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A COMBINATORIAL THEOREM ON THE EXISTENCE OF A SEPARATING ELEMENT AND ITS APPLICATIONS TO SEQUENCES AND σ-DERIVATIONS OF MEASURES x)

Pavel ČIHÁK, Praha

In the present paper we start with a combinatorial theorem on the existence of a separating element of a σ-complete Boolean algebra for a set of measures. We obtain immediately its applications to double sequences and limits of integrals. In order to obtain more general results, we introduce the notion of a σ-derivation of a measure and a σ-derivation of a set of measures. The considered interpretation of the σ-derivation of a measure in the Stone space is not used in the following main section. The purpose of this paper is to obtain some results which are essential generalizations of well-known theorems (Vitalli, Hahn, Saks and Nikodym) on sequences of measures and to present a combinatorial treatment.

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x) The theses of this paper have been communicated by the author in January 1967 on the topological seminar in Prague, directed by acad.Prof. M. Katětov.

Contents:

1. <u>A combinatorial theorem and its applications</u>

Let $\mathcal{O}\!\mathcal{L}$ be a Boolean algebra (see [3]) with operations $\cup, \cap, -$. Let $a(\mathcal{O}\!\mathcal{L})$ be a set of all finite non-negative additive functions defined on $\mathcal{O}\!\mathcal{L}$. Elements of the set $a(\mathcal{O}\!\mathcal{L})$ will be called measures.

Let w be a filter of the Boolean algebra $\mathcal{O}\!\mathcal{L}$.

Define $\check{m}(w) = \inf\{m(A); A \in w\}$ for $m \in a(\mathcal{O}\!\mathcal{L})$,

$\check{M}(w) = \inf\{\sup M(A); A \in w\}$ for $M \subset a(\mathcal{O}\!\mathcal{L}), M \neq \emptyset$,

where $M(A) = \{m(A); m \in M\}$ and $\hat{M}(w) = \sup\{\check{m}(A); m \in M\}$.

Then it follows immediately:

(1.1) <u>Lemma</u>. 1° $\check{M}(w) \geq \hat{M}(w) \geq 0$ for all $M \subset a(\mathcal{O}\!\mathcal{L})$, $M \neq \emptyset$.

2° If M is a finite subset of $a(\mathcal{O}\!\mathcal{L})$ then $\check{M}(w) = \hat{M}(w)$.

3° If $\check{M}(w) > \hat{M}(w)$, M_0 is a finite subset of M, $M_1 = M - M_0$ then $\check{M}_1(w) = \check{M}(w)$.

<u>Proof.</u> 1° If $A \in w$ then $\sup M(A) \geq \check{m}(w) \geq 0$ for all $m \in M$. Hence $\check{M}(w) \geq \hat{M}(w) \geq 0$.

2° Let $\varepsilon > 0$, $M = \{m_1, m_2, \dots, m_n\}$. Then there are elements $A_k \in w$ such that $m_k(A_k) \leq \hat{M}(w) + \varepsilon$. Put

$A = \bigcap_{k=1}^{n} A_k$. Then $A \in w$, $\check{M}(w) \leq \sup M(A) \leq \hat{M}(w) + \varepsilon$.

Hence $\check{M}(w) \leq \hat{M}(w)$. By 1° $\check{M}(w) = \hat{M}(w)$.

3° Clearly $\check{M}_1(w) \leq \check{M}(w)$. If $\check{M}_1(w) < \check{M}(w)$ then there is $A \in w$ such that $\sup M_1(A) < \check{M}(w)$. But M_0 is a finite set, w is a filter, hence there is $A_0 \in w$, $A_0 \subset A$ such that $\sup M_0(A_0) < \check{M}(w)$. We obtain $\check{M}(w) \leq \sup M(A_0) < \check{M}(w)$, i.e. a contradiction.

(1.2) An element $E \in \mathcal{O}\!\mathcal{l}$ is called to be <u>separating</u> for an infinite set M of measures iff there are two infinite subsets M^1 and M^2 of the set M such that $\inf M^1(E) > \sup M^2(E)$.

(1.3) <u>Theorem</u>. Let $\mathcal{O}\!\mathcal{l}$ be a σ-complete Boolean algebra. Let w be a filter of $\mathcal{O}\!\mathcal{l}$. Suppose M is a nonvoid subset of the set $a(\mathcal{O}\!\mathcal{l})$ and $\infty > \check{M}(w) > 2\hat{M}(w)$ (in particular, $\infty > \check{M}(w) > 0 = \hat{M}(w)$). Then there exists a separating element $E \in \mathcal{O}\!\mathcal{l}$ for the set M of a form $E = \bigcup_{n=1}^{\infty}(A_{2n-1} - A_{2n})$; $A_n \in w$, $A_n \supset A_{n+1}$ for $n = 1, 2, \dots$.

<u>Proof.</u> Put $d = \check{M}(w)$, $d_0 = \hat{M}(w)$, $\varepsilon = \frac{1}{6}(d - 2d_0)$. Hence $\varepsilon > 0$. There is an element $A_1 \in w$ such that $\sup M(A_1) \leq d + \varepsilon$. Put $M_1 = M$. There is a measure $m_1 \in M_1$ such that $m_1(A_1) \geq d - \varepsilon$. Since $\check{m}_1(w) \leq d_0$, there is $A_2 \in w$, $A_2 \subset A_1$ such that $m_1(A_2) \leq d_0 + \varepsilon$.

Put $M_2 = M_1 \dot{-} (m_1)$. It follows from (1.1) that $\check{M}_2(w) = \check{M}_1(w) = d$, etc.

Putting $M_n = M_{n-1} \dot{-} (m_{n-1})$, we obtain

that $\check{M}_n(w) = d$. Hence there is $m_n \in M_n$ such that $m_n(A_n) \geq d - \varepsilon$. Since $\check{m}_n(w) \leq d_0$, there is $A_{n+1} \in w$ such that $A_{n+1} \subset A_n$, $m_n(A_{n+1}) < d_0 + \varepsilon$.
We obtain two infinite sequences:

$$A_1 \supset A_2 \supset \ldots, \quad A_n \in w \qquad \text{for } n = 1, 2, \ldots \text{ and}$$
$$m_n \in M, \quad m_n \neq m_{n'} \qquad \text{for } n \neq n'.$$

Now, put $E_n = A_{2n-1} - A_{2n}$ for $n = 1, 2, \ldots$,

$E = \bigcup_{n=1}^{\infty} E_n$, $M^1 = \{m_{2n-1}\}_{n=1}^{\infty}$, $M^2 = \{m_{2n}\}_{n=1}^{\infty}$. We obtain the following inequalities:

$$m_{2n-1}(E) \geq m_{2n-1}(E_n) = m_{2n-1}(A_{2n-1}) - m_{2n-1}(A_{2n}) \geq$$
$$\geq d - \varepsilon - d_0 - \varepsilon = d - d_0 - 2\varepsilon, \quad E \subset (A_1 - A_{2n}) \cup A_{2n+1},$$
$$m_{2n}(E) \leq m_{2n}(A_1) - m_{2n}(A_{2n}) + m_{2n}(A_{2n+1}) \leq d + \varepsilon - d + \varepsilon +$$
$$+ d_0 + \varepsilon = d_0 + 3\varepsilon.$$

Hence $\inf M^1(w) - \sup M^2(w) \geq d - 2d_0 - 5\varepsilon = \varepsilon > 0$.

Now we intend to show how the theorem (1.3) on the existence of a separating element E can be applied to a double sequence lemma and to a limit of integrals lemma.

(1.4) <u>Lemma.</u> Let $(e_{jk};\ j = 1, 2, \ldots,\ k = 1, 2, \ldots)$ be a double sequence of nonnegative numbers. Suppose

$$\sup \{\sum_{k=1}^{\infty} e_{jk};\ j = 1, 2, \ldots\} < \infty,$$

$$\lim_{j \to \infty} e_{jk} \quad \text{exists for } k = 1, 2, \ldots .$$

Then

$$\lim_{j \to \infty} \sum_{k=1}^{\infty} e_{jk} \geq \sum_{k=1}^{\infty} \lim_{j \to \infty} e_{jk}.$$

Moreover, if $\lim_{j \to \infty} \sum_{k \in K} e_{jk}$ exists for each subset K of the set N of all positive integers, then

$$\lim_{j\to\infty}\sum_{k=1}^{\infty} e_{jk} = \sum_{k=1}^{\infty}\lim_{j\to\infty} e_{jk} .$$

Proof. 1. Clearly $\lim_{j\to\infty}\sum_{k=1}^{\infty} e_{jk} \geq \lim_{j\to\infty}\sum_{k=1}^{n} e_{jk} = \sum_{k=1}^{n}\lim_{j\to\infty} e_{jk}$ for all $n \in N$. Hence the first inequality holds.

2. The second statement follows from the theorem (1.3) if we put $\mathcal{U} = \exp N$, w the Fréchet filter of $\exp N$, $M = \{m_j \in a(\mathcal{U});\ m_j(A) = \sum\{e_{jk};\ k \in A\}$ for each $A \in \exp N$, $j \in N\}$. We have $\hat{M}(w) = 0$, $\lim_{j\to\infty} m_j(K)$ exists for each $K \in \exp N$ and $\check{M}(w) \leq < \infty$. If $\check{M}(w) > 0$ then there is a separating element E, which is impossible. Hence $\check{M}(w) = 0$. Let $\varepsilon > 0$. Then there is an element $A \in w$ such that $\sup M(A) \leq \varepsilon$. There is a number $n \in N$ such that $A \supset \{n+1, n+2, \ldots\}$. Hence

$$\lim_{j\to\infty}\sum_{k=1}^{\infty} e_{jk} - \sum_{k=1}^{\infty}\lim_{j\to\infty} e_{jk} = \lim_{j\to\infty}\sum_{k=n+1}^{\infty} e_{jk} - \sum_{k=n+1}^{\infty}\lim_{j\to\infty} e_{jk} \leq \varepsilon ,$$

$$\lim_{j\to\infty}\sum_{k=1}^{\infty} e_{jk} = \sum_{k=1}^{\infty}\lim_{j\to\infty} e_{jk} .$$

(1.5) Lemma. Let λ be a nonnegative finite σ-additive measure on a σ-complet field $\mathcal{X}$ of subsets of a nonvoid set X. Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of nonnegative λ-integrable functions on X such that

$f_n \longrightarrow 0$ λ-almost everywhere on X and

$\lambda(f_n)$ converges to a positive number γ for $n \longrightarrow \infty$.

Then there exists a separating element $E \in \mathcal{X}$ for $M = \{f_n \cdot \lambda\}_{n=1}^{\infty}$ so that the sequence $\lambda(f_n \cdot \chi_E)$ does not converge for $n \longrightarrow \infty$, where χ_E is the characteristic function of the set E .

<u>Proof</u>. Let ε be a positive number, $\varepsilon\lambda(X) < \gamma$. Put $B_k = \{x \in X;\, f_n(x) \leq \varepsilon$ for all $n \geq k\}$, $A_k = X \dot{-} B_k$ for $k \in N$. Clearly, $A_k \supset A_{k+1}$ $\lambda(A_k) \to \lambda(\bigcap_{k=1}^{\infty} A_k) = 0$, $\lambda(f_n \cdot \chi_{B_k}) \leq \varepsilon \cdot \lambda(B_k) \leq \varepsilon \cdot \lambda(X)$ for $n \geq k$, $\lambda(f_n \cdot \chi_{A_k}) = \lambda(f_n) - \lambda(f_n \cdot \chi_{B_k}) \geq \lambda(f_n) - \varepsilon\lambda(X)$ for $n \geq k$.

Put $m_n = f_n \cdot \lambda$, $M = \{m_n\}_{n=1}^{\infty}$, $w = \{A_k\}_{k=1}^{\infty}$. Then $M \subset a(\mathcal{M})$, $m_n(A_k) \geq \lambda(f_n) - \varepsilon \cdot \lambda(X)$ for $n \geq k$, $\sup M(A_k) \geq \gamma - \varepsilon \cdot \lambda(X)$ for all $k \in N$, $\infty > \check{M}(w) \geq \gamma - \varepsilon \cdot \lambda(X) > 0$, $\check{m}_n(w) = 0$ for all $n \in N$, $\hat{M}(w) = 0$. By the theorem (1.3) there exists a separating element $E \in \mathfrak{X}$ for M . Hence $\lambda(f_n \cdot \chi_E) = m_n(E)$ does not converge for $n \to \infty$.

(1.6) Note. If, moreover, X is a topological space, λ is a Borel measure and f_n are continuous functions, then A_k are open sets and theorem (1.3) implies that E is of a form $E = \bigcup_{k=1}^{\infty} (A'_{2k-1} - A'_{2k})$ where A'_k is a subsequence of the sequence A_k .

(1.7) <u>Examples</u>. Let λ be the Lebesgue's measure on the interval $X = \langle 0, 1 \rangle$.

1° Let $f_n = n \cdot \chi_{\langle 0, \frac{1}{n} \rangle}$ for each $n \in N$. Then $f_n \to 0$ $\lambda - a.e$ and $\lambda(f_n) = 1$ for each $n \in N$. By the lemma (1.5) there exists a sequence $\{a_k\}_{k=1}^{\infty}$ of numbers of the interval $(0, 1\rangle$ which converges monotonic to zero, such that the set

$E = \bigcup_{k=1}^{\infty} (a_{2k}, a_{2k-1}\rangle$ is a separating element for $M = \{f_n \cdot \lambda\}_{n=1}^{\infty}$. Hence the sequence $\lambda(f_n \cdot \chi_E)$ does not converge for $n \to \infty$.

2^0 Let $f_n(x) = 2 \cdot \sqrt{\frac{n}{\pi}}\ e^{-nx^2}$ for each $n \in N$. Then $f_n \to 0$ $\lambda - a \cdot e$, $\lambda(f_n) \to 1$ for $n \to \infty$. Hence by (1.5) there exists a separating element E of the previous form.

(1.8) Corollary. Suppose λ is a nonnegative finite σ-additive measure on a σ-complete field of subsets of a nonvoid set X, $\{f_n\}_{n=1}^{\infty}$ is a sequence of nonnegative λ-integrable functions on X such that

$f_n \to 0$ λ- almost everywhere on X and

$\lambda(f_n \cdot \chi_E)$ converges to a finite number for $n \to \infty$ for each $E \in \mathcal{X}$.

Then $\lim_{n\to\infty} \lambda(f_n) = 0$.

2. σ-derivation of measures

Let $\mathcal{O}\!\mathcal{L}$ be a Boolean algebra. Let 0 be a zero element of $\mathcal{O}\!\mathcal{L}$. Let W_0 be a set of all filters (all bases for all filters) such that each element $w \in W_0$ has a countable basis and

$$\bigcap\{A;\ A \in w\} = 0 .$$

Put $W_0 | E = \{w \in W_0;\ E \in w\}$ for each $E \in \mathcal{O}\!\mathcal{L}$. Clearly $W_0 | E_1 \subset W_0 | E_2 \subset W_0$ for E_1, $E_2 \in \mathcal{O}\!\mathcal{L}$, $E_1 \subset E_2$.

(2.1) Definition. 1^0 Let $m \in a(\mathcal{O}\!\mathcal{L})$. Put $\partial m(E) = \sup\{\check{m}(w); w \in W_0 | E\}$ for all $E \in \mathcal{O}\!\mathcal{L}$ and $|\partial m| = \partial m(-0)$.

Then the nonnegative function ∂m defined on $\mathcal{O}$ will be called the σ-derivation of the measure m.

2^o Let M be a nonvoid subset of the set $a(\mathcal{O})$. Put $\partial M(E) = \sup\{\check{M}(w); w \in W_0 \mid E\}$ for all $E \in \mathcal{O}$ and $|\partial M| = \partial M(-0)$.

Then the nonnegative function ∂M defined on $\mathcal{O}$ will be called the σ-derivation of the set M of measures.

If we prove that the σ-derivation ∂m is an element of $a(\mathcal{O})$ then we can define the second σ-derivation $\partial^2 m$ of the measure m:

$$\partial^2 m = \partial(\partial m) .$$

(2.2) <u>Theorem.</u> Let $m \in a(\mathcal{O})$. Then

1^o $0 \leq \partial m \leq m$,

2^o $\partial m \in a(\mathcal{O})$,

3^o m is a σ-additive function on the subalgebra $\mathcal{O} \mid E$ if and only if $\partial m(E) = 0$ for $E \in \mathcal{O}$,

4^o $\partial^2 m = \partial m$.

<u>Proof</u>. 1^o If $w \in W_0 \mid E$ then $E \in w$, $0 \leq \check{m}(w) \leq m(E)$ for $E \in \mathcal{O}$.

2^o Suppose $E_1, E_2 \in \mathcal{O}$, $E_1 \cap E_2 = 0$. If $w_i \in W_0 \mid E_i$ for $i = 1, 2$ then $w = \{A_1 \cup A_2; A_i \in w_i, i = 1, 2\} \in W_0 \mid E_1 \cup E_2$ and $\check{m}(w_1) + \check{m}(w_2) \leq \check{m}(w) \leq \partial m(E_1 \cup E_2)$. Hence

$$\partial m(E_1) + \partial m(E_2) \leq \partial m(E_1 \cup E_2) .$$

If $w \in W_0 \mid E_1 \cup E_2$ then $w_i = w \mid E_i \in W_0 \mid E_i$ for $i = 1, 2$

and $\partial m(E_1)+\partial m(E_2) \geq \check{m}(w_1)+\check{m}(w_2) \geq \check{m}(w)$.

Hence

$$\partial m(E_1)+\partial m(E_2) \geq \partial m(E_1 \cup E_2) .$$

3^o If the measure m is σ-additive on $\mathcal{O}\!\mathcal{l}\,|\,E$ then $\check{m}(w)=0$ for all $w \in W_0\,|\,E$. Hence $\partial m(E)=0$.

On the other hand, let $\partial m(E)=0$ and let $\{E_k\}_{k=1}^{\infty}$ be a disjoint family of elements of $\mathcal{O}\!\mathcal{l}\,|\,E$, such that $\bigcup_{k=1}^{\infty} E_k$ exists in $\mathcal{O}\!\mathcal{l}\,|\,E$. Then $w=\{\bigcup_{k=n+1}^{\infty} E_k\}_{n=1}^{\infty} \in$ $\in W_0\,|\,E$ and

$0=\partial m(E)=\check{m}(w)=\lim_{n\to\infty} m(\bigcup_{k=n+1}^{\infty} E_k)=m(\bigcup_{k=1}^{\infty} E_k)-\sum_{k=1}^{\infty} m(E_k)$.

4^o Put $\eta=\partial m$. It is sufficient to show that $\check{\eta}(w) \geq \check{m}(w)$ for all $w \in W_0$, since $\partial\eta \leq \partial m$ by 1^o. If $A \in w \in W_0$ then $\eta(A)=\partial m(A)=\sup\{\check{m}(w_0);\, w_0 \in W_0\,|\,A\}$. But $w \in W_0\,|\,A$. Hence $\partial m(A) \geq \check{m}(w)$,

$\check{\eta}(w)=\inf\{\partial m(A);\ A \in w\} \geq \check{m}(w),\ \partial\eta \geq \partial m$.

(2.3) <u>Lemma.</u> Let $m \in a(\mathcal{O}\!\mathcal{l})$. Then $m-\partial m \in a(\mathcal{O}\!\mathcal{l})$, $m-\partial m$ is a σ-additive measure having the following property:

if $\lambda \in a(\mathcal{O}\!\mathcal{l}),\ \lambda \leq m,\ \partial\lambda=0$ then $\lambda \leq m-\partial m$.

<u>Proof</u>. $\partial(m-\partial m)=\partial m-\partial^2 m=0$ by (2.2, 4^o), hence $m-\partial m$ is a σ-additive element of $a(\mathcal{O}\!\mathcal{l})$ by (2.2,1^o) and (2.2,3^o).

If $E \in \mathcal{O}\!\mathcal{l}$ then $\check{\lambda}(w)=0,\ \lambda(E)=\lambda(E)-\check{\lambda}(w)=$ $=\sup\{\lambda(E-A);\ A \in w\} \leq \sup\{m(E-A);\ A \in w\}=m(E)-\check{m}(w)$

for all $w \in W_0\,|\,E$. Hence $\lambda(E) \leq m(E)-\sup\{\check{m}(w);$

$w \in W_o \mid E\} = m(E) - \partial m(E)$, $\lambda \leq m - \partial m$.

(2.4) Theorem. Let M be a nonvoid subset of $a(\mathcal{U})$. Then 1° $o \leq \partial M \leq \sup M$,

2° $\partial M(E_1) + \partial M(E_2) \geq \partial M(E_1 \cup E_2)$ for $E_1, E_2 \in \mathcal{U}$.

3° The measures of M are evenly σ-additive on $\mathcal{U} \mid E$ if and only if $\partial M(E) = o$ for $E \in \mathcal{U}$.

Proof. 1° If $A \in w \in W_o \mid E$ then $o \leq \sup M(A) \leq \leq \sup M(E)$. Hence

$$o \leq \check{M}(w) \leq \sup M(E), \quad \partial M(E) \leq \sup M(E) .$$

2° Let $w \in W_o \mid E_1 \cup E_2$. Put $w_i = w \mid E_i$ for $i = 1, 2$.

If $A_1 \in w_1$, $A_2 \in w_2$ then there is $A \in w$ such that $A_i \supset A \cap E_i$ for $i = 1, 2$. Hence

$$\sup M(A_1) + \sup M(A_2) \geq \sup M(A \cap E_1) + \sup M(A \cap E_2) \geq \sup M(A) \geq \check{M}(w) ,$$

$$\check{M}(w_1) + \check{M}(w_2) \geq \check{M}(w) ,$$

$$\partial M(E_1) + \partial M(E_2) \geq \partial M(E_1 \cup E_2) .$$

3° Suppose $E \in \mathcal{U}$, $\partial M(E) = o$ and $\{E_k\}_{k=1}^{\infty}$ is a disjoint subfamily of $\mathcal{U} \mid E$. Put $A_n = \bigcup\{E_k ; k = n, n+1, \dots\}$, $w = \{A_n\}_{n=1}^{\infty}$. Then $w \in W_o \mid E$, $\check{M}(w) = o$. Hence if $\varepsilon > 0$ then there is a number $n \in N$ such that $\sup M(A_n) \leq \varepsilon$, i.e.

$$m(\bigcup\{E_k ; k \geq 1\}) - \sum_{k=1}^{n-1} m(E_k) = m(\bigcup\{E_k ; k \geq n\}) = m(A_n) \leq \varepsilon$$

for all $m \in M$. Hence the measures of M are evenly σ-additive.

Conversely, suppose the measures of M are evenly σ-additive on $\mathcal{A} \mid E$ and $w \in W_0 \mid E$. Put $E_n = A_n - A_{n+1}$, where $\{A_n\}_{n=1}^{\infty}$ is a countable monotone basis for w.

Then $\{E_k\}_{k=1}^{\infty}$ is a disjoint subfamily of $\mathcal{A} \mid E$, $\bigcup_{k=1}^{\infty} E_k = A_1$. If $\varepsilon > 0$ then there is a number $n \in N$ such that

$m(\bigcup\{E_k; k \geq 1\}) - \sum_{k=1}^{n-1} m(E_k) \leq \varepsilon$ for all $m \in M$. Hence

$m(A_n) = m(A_1) - \sum_{k=1}^{n-1} (m(A_{k+1}) \leq \varepsilon$ for all $m \in M$,

$\check{M}(w) = 0$, $\partial M(E) = 0$.

3. Interpretation of σ-derivations in the Stone space

Let X be the Stone space of a Boolean algebra $\mathcal{A}$. Let h be the natural isomorphism on $\mathcal{A}$ into $\exp X$. Put $\mathcal{A}' = \{A' = h(A); A \in \mathcal{A}\}$.

(3.1) Lemma. Let $m \in a(\mathcal{A})$. Put $m'(A') = m(A)$ for all $A' = h(A), A \in \mathcal{A}$. Then $m' \in a(\mathcal{A}')$ and $|\partial m'| = 0$. Hence the measure m' is σ-additive on the field $\mathcal{A}'$.

Proof. Let w' be a filter of $\mathcal{A}'$ which has a countable basis and void intersection in the Stone space X. Then $\emptyset \in w'$, $\check{m}'(w') = 0$ since each element of w' is compact. Hence $|\partial m'| = 0$.

(3.2) Let $\mathfrak{X}$ be a σ-field of all Borel subsets of the Stone space X. Let $m \in a(\mathcal{A})$. Then the measure m'

has one and only one σ-additive extension on $\mathfrak{X}$ (by Hahn's theorem [1]).

(3.3) Theorem. Let h' be a map from W_o to $\mathfrak{X}$ such that $h'(w)=\bigcap\{h(A); A\in w\}$ for each $w\in W_o$.

Let $\mathcal{D}=\{D=h'(w); w\in W_o\}$. Then

1° $\partial m(E)=\sup\{m'(D); D\in\mathcal{D}, D\subset h(E)\}$ for $E\in\mathcal{A}$.

2° Each set $D\in\mathcal{D}$ is closed and nowhere dense in X.

Proof. 1° $w\in W_o\,|\,E$ if and only if $D=h'(w)\subset \subset h(E)$ for $E\in\mathcal{A}$. Hence
$\check{m}(w)=\inf\{m(A); A\in w\}=\inf\{m'(A'); A'=h(A), A\in w\}=m'(D)$
(by (3.2)).

2° If $A'_o\in\mathcal{A}'$, $A'_o\subset D=h'(w)$, $w\in W_o$ then $A_o=h'^{-1}(A'_o)\subset\bigcap w$. Hence $A_o=0$, $A'_o=\emptyset$ and D is nowhere dense.

(3.4) Example. Let $\mathcal{A}=\exp N$, let βN be the Čech-Stone compactification of N, let $N^*=\beta N\dot{-}N$ and let $m\in a(\mathcal{A})$. Then $\partial m(E)=m'(E'\cap N^*)$ for each $E'=h(E)$, $E\in\mathcal{A}$.

Proof. Let w_o be the Fréchet filter of $\mathcal{A}$. Since each filter $w\in W_o\,|\,E$ minorizes the filter $w_o\,|\,E$, we get $\check{m}(w)\le\check{m}(w_o\,|\,E)$.

Hence $\partial m(E)=\check{m}(w_o\,|\,E)=m'(h'(w_o\,|\,E))=m'(E'\cap N^*)$.

Now, we want to obtain some analogous results in a more general case.

(3.5) Theorem. Let $\mathcal{O}\!\mathcal{C}$ be an atomic Boolean algebra. Let $m \in a(\mathcal{O}\!\mathcal{C})$. Then there is a meager subset Y of X, $Y \subset \bigcup \mathcal{D}$ such that $\partial m(E) = m'(E' \cap Y)$ for each $E' = h(E)$, $E \in \mathcal{O}\!\mathcal{C}$.

Proof. Since $\mathcal{O}\!\mathcal{C}$ is atomic, $D' \cup D'' \in \mathcal{D}$ for D', $D'' \in \mathcal{D}$. Hence there is a sequence $\{D_n\}$ of elements of $\mathcal{D}$ such that $D_1 \subset D_2 \subset \dots$, $|\partial m| - \frac{1}{n} \leq m'(D_n)$ for each $n \in N$. Since the measure m' is σ-additive, we get $|\partial m| = m'(Y)$, where $Y = \bigcup_{n=1}^{\infty} D_n$. From (3.3) follows that the set Y is meager in X.

Now let $E \in \mathcal{O}\!\mathcal{C}$. Clearly the number $\varepsilon = \partial m(E) - m'(E' \cap Y)$ is positive or zero. If $\varepsilon > 0$ then there is an element $D_0 \in \mathcal{D}$ such that $\partial m(E) - \varepsilon < m'(D_0)$, $D_0 \subset E'$. It follows that

$$m'(D_0 \dot{-} Y) \geq m'(D_0) - m'(E' \cap Y) > 0 \text{ . Hence}$$

$$|\partial m| \geq \lim_{n \to \infty} m'(D_n \cup D_0) = m'(Y \cup D_0) = m'(Y) + m'(D_0 \dot{-} Y) > m'(Y) = |\partial m| ,$$

which is a contradiction. Hence

$$\partial m(E) = m'(E' \cap Y) .$$

(3.6) Theorem. Let $\mathcal{O}\!\mathcal{C}$ be a σ-complete Boolean algebra and let $m \in a(\mathcal{O}\!\mathcal{C})$. Then there exists a sequence of elements $Z_k \in \mathcal{O}\!\mathcal{C}$ such that

$$Z_k \supset Z_{k+1} , \ \partial m(Z_k) = |\partial m| , \ m(Z_k) - \partial m(Z_k) < \frac{1}{2^k}$$

for all $k \in N$. If $Y = \bigcap_{k=1}^{\infty} h(Z_k)$ then Y is a closed G_σ subset of the Stone space X and $\partial m(E) = m'(E' \cap Y)$ for each $E \in \mathcal{O}\!\mathcal{C}$, $E' = h(E)$.

Proof. Let $\{w_n\}_{n=1}^{\infty}$ be a sequence such that $w_n \in$ $\in W_0$, $|\partial m| - \frac{1}{n} < \check{m}(w_n)$. If $A_n \in w_n$, $Z = \bigcup_{n=1}^{\infty} A_n$ then $|\partial m| = \partial m(Z)$, $\partial m(-Z) = 0$. Indeed, let $u \in W_0 | -Z$. Then $u_n = \{B \cup A;\ B \in u,\ A \in w_n\} \in W_0$ for all n and $\check{m}(u_n) = \check{m}(u) + \check{m}(w_n)$, $0 \le \check{m}(u) = \check{m}(u_n) -$ $- \check{m}(w_n) \le |\partial m| + \frac{1}{n} - |\partial m| = \frac{1}{n}$. Hence

$$\check{m}(u) = 0,\ \partial m(-Z) = 0,\ \partial m(Z) = |\partial m| - \partial m(-Z).$$

Now, choose $A_{n,k} \in w_n$ such that $\{A_{n,k}\}_{k=1}^{\infty}$ is a monotone basis for w_n and $m(A_{n,k}) \le \check{m}(w_n) + \frac{1}{2^{n+k}}$ for all $n, k \in N$.

Put $Z_k = \bigcup_{n=1}^{\infty} A_{n,k}$, $E_{n,k} = A_{n,k} - A_{n,k+1}$ for all $n, k \in N$. Then

$$\partial m(-Z_k) = 0, \sum_{i=k}^{\infty} m(E_{n,i}) = \sum_{i=k}^{\infty} (m(A_{n,i}) - m(A_{n,i+1})) =$$

$$= m(A_{n,k}) - \check{m}(w_n) \le \frac{1}{2^{n+k}}, \quad Z_k = \bigcup_{n=1}^{\infty} \bigcup_{i=k}^{\infty} E_{n,i},$$

$$\sum_{n=1}^{\infty} \sum_{i=k}^{\infty} m(E_{n,k}) \le \sum_{n=1}^{\infty} \frac{1}{2^{n+k}} = \frac{1}{2^k}.$$

Hence $0 \le m(Z_k) - \partial m(Z_k) \le \sum_{n=1}^{\infty} \sum_{i=k}^{\infty} m(E_{n,i}) \le \frac{1}{2^k}$ for all $k \in N$.

Now, let $E \in \mathcal{O}\!\mathcal{l}$, $E' = h(E)$. By Theorem (2.2) $m - \partial m \in a(\mathcal{O}\!\mathcal{l})$. Hence $0 \le m(E \cap Z_k) - \partial m(E \cap Z_k) \le \frac{1}{2^k}$ for each $k \in N$, $0 \le m'(E' \cap h(Z_k)) - \partial m(E) \le \frac{1}{2^k}$ for each k. Using σ-additivity of m' we obtain an equality: $m'(E' \cap Y) = \partial m(E)$, $m'(Y) = |\partial m|$.

The set Y is an intersection of a countable family of clopen sets, hence closed and G_δ

4. Sequences of measures

In this section we present applications of the theorem on existence of a separating element to sequences of measures,

(4.1) Definition. Let $\mathfrak{A}$ be a Boolean algebra. A sequence $\{m_n\}_{n=1}^{\infty}$ of measures $m_n \in a(\mathfrak{A})$ converges to a measure $\mu \in a(\mathfrak{A})$ iff $\lim_{n\to\infty} m_n(A) = \mu(A)$ for each $A \in \mathfrak{A}$. (This property will be denoted by the symbol $m_n \to \mu$.)

(4.2) Lemma. Let $m_n \to \mu$, let $M = \{m_n; n \in N\}$ and let w be a filter of $\mathfrak{A}$. Then

1° $\check{\mu}(w) \leq \check{M}(w)$.

2° If $\hat{M}(w) \leq \check{\mu}(w)$ then $\check{\mu}(w) = \check{M}(w)$.

3° If the Boolean algebra $\mathfrak{A}$ is σ-complete then $\check{M}(w) \leq 2 \cdot \hat{M}(w)$ and $\check{\mu}(w) \leq 2 \cdot \overline{\lim_{n\to\infty}} \check{m}(w)$.

Proof. 1° If $\varepsilon > 0$ then there is $A \in w$ such that $m_n(A) \leq \check{M}(w) + \varepsilon$ for all $n \in N$. Hence $\check{\mu}(w) \leq \mu(A) = \lim_{n\to\infty} m_n(A) \leq \check{M}(w) + \varepsilon$, $\check{\mu}(w) \leq \check{M}(w)$.

2° If $\varepsilon > 0$ then there is $A \in w$ such that $\mu(A) < \check{\mu}(w) + \varepsilon$. Since $m_n(A) \to \mu(A)$ there is $n_0 \in N$ such that $m_n(A) \leq \check{\mu}(w) + \varepsilon$ for all $n > n_0$. From the inequality $\hat{M}(w) \leq \check{\mu}(w)$ it follows that there is $A_1 \in w$, $A_1 \subset A$ such that $m_n(A_1) \leq \check{\mu}(w) + \varepsilon$ for $n = 1, 2, \ldots, n_0$. Hence

$$\check{M}(w) \leq \sup M(A_1) \leq \check{\mu}(w) + \varepsilon, \quad \check{M}(w) \leq \check{\mu}(w) .$$

3° If $2 \cdot \hat{M}(w) < \check{M}(w)$ then by Theorem (1.3) there exists a separating element E in the σ-complete Boolean algebra $\mathcal{O}\!\mathcal{l}$. But $m_n(E) \to p(E)$, which is a contradiction. Hence $\check{M}(w) \leq 2 \cdot \hat{M}(w)$. Put $M_n = \{m_i ; i \geq n\}$ for $n \in N$. Clearly, $\check{M}_n(w) \leq 2 \cdot \hat{M}_n(w)$, hence $\check{p}(w) \leq$ $\leq \check{M}_n(w) \leq 2 \cdot \hat{M}_n(w) = 2 \cdot \sup_{i \geq n} \check{m}_i(w)$ for all $n \in N$,

$$\check{p}(w) \leq 2 \cdot \overline{\lim_{n \to \infty}} \check{m}_n(w) .$$

(4.3) Lemma. Let $\mathcal{O}\!\mathcal{l}$ be a σ-complete Boolean algebra, let M be a sequentially compact nonvoid subset of $a(\mathcal{O}\!\mathcal{l})$ and let w be a filter of $\mathcal{O}\!\mathcal{l}$ with a countable basis. Then $\check{M}(w) \leq 2 \cdot \hat{M}(w)$.

Proof. If there is a number d such that $2 \cdot \hat{M}(w) < d <$ $< \check{M}(w)$ and if $\{A_k\}_{k=1}^{\infty}$ is a monotone basis for the filter w then the set $M_k = \{m \in M ; m(A_k) > d\}$ is infinite for each $k \in N$. Indeed, if M_k is finite set then $\check{M}_k(w) = \hat{M}_k(w) \leq 2 \hat{M}_k(w)$, hence there is a set $A \in w$, $A \subset A_k$ such that $\sup M_k(A) \leq d$. Then $\check{M}(w) \leq$ $\leq \sup M(A) \leq d$, which is a contradiction.

Now choose $m_k \in M_k$. If $M_0 = \{m_{k_i}\}_{i=1}^{\infty}$ is a subsequence of the sequence $\{m_k\}_{k=1}^{\infty}$ then $\sup M_0(A_k) \geq d$ for each $k \in N$. Hence $\check{M}_0(w) \geq d > 2 \cdot \hat{M}(w) \geq 2 \cdot \hat{M}_0(w)$,

$\check{M}_0(w) > 2 \cdot \hat{M}_0(w)$. From Theorem (1.3) it follows that there is a separating element in $\mathcal{O}\!\mathcal{l}$ for the set M_0. Hence the subsequence $\{m_{k_i}\}_{i=1}^{\infty}$ does not converge and the set M is not sequentially compact, which is a contradiction.

(4.4) Theorem. Let $\mathcal{O}\!\mathcal{l}$ be a σ-complete Boolean algebra. Let L be a nonvoid subset of $a(\mathcal{O}\!\mathcal{l})$ such that $|\partial L| = 0$ and let M be a sequentially compact subset of $a(\mathcal{O}\!\mathcal{l})$ (in particular, let M be a sequence which converges). Suppose $\check{m}(w) = 0$ for each $m \in M$ and for each filter w of $\mathcal{O}\!\mathcal{t}$ such that $\check{L}(w) = 0$.

Then for each $\varepsilon > 0$ there is $\sigma > 0$ such that if $E \in \mathcal{O}\!\mathcal{t}$, $\lambda(E) \leq \sigma$ for all $\lambda \in L$ then $m(E) \leq \varepsilon$ for all $m \in M$.

Proof. If there is a number $\varepsilon > 0$ and elements $E_k \in \mathcal{O}\!\mathcal{t}$ such that $\sup L(E_k) \leq \frac{1}{2^k}$ and $\sup M(E_k) \geq \varepsilon$, then $w = \{A_k\}_{k=1}^{\infty} \in W_0$, where $A_k = \bigcup_{i=k}^{\infty} E_i$, and $0 = |\partial L| \geq \check{L}(w) \geq 0$. Hence $\hat{M}(w) = 0$. From lemma (4.3) it follows that $\check{M}(w) \leq 2 \cdot \hat{M}(w)$, i.e. $\check{M}(w) = 0$. Hence there is a number $k \in N$ such that $\varepsilon > \sup M(A_k) \geq \sup M(E_k)$, which is a contradiction.

(4.4) Note. Theorem (4.4) is a generalization of a theorem due to Vitali, Hahn and Saks (see [1]) which is a particular case of (4.4) for $L = (\lambda)$, where λ is a σ-additive measure.

(4.5) Theorem. Let $\mathcal{O}\!\mathcal{t}$ be a σ-complete Boolean algebra. Let M be a nonvoid sequentially compact subset of $a(\mathcal{O}\!\mathcal{t})$ and let $E \in \mathcal{O}\!\mathcal{t}$. Suppose $\partial m(E) = 0$ for each $m \in M$. Then

$$\partial M(E) = 0 .$$

Proof. From lemma (4.3) it follows that $\check{M}(w) \leq 2 \cdot \hat{M}(w)$ for each $w \in W_0 | E$. But $\hat{M}(w) = \sup\{\check{m}(w); m \in M\} \leq \sup\{\partial m(E); m \in M\} = 0$ for each $w \in W_0 | E$. Hence $\partial M(E) = 0$.

(4.6) Note. Using (4.4), (2.2) and (2.4) we obtain the following statement:

If M is a sequentially compact subset of $a(\mathcal{O})$ such that each measure $m \in M$ is σ-additive then the measures of M are evenly σ-additive.

(4.7) Theorem. Let $\mathcal{O}$ be a σ-complete Boolean algebra, let $M = \{m_n\}_{n=1}^{\infty}$, $m_n \in a(\mathcal{O})$ and let $m_n \to p$. Then

$$\partial M(E) \leq 2 \cdot \sup\{\partial m(E); m \in M\} \quad \text{and}$$

$$\partial p(E) \leq 2 \cdot \overline{\lim_{n \to \infty}} \, \partial m_n(E) \quad \text{for each } E \in \mathcal{O} .$$

Proof. 1° From lemma (4.2) it follows that $\check{M}(w) \leq 2 \cdot \hat{M}(w) \leq 2 \cdot \sup\{\partial m(E); m \in M\}$ for each $w \in W_0 | E$. Hence $\partial M(E) \leq 2 \cdot \sup\{\partial m(E); m \in M\}$.

2° From (4.2) it follows that $\check{p}(w) \leq 2 \cdot \overline{\lim_{n \to \infty}} \check{m}_n(w) \leq 2 \cdot \overline{\lim_{n \to \infty}} \partial m_n(E)$, $w \in W_0 | E$. Hence $\partial p(E) \leq 2 \cdot \overline{\lim_{n \to \infty}} \partial m_n(E)$ for $E \in \mathcal{O}$.

(4.8) Note. Theorem (4.7) is a generalization of a theorem due to Nikodym O.M., which is a particular case of (4.7), when all measures m_n, $n \in N$, are σ-additive, i.e. $|\partial m| = 0$ for all $n \in N$. Then from (4.7) it follows that p and $\{m_n\}_{n=1}^{\infty}$ are evenly σ-additive measures.

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