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REMARKS ON NONLINEAR OPERATORS AND FUNCTIONALS

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Introduction. The purpose of this note is to establishsome conditions under which a mapping F or a subadditive functional f is additive, linear or positive homogeneous on a linear space X. A typical result (Theorem 4) is as follows. Let f be a subadditive functional on X. Assume there exist a neighborhood $V(0)$ of 0 and a functional g defined on $V(0)$ so that $g(0)=0$ and $f(u) \leq g(u)$ for each $u \in V(0)$. If g possesses a linear Gâteaux differential $Dg(0,h)$ at 0, then f is linear on X. Theorem 7 deals with the boundedness property of even subadditive functionals, while Theorem 8 concerns the uniform boundedness of the Gâteaux derivative $f'(u)$ of a convex subadditive functional.

§ 1. Terminology and notations. Let X, Y be real linear normed spaces, X^* dual of X, E_1 1-dimensional Euclidean space. A mapping $F: X \longrightarrow Y$ is said to be

(a) additive on X if $F(u_1+u_2) = F(u_1)+F(u_2)$ for every $u_1, u_2 \in X$.

(b) homogeneous (positively homogeneous) if $F(tu) = t\,F(u)$ for every $t \in E_1$ (for each $t \geq 0$) and every $u \in X$.

(c) linear if F is additive and homogeneous

(d) bounded if for each bounded subset $M \subset X$ $F(M)$ is bounded in Y.

A functional f defined on X is called

(1) subadditive if $f(u_1+u_2) \leqq f(u_1)+f(u_2)$ for every $u_1, u_2 \in X$;

(2) convex on a convex set $M \subset X$ if

$$f(tu+(1-t)v) \leqq tf(u)+(1-t)f(v)$$

for each $t \in \langle 0,1 \rangle$ and each $u, v \in M$.

(3) odd (even) on X if $f(-u)=-f(u)$ $(f(-u)=f(u))$ for every $u \in X$.

We shall say that a mapping $F: X \rightarrow Y$ possesses the Baire property in the set $M \subset X$ of the second category in X if there exists a subset $N \subset M$ of the 1st category in M such that the restriction $F/_{M-N}$ of F to $M-N$ is continuous. A set $N \subset X$ is said to be a Baire set in X if there exists an open set $G \subset X$ so that $G-N$, $N-G$ are both the sets of the 1st category in X. For Gâteaux differentials and Gâteaux derivative we use the notions and notations given in the Vajnberg's book [1,chapt.I]. By the one-sided Gâteaux differential $V_+ f(u_0, h)$ of a convex functional f at $u_0 \in X$ we mean the limit

$$\lim_{t \to 0_+} \frac{1}{t}[f(u_0+th)-f(u_0)] = V_+ f(u_0,h), \quad h \in X.$$

If f is convex and finite on X, the one-sided Gâteaux differential $V_+ f(u,h)$ exists for every $u, h \in X$ and it is subadditive positive homogeneous functional in $h \in X$ for every (but fixed) $u \in X$ [2,chapt.10]. Therefore

$$f(u+th)-f(u) = V_+ f(u,th)+\omega_+(u_0,th), \quad u, h \in X,$$

where
$$\lim_{t \to 0_+} \frac{\omega_+(u,th)}{t} = 0.$$

§ 2. We start with the following

Theorem 1. Let X, Y be linear normed spaces, $F: X \longrightarrow Y$ so that $F(tu) = tF(u)$ for every $u \in X$ and $t \in (0, t_0)$, where $t_0 < 1$. Under this assumption the following assertions are valid:

(a) F is positively homogeneous on X.

(b) If F possesses a linear Gâteaux differential $DF(0,h)$ at 0, then F is linear on X.

(c) If F has a Gâteaux derivative $F'(0)$ at 0, then F is linear and continuous on X.

(d) If F has a linear Gâteaux differential $DF(u,h)$ on the segment $(0, t_0 v_0) = \{u \in X : u = tv_0, 0 < t < t_0\}$ for some $0 \neq v_0 \in X$ and $\lim_{t \to 0_+} \| DF(tv_0, v_0) \| = 0$, $\lim_{t \to 0_+} \frac{1}{t} \| \omega(tv_0, th) \| = 0$ for an arbitrary (but fixed) $h \in X$, then F is linear on X.

Proof. (a) By our hypothesis there exists

$$\lim_{t \to 0_+} \frac{F(0+th)-F(0)}{t} \overset{def}{=} V_+F(0,h) = \lim_{t \to 0_+} \frac{F(th)}{t} = F(h)$$

for every $h \in X$. As $V_+ F(0, h)$ is positively homogeneous in $h \in X$, $F(h)$ has the same property.

(b) is a strengthening of Th.1 [3]. It can be proved more simply as follows: $V_+ F(0,h)$ exists and $V_+ F(0,h) = F(h), h \in X$. Since F has $DF(0,h)$ at 0, then $V_+ F(0,h) = DF(0,h) = F(h)$, $h \in X$ and hence F must be linear in $h \in X$.

(c) is clear. (d) is a slight generalization of Th.2 [3].

Theorem 2. Let $F: X \rightarrow Y$ be a mapping of X into Y so that $F(0) = 0$. Assume F possesses the Gâteaux differential $VF(0,h)$ at 0. Then F is homogeneous on $X \Longleftrightarrow$ the remainder $\omega(0,h)$ of

$VF(0,h)$ is homogeneous on X. Moreover, let F possess a linear Gâteaux differential $DF(0,h)$ at 0. Then its remainder $\omega(0,h)$ is homogeneous in $h \in X \Longleftrightarrow$ $\Longleftrightarrow F$ is linear on X.

Proof. Since $F(0)=0$ and F possesses the Gâteaux differential $VF(0,h)$ at 0, we have that

$$F(tu) = VF(0,tu) + \omega(0,tu),$$

$$tF(u) = tVF(0,u) + t\cdot\omega(0,u)$$

for every $u \in X$ and $t \in E_1$. Being $VF(0,u)$ homogeneous in $u \in X$,

$$F(tu) - tF(u) = \omega(0,tu) - t\omega(0,u).$$

Hence $F(tu) = tF(u)$, $u \in X$, $t \in E_1 \Longleftrightarrow \omega(0,tu) = t\omega(0,u)$, $u \in X$, $t \in E_1$. The second assertion follows at once from the first part of Th.2 and from the results (a),(b) of Th.1. Theorem is proved.

Theorem 3. Let X be a linear normed space, f a convex finite functional on X. Under this assumption the following assertions are valid:

(a) If $f(tu) = tf(u)$ for every $u \in X$ and each $t \in (0, t_0)$, where $t_0 < 1$, then f is subadditive and positive homogeneous on X. Moreover, if f possesses the Gâteaux differential $Vf(0,h)$ at 0, then f is linear on X.

(b) If $f(0)=0$ and $\omega_+(0,h)$ is subadditive in $h \in X$, then f is subadditive on X.

(c) If $f(0)=0$, then f is positive homogeneous on $X \Longleftrightarrow \omega_+(0,h)$ is positively homogeneous on X.

(d) If f is continuous subadditive functional on X

and $f(0)=0$, then $f(tu) \leqq tf(u)$ for every $u \in X$ and each $t \geqq 0$.

<u>Proof.</u> (a) Being f convex, $V_+ f(u, h)$ (for fixed $u \in X$) is subadditive and positive homogeneous on X. As $f(0)=0$, we have for $u, v \in X$, $t \in (0, t_0)$ that

$$f(t(u+v)) = V_+ f(0, t(u+v)) + \omega_+(0, t(u+v)),$$

$$(1) \quad f(tu) = V_+ f(0, tu) + \omega_+(0, tu),$$

$$f(tv) = V_+ f(0, tv) + \omega_+(0, tv).$$

Since f is convex, $\omega_+(0, h) \geqq 0$ for every $h \in X$ (Lemma 2 [4]). In view of subadditivity of $V_+ f(u, h)$ and our hypothesis

$$f(u+v) - f(u) - f(v) \leqq \frac{1}{t}\omega_+(0, t(u+v)) -$$

$$-\frac{1}{t}\omega_+(0, tu) - \frac{1}{t}\omega_+(0, tv) \leqq \frac{1}{t}\omega_+(0, t(u+v))$$

for every $u, v \in X$, $t \in (0, t_0)$. As

$\frac{1}{t}\omega_+(0, t(u+v)) \longrightarrow 0$ whenever $t \to 0_+$,

$f(u+v) \leqq f(u) + f(v)$ for every $u, v \in X$

The second assertion of a) is an immediate consequence of The.1(a),(b) and Remark 2 [3].

(b) To prove (b) write (1) without t on the left and right sides and use the property that $V_+ f(0, h)$, $\omega_+(0, h)$ are both subadditive on X

(c) The one-sided differential $V_+ f(0, h)$ of f at 0 is positively homogeneous on X. Hence the assertion (c) is a consequence of Theorem 2.

(d) Convexity of f implies ($u \in X$, $t \in \langle 0, 1 \rangle$) that

$$f(tu)=f(tu+(1-t)0) \leqq tf(u)+(1-t)f(0)=tf(u).$$

Hence $f(tu) \leqq t f(u)$ for each $u \in X$ and $t \in \langle 0,1 \rangle$. Let $r = \frac{m}{n}$ be a rational number (m, n are positive integers). In view of subadditivity of f and the last inequality we have that

$$f(\tfrac{m}{n}u) \leqq m f(\tfrac{1}{n}u) \leqq \tfrac{m}{n} f(u).$$

Let t be a positive irrational number. Then there exists a sequence of rational numbers $r_n > 0$ so that $r_n \to t$. Continuity of f gives

$$f(tu)=\lim_{n\to\infty} f(r_n u) \leqq \lim_{n\to\infty} r_n f(u) = tf(u),$$

which proves c). This concludes the proof of our theorem.

Remark 1. The assertion (a) of Th.3 one may prove simpler using the properties of $V_+ f(0,h)$. But we gave preference to the given proof (a) because the proof of the assertion (b) is based on the same arguments as (a).

Theorem 4. Let f be a subadditive functional on X. Assume there exist a neighborhood $V(0)$ of 0 and a functional g defined on $V(0)$ so that $g(0)=0$ and $f(u) \leqq g(u)$ for each $u \in V(0)$. If g possesses a linear Gâteaux differential $Dg(0,h)$ at 0, then f if linear on X.

Proof. Subadditivity of f implies that $f(0) \leqq 2f(0)$. Hence $f(0) \geqq 0$. But $0 \leqq f(0) \leqq g(0) = 0 \Rightarrow f(0)=0$. Suppose $u \in X$, $h \in X$, $t > 0$. Then

$$f(u) = f(u+h-h) \leqq f(u+h) + f(-h)$$

and subadditivity of f implies that

(2) $\quad -f(-h) \leqq f(u+h) - f(u) \leqq f(h)$.

For sufficiently small $t > 0$, $th \in V(0) \Longrightarrow$ $\Rightarrow f(th) \leqq g(th)$. Replace in (2) th for h and divide it by $t > 0$, we have for $t > 0$ small enough that

$$-\frac{g(-th)}{t} \leqq -\frac{f(-th)}{t} \leqq \frac{f(u+th)-f(u)}{t} \leqq \frac{f(th)}{t} \leqq \frac{g(th)}{t} .$$

As $g(th) = Dg(0,th) + \omega(0,th)$, we obtain that

$$Dg(0,h) - \frac{1}{t}\,\omega(0,t(-h)) \leqq \frac{1}{t}\,[f(u+th)-f(u)] \leqq$$

(3)

$$\leqq Dg(0,h) + \frac{1}{t}\,\omega(0,th) .$$

Since the limits on the left and the right side of (3) exist and are equal to $Dg(0,h)$, we conclude that

$$\lim_{t\to 0_+} \frac{1}{t}\,[f(u+th)-f(u)] = V_+ f(u,h) = Dg(0,h)$$

for each $h \in X$ and $u \in X$. Hence $V_- f(u,h) = -V_+ f(u,-h) = Dg(0,h)$ for every $u \in X$ and $h \in X$. Therefore f possesses the Gâteaux differential $Vf(u,h)$ for every $u \in X$ and $Vf(u,h) = Dg(0,h)$ for every $u \in X$, $h \in X$. As $f(0) = 0$, by the mean-value theorem

$f(u) = Vf(\tau u, u) = Dg(0,u)$, $u \in X$, $0 < \tau < 1$.

Since $Dg(0,u)$ is linear on X , our theorem is proved.

<u>Remark 2</u>. The final part of the proof of Th.4 may be done as follows: From (3) it follows that

$$\left|\frac{f(u+th)-f(u)}{t}-Dg(0,h)\right|\leq \max\left(\left|\frac{\omega(0,th)}{t}\right|,\left|\frac{\omega(0,-th)}{t}\right|\right).$$

As $t \to 0$, the term on the right side tends to 0. Hence f possesses a Gâteaux differential $Vf(u,h)$ at every point $u \in X$ and $Vf(u,h) = Dg(0,h)$, $h \in X$, $u \in X$. By the mean-value theorem $f(u) = Vf(\tau u, u) = Dg(0,u)$, $u \in X$.

Corollary 1. Let f be a subadditive functional on X so that $f(0) = 0$. Assume f possesses a linear Gâteaux differential $Df(0,h)$ at 0. Then f is linear on X.

Corollary 2. Let f be a convex finite functional on X. Assume there exist a neighborhood $V(0)$ of $0 \in X$ and a functional g defined on $V(0)$ so that $g(0) = 0$ and $V_+ f(u_0, h) \leq g(h)$ for each $h \in V(0)$ and for some $u_0 \in X$. If g has a linear Gâteaux differential $Dg(0,h)$ at 0, then f possesses a linear Gâteaux differential $Df(u_0,h)$ at u_0. Moreover, if f is continuous, then f possesses the Gâteaux derivative $f'(u_0)$ at u_0.

Proof. The one-sided Gâteaux differential $V_+ f(u_0,h)$ is subadditive functional on X. From Theorem 4 it follows that $V_+ f(u_0,h)$ is linear in $h \in X$ on X. Hence $V_+ f(u_0,h) = V_- f(u_0,h)$ for every $h \in X$. This shows that f has the Gâteaux differential $Vf(u_0,h)$ at u_0. But convexity of f implies that $Vf(u_0,h) = Df(u_0,h)$, $h \in X$ (Remark 2 [3]). If f is continuous, using Proposition

6 [5], we obtain that $Df(u_0, h) = f'(u_0)h$, $h \in X$. This completes the proof.

Corollary 3. Let f be a continuous subadditive functional on X. If $f(tu) \leqq \varphi(t)f(u)$ for every $u \in X$ and $t \in (0, t_0)$, where $t_0 < 1$, φ is a real function on $(0, t_0)$ so that $\lim\limits_{t \to 0_+} \frac{\varphi(t)}{t} = 0$, then $f(u) = 0$ for every $u \in X$.

Proof. First of all, f has the Gâteaux derivative $f'(u)$ on X and $f'(u) = 0$ for every $u \in X$. By Theorem 8.6.1 [6],Chapt.VIII (here we must point out that this theorem is valid even for mappings which have the Gâteaux derivative only) we get that $f(u) = c = const.$ for every $u \in X$. Since $f(0) \geqq 0$, $c \geqq 0$. Suppose that $c > 0$. Then we have $c = f(tu) \leqq \varphi(t)f(u) = c\,\varphi(t)$ for every $u \in X$ and each $t \in (0, t_0)$. Hence $\frac{1}{t} \leqq \frac{\varphi(t)}{t}$ for each $t \in (0, t_0)$ which contradicts with the fact that $\lim\limits_{t \to 0_+} \frac{1}{t}\varphi(t) = 0$. Therefore $c = 0$ and $f(u) = 0$ for every $u \in X$. This completes the proof. Corollaries 1,3 show that functionals considered inTheorem 1 [7], Th.1 [8], Th. 4 [3] are linear.

Theorem 5. Let f be an odd subadditive functional on X. Suppose f is continuous at 0. Then f is linear and continuous on X.

Proof. The inequality $2f(0) \geqq f(0)$ implies that $f(0) \geqq 0$. On the other hand we have for $u \in X$ that $0 \leqq f(0) = f(u - u) \leqq f(u) +$

$+f(-u) = f(u) - f(u) = 0$. Thus $f(0) = 0$ and f being continuous at 0, it is continuous on X . For arbitrary $u, v \in X$ we obtain

$$f(v) = f((u+v)-u) \leqq f(u+v) + f(-u) =$$
$$= f(u+v) - f(u) \leqq f(v) .$$

Hence $f(v) \leqq f(u+v) - f(u) \leqq f(v)$ implies the equalities among these terms. This means that f is additive on X and being continuous on X , f is linear on X . This completes the proof.

Theorem 6. Let f be a subadditive functional on X having a linear Gâteaux differential $Df(u_0, h)$ at some point $u_0 \in X$. If $f(-u_0) = -f(u_0)$, then f is linear on X .

Proof. From $0 \leqq f(0) = f(u_0 - u_0) \leqq f(u_0) + +f(-u_0) = 0$ it follows that $f(0) = 0$. If $u_0 = 0$, then f is linear by Corollary 1. Suppose that $u_0 \neq 0$ and that $h \in X$ is an arbitrary element of X . From $f(u_0) = f(u_0 - h + h) \leqq f(u_0 - h) + f(h)$ and using our hypothesis we have that

$$f(u_0) - f(u_0 - h) \leqq f(h) = f((u_0 + h) - u_0) \leqq$$
$$\leqq f(u_0 + h) + f(-u_0) = f(u_0 + h) - f(u_0) .$$

Consider $t > 0$, replace in these inequalities h by th and then divide by $t > 0$, we get that

$$\frac{1}{t}[f(u_0) - f(u_0 - th)] \leqq \frac{1}{t} f(th) \leqq$$
$$\leqq \frac{1}{t}[f(u_0 + th) - f(u_0)] .$$

Since f possesses a linear Gâteaux differential $Df(u_0, h)$ at u_0, we obtain $(t > 0)$

$$Df(u_0, h) - \frac{\omega(u_0, t(-h))}{t} \leqq \frac{f(th)}{t} \leqq$$
$$\leqq Df(u_0, h) + \frac{\omega(u_0, th)}{t} .$$

These inequalities imply that there exists

$\lim\limits_{t \to 0_+} \frac{f(th)}{t}$ and that this limit is equal to $Df(u_0, h)$ for every $h \in X$. From this fact we conclude that f possesses a linear Gâteaux differential $Df(u, h)$ on X and that $Df(u, h) = Df(u_0, h)$ for every $u, h \in X$. According to the mean-value theorem $f(u) = Df(\tau u, u) = Df(u_0, h)$, $u \in X$, $(0 < \tau < 1)$ which proves our theorem.

Corollary 4. Let f be subadditive functional on X. If f is linear on some open subset $M \neq 0$ of X, then f is linear on X.

Theorem 7. Let X be a linear normed space of the 2nd category in itself, f a subadditive functional on X. Let one of the following three conditions be fulfilled: (a) f is even and upper-bounded on a Baire subset of the 2nd category in X; (b) f is nonnegative on X and it is upper-bounded on a symmetric Baire subset of the 2nd category in X; (c) f is even and there exist an open subset $M \neq 0$ of X, a functional g defined on M so that g possesses a Baire property in M and $f(u) \leqq g(u)$ for each $u \in M$.

Then f is bounded in X.

Proof. Assume (a). Then $0 \leqq f(0) = f(u - u) \leqq$ $\leqq f(u) + f(-u) = 2f(u)$, $u \in X \Longrightarrow f(u) \geqq 0$ for every $u \in X$. By our hypothesis there exist a Baire subset B of the 2nd category in X and a constant $C > 0$ so that $f(u) \leqq C$ for each $u \in B$. Then the set W of all differences $w = u - v$, where $u, v \in B$ is a neighborhood of 0 in X. Hence there exists $\delta_0 > 0$ such that $\|w\| < \delta_0 \Rightarrow$ $\Rightarrow w \in W$. For any $w \in W$ with $\|w\| < \delta_0$ we have $(w = u - v,\ u, v \in B)$ $0 \leqq f(w) =$ $= f(u-v) \leqq f(u) + f(v) \leqq 2C$. Let u be an arbitrary point of the ball $\|u\| \leqq R$. Then there exists an integer n_0 so that $R n_0^{-1} \leqq \delta_0$. We obtain that $0 \leqq f(u) = f(\frac{u}{n_0} \cdot n_0) \leqq n_0 f(\frac{u}{n_0}) \leqq 2Cn_0$.

This shows that f is bounded in X. The proof of (b) is similar to that of (a).

Assuming (c) we see that M is a set of the 2nd category in X. By our hypothesis there exists $u_0 \in$ $\in M - A$, where A is a set of the 1st category in M, so that the restriction $g/_{M-A}$ of g to $M - A$ is continuous at u_0. Hence there exists a non-empty open subset $N \subset M$ so that $u_0 \in N$ and $u \in N - A \Rightarrow g(u) \leqq g(u_0) + 1$. The set $B =$ $= N - A$ is a Baire set of the 2nd category in X. Hence $u \in B \Rightarrow f(u) \leqq g(u_0) + 1$. The rest results at once from (a) of our theorem. Theorem is proved.

Corollary 5. Let X be a linear normed space of the second category in itself, f a subadditive even functional on X. If f is upper semicontinuous at some point $u_0 \in X$, then f is bounded in X.

In the sequel we shall use the so-called Banach-Steinhaus uniform-boundedness principle: Let X, X_1 be linear normed spaces, A be a set of the 2nd category in X, $\mathfrak{M}$ a set of linear continuous operators of X into X_1. If $x \in A \Longrightarrow \sup_{U \in \mathfrak{M}} \| U(x) \| < \infty$, then $\sup_{U \in \mathfrak{M}} \| U \| < \infty$.

We prove the following

<u>Theorem 8</u>. Let X be a linear normed space of the second category in itself, f a convex continuous subadditive and finite functional on X. Assume f possesses the Gâteaux differential $Vf(u,h)$ on the set $N \subset X$, $N \neq \emptyset$.

Then there exists a constant $C > 0$ so that $\| f'(u) \| \leqq C$ for each $u \in N$, where $f'(u)$ denotes the Gâteaux derivative $f'(u)$ of f at u. In particular, if N is convex, then f is Lipschitzian on N with constant C.

<u>Proof</u>. If $0 \in N$, then f is additive on X according to Theorem 3 a). Being f continuous it is homogeneous and hence linear on X. Therefore our conclusions are trivially fulfilled.

Assume that $0 \notin N$. By Proposition 6 [5] $Vf(u,h) = f'(u)h$ for each $u \in N$ and every $h \in X$. Using lemma 2 [4] and subadditivity of f we obtain

$$-f(-h) \leqq f(u) - f(u-h) \leqq f'(u)h \leqq f(u+h) - f(u) \leqq f(h)$$

for each $u \in N$ and $h \in X$. Hence

$$|f'(u)h| \leqq \max(|f(h)|, |f(-h)|)$$

for each $u \in N$ and $h \in X$. By theorem 2.5.3 [9] $|f(h)| \leqq M_f(\|h\| + 1)$ for every $h \in X$, where $0 \leqq M_f = \sup_{\|h\| \leqq 1} f(h) < +\infty$. Hence $h \in X \Longrightarrow$ $\longrightarrow \sup_{u \in N} |f'(u)h| \leqq M_f(\|h\|+1)$. **According to Banach-**Steinhaus principle there exists a constant $C > 0$ so that $\sup_{u \in N} \|f'(u)\| \leqq C$. Thus the first part of out theorem is proved. To prove the second assertion it is sufficient to use the above fact and the mean-value theorem. This concludes the proof.

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