István Juhász

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A GENERALIZATION OF TYCHONOFF'S THEOREM

I. JUHÁSZ, Budapest

A well-known theorem of Alexander says that ordinary compactness of a space R is equivalent to the following property:

R possesses an open subbase S such that any covering of R consisting of members of S has a finite subcovering. In Kelley's book [1] this fact is used to prove Tychonoff's theorem. Using this method, however, one can arrive at a rather striking generalization of Tychonoff's product theorem for a certain "subbase-modification" of almost all compactness properties. This also shows that if "Alexander's theorem holds" for one of these compactness properties, then "Tychonoff's theorem holds" for it too. I wonder whether the converse of this last statement is true.

In what follows, capital Greek letters: Γ, Λ, Π, ... will denote open coverings of topological spaces, while small Greek letters: γ, λ, $\varkappa$, ... will be used for denoting systems of open coverings. We shall write $\Gamma < \Lambda$ if Γ is a refinement of Λ, i.e. for each $G \in \Gamma$ there exists an $L \in \Lambda$ such that $G \subset L$. $\mathcal{T}$ denotes the class of all topological spaces.

<u>Definition</u>: A function K is called a compactness function, iff its domain is $\mathcal{T}$, and its values are pairs in

the form

$$K(R) = [\mathscr{K}_1(R), \mathscr{K}_2(R)] \qquad (R \in \mathscr{T}) ,$$

where $\mathscr{K}_1(R)$ and $\mathscr{K}_2(R)$ are systems of open coverings of R. (Sometimes we simply write $K = [\mathscr{K}_1, \mathscr{K}_2]$.) In what follows, we always assume that, if Γ_0 is a subcovering of $\Gamma \in \mathscr{K}_1(R)$, then $\Gamma_0 \in \mathscr{K}_1(R)$, too.

Definition: If K is a compactness function, a space $R \in \mathscr{T}$ is called K-compact, iff for any $\Gamma_1 \in \mathscr{K}(R)$ there exists a $\Gamma_2 \in \mathscr{K}_2(R)$ such that $\Gamma_2 < \Gamma_1$. (A general compactness definition actually equivalent to the above one can be found in [2].)

Let $R \in \mathscr{T}$ and

$\gamma(R)$: the system of all open coverings of R ;

$\gamma_m(R)$: the system of all open coverings Γ of R, for which $|\Gamma| < m$, where m is an arbitrary (finite or infinite) cardinal number;

$\lambda(R)$: the system of all locally finite (open) coverings of R ;

$\mu(R)$: the system of all pointwise finite coverings of R;

$\pi(R)$: the system of all star-finite coverings of R (a covering Γ is called star-finite, iff any member of Γ meets only a finite number of members of Γ.

By means of these functions $\gamma, \gamma_m, \lambda, \pi$ and μ almost all of the usual compactness properties can be formulated:

If $C = [\gamma, \gamma_{\aleph_0}]$, then C-compactness is ordinary compactness.

If $C_m = [\gamma_m, \gamma_{\aleph_0}]$ $(m \geq \aleph_0)$, then C_{m^+}-compactness is m-compactness in the sense of [3], p. 81. (Here m^+ is the smallest cardinal greater than m.)

If $C_m^n = [\gamma_m, \gamma_n]$, then C_m^n-compactness is compactness in a given interval of cardinal numbers, as it was defined by Ju.M. Smirnov in [4].

If $L = [\gamma, \lambda]$; $M = [\gamma, \mu]$ and $P = [\gamma, \pi]$, respectively, then L-, M-, and P-compactness coincide with paracompactness, metacompactness (or weak paracompactness), and strong paracompactness, respectively.

If $L_m = [\gamma_m, \lambda]$, then L_{m^+}-compactness is m-paracompactness, see e.g. [6].

Even pseudocompactness (we recall that a space R is pseudocompact iff any continuous real function on R is bounded) can be defined this way, since it is well-known (see e.g. [5], Th.11) that R is pseudocompact, iff any locally finite open covering of R has a finite subcovering, hence evidently pseudocompactness coincides with L^*-compactness, where

$$L^* = [\lambda, \gamma_{\aleph_0}] .$$

Now we are able to define the subbase modification of a compactness property, that was mentioned in the introduction.

<u>Definition</u>: Let $K = [\mathscr{H}_1, \mathscr{H}_2]$ be an arbitrary compactness function. Then a space R is called subbase K-compact, or briefly SK-compact, iff R possesses an open subbase S such that for any $\Gamma_1 \in \mathscr{H}_1(R)$ with $\Gamma_1 \subset S$ there exists a $\Gamma_2 \in \mathscr{H}_2(R)$ such that $\Gamma_2 < \Gamma_1$.

Thus Alexander's theorem can be formulated as follows: C-compactness coincides with SC-compactness.

As another example we can consider the compactness function $C^3 = [\gamma, \gamma_3]$, which is completely uninteresting in itself, but for which SC^3-compactness coincides with the supercompactness property, introduced by J. de Groot.

<u>Definition</u>: The compactness function $K = [\varkappa_1, \varkappa_2]$ will be called projective, iff the following condition is fulfilled:

If $R = \times_{\alpha \in A} R_\alpha$ is an arbitrary product of topological spaces, then for each $\alpha \in A$, $\Gamma \in \varkappa_i(R_\alpha)$ if and only if $\pi_\alpha^{-1}(\Gamma) \in \varkappa_1(R)$, $(i = 1, 2)$; here π_α denotes the canonical projection $\pi_\alpha : R \to R_\alpha$, and

$$\pi_\alpha^{-1}(\Gamma) = \{\pi_\alpha^{-1}(G) : G \in \Gamma\} .$$

<u>Proposition 1</u>: All the compactness functions defined above are projective.

<u>Proof</u>: This is obvious, if $\varkappa_1 = \gamma$ or $\varkappa_i = \gamma_m$ for some cardinal m.

If $\varkappa_i = \lambda$, and $\Gamma \in \lambda(R_\alpha)$, let $x \in R = \times_{\alpha \in A} R_\alpha$ be an arbitrary point of the product space R. Since Γ is locally finite, there exists such a neighborhood V^α of the point $\pi_\alpha(x)$, which only meets finitely many members of Γ. Then, however, $\pi_\alpha^{-1}(V^\alpha)$ is a neighborhood of X meeting finitely many elements of the covering $\pi_\alpha^{-1}(\Gamma)$ only. This shows that $\pi_\alpha^{-1}(\Gamma)$ is locally finite, indeed.

On the other hand, if $\pi_\alpha^{-1}(\Gamma)$ is locally finite, and U is such an open neighborhood of $x \in R$, which only meets finitely many members of $\pi_\alpha^{-1}(\Gamma)$, then $\pi_\alpha(\mathcal{U})$ is a neighborhood of $\pi_\alpha(x)$, which has the same property with regard to Γ. Indeed, if $\mathcal{U} \cap \pi_\alpha^{-1}(G) = \emptyset$ for some $G \in \Gamma$, then $\pi_\alpha(\mathcal{U}) \cap G = \emptyset$ obviously.

The cases $\varkappa_i = \mathcal{U}$ and $\varkappa_i = \pi$ can be handled by analogy.

<u>Proposition 2:</u> Let $K = [\varkappa_1, \varkappa_2]$ be an arbitrary projective compactness function. Then any product of SK-compact spaces is also SK-compact.

<u>Proof</u>: Let $R = \times_{\alpha \in A} R_\alpha$, where R_α is SK-compact for each $\alpha \in A$. Thus for each $\alpha \in A$ there exists an open subbase S_α for R_α such that $\Gamma_1 \in \varkappa_1(R_\alpha)$ and $\Gamma_1 \subset S_\alpha$ implies the existence of a covering $\Gamma_2 \in \varkappa_2(R_\alpha)$, for which $\Gamma_2 < \Gamma_1$.

It is easy to see that the family $S = \{\pi_\alpha^{-1}(G_\alpha) : G \in S_\alpha, \alpha \in A\}$ constitutes a subbase for the product space R. Using this subbase of R we shall show that R is SK-compact.

Indeed, let $\Gamma \in \varkappa_1(R)$ be a covering with $\Gamma \subset S$. Then any member $G \in \Gamma$ has the form $G = \pi_\alpha^{-1}(G_\alpha)$ for some $\alpha \in A$ and $G_\alpha \in S_\alpha$. If $\alpha \in A$, let

$$\mathcal{G}_\alpha = \{G_\alpha \in S_\alpha : \pi_\alpha^{-1}(G_\alpha) \in \Gamma\},$$

and

$$T_\alpha = \cup \mathcal{G}_\alpha .$$

We shall prove that there exists such an index $\alpha_0 \in A$, for which

$$T_{\alpha_0} = R_{\alpha_0} .$$

Assume, on the contrary, that $T_\alpha \neq R_\alpha$ for every $\alpha \in A$. Then we can choose an element $x_\alpha \in R_\alpha \setminus T_\alpha$

for each $\alpha \in A$. Thus a point $x \in R$ can be defined such that $\pi_\alpha(x) = x_\alpha$ for each $\alpha \in A$. But then this point x cannot belong to any member of Γ, since $x \in \pi_\alpha^{-1}(G_\alpha)$ would imply $x_\alpha \in G_\alpha \subset T_\alpha$, which is a contradiction. Thus we can really find such an index $\alpha_0 \in A$, for which $T_{\alpha_0} = R_{\alpha_0}$.

This means, however, that

$\Gamma_0 = \{\pi_{\alpha_0}^{-1}(G_{\alpha_0}) : G_{\alpha_0} \in \mathcal{G}_{\alpha_0}\} = \pi_{\alpha_0}^{-1}(\mathcal{G}_{\alpha_0})$

is a subcovering of Γ, since $\mathcal{G}_{\alpha_0}$ is a covering of R_{α_0}. But $\Gamma_0 \subset \Gamma$ implies $\Gamma_0 \in \mathcal{H}_1(R)$, hence $\mathcal{G}_{\alpha_0} \in \mathcal{H}_1(R_{\alpha_0})$, because K is projective. But

$$\mathcal{G}_{\alpha_0} \subset S_{\alpha_0}$$

holds, too, consequently there exists a covering $\Gamma_2 \in \mathcal{H}_2(R_\alpha)$, for which

$$\Gamma_2 < \mathcal{G}_{\alpha_0} .$$

But then

$$\pi_{\alpha_0}^{-1}(\Gamma_2) < \pi_{\alpha_0}^{-1}(\mathcal{G}_{\alpha_0}) = \Gamma_0 \subset \Gamma_1 ,$$

and

$$\pi_\alpha^{-1}(\Gamma_2) \in \mathcal{H}_2(R) ,$$

because K is projective, and this proves our proposition.

<u>Corollary 1</u>: If K-compactness coincides with SK-compactness, for some projective compactness function K, then any product of K-compact spaces is also K-compact. (This can also be expressed this way: If Alexander's theorem holds for such a K, then Tychonoff's theorem holds for K, too.)

<u>Corollary 2:</u> Any product of supercompact spaces is supercompact.

Problem: For what compactness functions (or properties) are Alexander's theorem and Tychonoff's theorem equivalent?

References

[1] J.L. KELLEY: General Topology, Van Nostrand, New York, 1955.

[2] P.S. ALEXANDROFF: On the Theory of Topological Spaces, ..., Uspechi Math., 15, 2(92)(1960), 25-95.

[3] H.J. KOWALSKY: Topologische Räume, Birkhauser, Basel-Stuttgart, 1961.

[4] Ju.M. SMIRNOV: On Topological Spaces ... , Izvestya, 14 (1950), 155-178.

[5] Ju.M. SMIRNOV: On the Completeness of Proximity Space, Trudy MMO 3(1954), 271-306.

[6] S.L. GULDEN: Equivalent forms of m-paracompactness, Prace Mat. (Comm. Mat.) 11(1968), 2, 265-278.

(Received