$
$$\langle x, y, n \rangle \cdot \langle x, y, n \rangle = x ;$$
$$x \cdot \langle x, y, n \rangle = y ;$$

$$\alpha \cdot \langle x, y, n \rangle = \alpha \quad \text{for all } \alpha \in F(A) \text{ satisfying } \alpha \neq \langle x, y, n \rangle \text{ and } \alpha \neq x .$$

<u>Lemma 2.</u> If $\varphi$ is a monomorphism of a groupoid $\mathbb{A}$ into a groupoid $\mathbb{B}$, then $F(\varphi)$ is a monomorphism of $\Phi(\mathbb{A})$ into $\Phi(\mathbb{B})$.

The proof is evident.

Lemma 3. If $\mathbb{A}$ is a groupoid, then $\Phi(\mathbb{A})$ is a simple groupoid.

Proof. Suppose that $\theta$ is a non-trivial congruence relation of $\Phi(\mathbb{A})$. There exist three different elements $x, y, z \in F(A)$ such that $\langle x, y\rangle \in \theta$ and $\langle x, z\rangle \notin \theta$. Let $n$ be the least natural number such that $y, z \in A_0 \cup \ldots \cup A_{n-1}$ ; we have $\langle y, z, n\rangle \in$ $\in A_n$. As $\langle x, y\rangle \in \theta$, we get $\langle x \cdot \langle y, z, n\rangle, y \cdot$ $\cdot \langle y, z, n\rangle\rangle \in \theta$. If $x = \langle y, z, n\rangle$, then $\langle x \cdot \langle y, z, n\rangle, y \cdot \langle y, z, n\rangle\rangle = \langle y, z\rangle$. If $x \neq \langle y, z, n\rangle$, then $\langle x \cdot \langle y, z, n\rangle, y \cdot \langle y, z, n\rangle\rangle = \langle x, z\rangle$. In both cases we get a contradiction.

Corollary. Every groupoid $\mathbb{A}$ is a subgroupoid of a simple groupoid $\mathbb{B}$. If $A$ is infinite, we may demand $\mathrm{Card}\, A = \mathrm{Card}\, B$.

Lemma 4. Let $\mathbb{A} \in \mathcal{G}'$. Then $\Phi(\mathbb{A})$ is a simple groupoid without idempotents. If $\alpha \in F(A)$, then $\alpha \cdot ((\alpha \cdot \alpha) \cdot \alpha) = \alpha$ if and only if $\alpha \notin A_0$.

Proof. It is sufficient to prove $\alpha \cdot ((\alpha \cdot \alpha) \cdot \alpha) =$ $= \alpha$ for all $\alpha \in F(A) - A_0$. We have

$$\langle x, y, n\rangle \cdot ((\langle x, y, n\rangle \cdot \langle x, y, n\rangle) \cdot \langle x, y, n\rangle) =$$

$$= \langle x, y, n\rangle \cdot (x \cdot \langle x, y, n\rangle) = \langle x, y, n\rangle \cdot y = \langle x, y, n\rangle.$$

Lemma 5. Let $\mathbb{A}, \mathbb{B} \in \mathcal{G}'$ and let $\psi$ be a monomorphism of $\Phi(\mathbb{A})$ into $\Phi(\mathbb{B})$. Then there exists a monomorphism $\varphi$ of $\mathbb{A}$ into $\mathbb{B}$ such that $\psi = F(\varphi)$.

Proof. If $x \in F(A)$ ($x \in F(B)$, resp.), then $x\cdot((x\cdot x)\cdot x) = x$ if and only if $x \notin A_0$ ($x \notin B_0$, resp.). From this it follows that the monomorphism $\psi$ maps $A_0$ into $B_0$ and $F(A) - A_0$ into $F(B) - B_0$. Define a mapping $\varphi$ of $A$ into $B$ by $\psi(\langle a, 0\rangle) = \langle \varphi(a), 0\rangle$. This $\varphi$ is evidently a monomorphism of $\mathbb{A}$ into $\mathbb{B}$. It is sufficient to prove that $\psi$ coincides with $F(\varphi)$ on $A_n$, for all $n$. This is evident for $n = 0$. Let $n > 0$ and let the assertion hold for all natural numbers smaller than $n$. Let $\langle x, y, n\rangle \in A_n$. Evidently, $\langle x, y, n\rangle$ is the only element $\alpha$ of $F(A)$ with the following properties: $\alpha\cdot((\alpha\cdot\alpha)\cdot\alpha) = \alpha$; $x\cdot\alpha = y$; $x \neq \alpha$. Similarly, $\langle\psi(x), \psi(y), n\rangle$ is the only element $\beta$ of $F(B)$ with the following properties: $\beta\cdot((\beta\cdot\beta)\cdot\beta) = \beta$; $\psi(x)\cdot\beta = \psi(y)$; $\psi(x) \neq \beta$. As $\psi$ is a monomorphism, $\psi(\langle x, y, n\rangle)$ has the three properties of $\beta$ and we get $\psi(\langle x, y, n\rangle) = \langle\psi(x), \psi(y), n\rangle = \langle F(\varphi)(x), F(\varphi)(y), n\rangle = F(\varphi)(\langle x, y, n\rangle)$.

Theorem. $\mu(\mathcal{G})$ is isomorphic to a full subcategory of $\mu(\mathcal{G}_2) = \mathfrak{h}(\mathcal{G}_2)$.

Proof. By Lemma 1, $\mu(\mathcal{G})$ is isomorphic to a full subcategory of $\mu(\mathcal{G}')$. By Lemmas 2,3,4 and 5, $\mu(\mathcal{G}')$ is isomorphic to a full subcategory of $\mu(\mathcal{G}_2) = \mathfrak{h}(\mathcal{G}_2)$.

## References

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(Oblatum 10.11.1969)