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In this paper we provide answers to four questions raised by the author in [1].

(Q_1) Does every symmetric generalized uniform space have an open base? No. (Q_2) Does there exist a symmetric generalized uniform space which has a convergent filter which is not Cauchy? Yes. (Q_3) Is there a proximity class of symmetric generalized uniformities with more than two totally bounded elements? Yes. (Q_4) Does there exist a totally bounded uniformity which is not p-correct or p-correct of degree n ? Yes.

Let R denote the reals, let I denote the positive integers. Let $\Delta = \{(x, y) \mid x = y \in R\}$. Let $\Delta^* = \{(x, y) \mid x = -y \in R\}$. Let $I_n = [-1/n, 1/n]$ for each $n \in I$. Let $B_n = ((I_n \times I_n) - \Delta^*) \cup \Delta$ for each $n \in I$. Let $\mathcal{B} = \{B_n \mid n \in I\}$.

<u>Theorem</u>. $\mathcal{B}$ is a base for a symmetric generalized uniformity $\mathcal{U}$ on R that has the following properties:

(1) For every U, V in $\mathcal{U}$ $(U \cap V) \in \mathcal{U}$.

(2) $(0, 0) \notin B_n^\circ$ for every $n \in I$; so

that $\mathcal{U}$ does not have an open base, and $\mathcal{U}$ is not p-correct.

(3) $(B_m \circ B_n) \cap ((R \times R) - B_m) \neq \emptyset$ for every m, n in I.

(4) The neighborhood system of 0 is a convergent filter in $(R, \mathcal{J}(\mathcal{U}))$ that is not Cauchy with respect to $\mathcal{U}$.

(5) $(R, \mathcal{U})$ is complete.

(6) $\mathcal{J}(\mathcal{U})$ is not compact; so that $\mathcal{U}$ is not totally bounded.

Proof. The proof that $\mathcal{B}$ is a base for a symmetric generalized uniformity on R is straightforward, for suppose $b \in B_n[A] \cap B$. If $b \neq 0$, then there exists $m \in I$ such that $B_m[b] = b$ (choose m such that $1/m < |b|$). If $b = 0$ then there exists $a_n \in A \cap [-1/n, 1/n]$ such that $(a_n, 0) \in B_n$. If $a_n = 0$, then $B_n[0] \subset B_n[A]$. If $a_n \neq 0$, then for any $m \in I$ such that $1/m < |a_n|$ we have that $B_m[0] \subset B_n[A]$.

Proof 1. $(B_n \cap B_m) = B_m$ if $m \geq n$.

Proof 2. $(O_1 \times O_2) \cap (\Delta^* - (0,0)) \neq \emptyset$ for every O_1, O_2 in $\mathcal{N}(0)$, the neighborhood system of 0.

Proof 3. $(B_n \circ B_n) \cap (\Delta^* - (0,0)) \neq \emptyset$ for every $n \in I$.

Proof 4. Same as proof of 2.

Proof 5. Let $\mathcal{F}$ be a weakly Cauchy filter in $(R, \mathfrak{J}(\mathcal{U}))$. For every $n \in I$ there exists $x_n \in R$ such that $B_n(x_n) \in \mathcal{F}$. Suppose for some $m \in I$ $x_m \in (X - [-1/m, 1/m])$. Then $\mathcal{F} \supset \mathcal{N}(x_m)$, the neighborhood system of x_m. Suppose for every $n \in I$ we have that $-1/n \leq x_n \leq 1/n$. Then 0 is a cluster point for $\mathcal{F}$.

Proof 6. $\{x\}$ is open if $x \neq 0$, and for each $n \in I$ $(-1/n, 1/n)$ is an open neighborhood of 0.

By a suitable modification of the above construction it is possible to prove the following

Theorem. There exists a symmetric generalized uniformity $\mathcal{U}$ on R without an open base that generates the usual topology for R such that for every U, V in $\mathcal{U}$ $(U \cap V) \in \mathcal{U}$.

Theorem. There exists a totally bounded symmetric generalized uniform space that is not p-correct or p-correct of degree n for every $n \in I$.

References

[1] C.J. MOZZOCHI: Symmetric Generalized Uniform and Proximity Spaces, a publication of the Department of Mathematics, Trinity College, Hartford, Connecticut, October 1968.

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