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On $\mathcal{B}$-SPACES

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This is a summary of author's paper "Separation theorem and applications to Borel sets" that will appear elsewhere. The results were included in a series of the author's lectures at the University of Bari in the spring semester of 1967-68 academic year. All spaces are assumed to be completely regular (i.e. separated and uniformizable). The notation of [1] will be used throughout.

1. Denote by N the set and the discrete space of natural numbers. The product space N^N is denoted by Σ . The space Σ is known to be homomorphic to the space of all irrational numbers on the real line. The space Σ plays an important role in the classical theory. The method of correspondences introduced in [3] allows us to preserve the prominent role of Σ from the classical separable descriptive theory in the separable theory in the class of all completely regular spaces. If $\mathcal{M}$ is a collection of sets we denote by $\mathcal{B}(\mathcal{M})$ the smallest collection $\mathcal{N} \supset \supset \mathcal{M}$ that is closed under countable unions and countable intersections, and we define $\mathcal{B}_{\alpha}(\mathcal{M})$ just replacing unions by disjoint unions. For a space P denote by zero(P) the set of all zero-sets in P , and by cozero(P) the set of

all cozero-sets in P . The Baire sets in P are the elements of $\mathcal{B}$ (zero(P)); it is easy to show that $\mathcal{B}$ (zero(P)) = $\mathcal{B}_{\sigma}$(cozero(P)) .

If $\mathcal{M}$ and $\mathcal{N}$ are collections of sets, we denote by $[\mathcal{M}] \cap [\mathcal{N}]$ the set of all $M \cap N$ with M in $\mathcal{M}$ and N in $\mathcal{N}$. If $\mathcal{M}$ is a collection of subsets of P , then compl($\mathcal{M}$) consists of the complements in P of elements of $\mathcal{M}$.

2. An usco-compact correspondence of a space P into a space Q is an upper semi-continuous correspondence (many-valued mapping) with compact values, that means, the preimages of closed sets are closed, and the values are compact. If in addition the values at distinct points are disjoint, then the correspondence is called dusco-compact. Usco-compact images of Σ are called analytic spaces, dusco-compact images of Σ are called Borelian spaces, see [3], Sections 2 and 3 . It should be remarked that Borelian spaces are called descriptive Borel by C.A.Rogers [7].

The graph of an usco-compact correspondence into a separated space is closed. The Souslin sets in a space are defined to be images of Σ under closed-graph correspondences. Thus every analytic subspace of a space P is a Souslin set in P , and analytic spaces are just the absolute Souslin sets.

The Borelian spaces are characterized as absolute $([\text{closed}] \cap [\text{Baire}])_{\sigma_d \sigma}$

where σ_d is for disjoint countable union. In the case of metrizable spaces analytic spaces are precisely classical analytic sets, and Borelian spaces are precisely classical separable absolute Borel sets. In this paper we introduce a wider generalization of classical absolute Borel sets, the so called B-spaces.

Definition 1. A correspondence of Σ into something is called a boxed correspondence if the preimages of points are boxes in Σ, i.e. product sets. A boxed usco-compact correspondence is called busco-compact. Busco-compact images of Σ are called B-spaces.

Any disjoint correspondence on Σ is boxed, and hence Borelian spaces are B-spaces.

Theorem 1. The class of all B-spaces contains all $(\text{compact})_{\sigma\delta}$ sets, and it is closed under dusco-compact correspondences. The set of all B-spaces $P \subset Q$ is closed under countable intersections and countable disjoint unions.

It should be remarked that a σ-compact space need not be Borelian (see [3], Remark following Theorem 10), whereas every σ-compact space is a B-space by Theorem 1.

3. Complete sequences. Let $\mu = \{\mathfrak{M}_n\}$ be a sequence of coverings of a space P. A μ-Cauchy filter is a filter $\mathfrak{M}$ on P such that $\mathfrak{M} \cap \mathfrak{M}_n \neq \emptyset$ for all n. Finally, μ is complete if $\bigcap cl [\mathfrak{M}] \neq \emptyset$ for each μ-Cauchy filter on P.

Theorem 2. A space P is analytic if and only if there exists a complete sequence of countable coverings of P.

It should be remarked that spaces admitting a complete sequence of countable coverings were studied in [4] under the name B-spaces. Here the term B-space is used in much more restrictive sense.

Definition 2. A B-structure on a space P is a complete sequence $\mu = \{\mathcal{M}_n\}$ of countable coverings of P such that

$$\bigcap M_n = \bigcap \{cl \bigcap \{M_k \mid k \leq n\} \mid n \in N\}$$

for any M_n in $\mathcal{M}_n$. A Borelian structure is a B-structure such that the coverings are disjoint.

Recall [3] that a space P is Borelian if and only if there exists a Borelian structure on P. In the same lines:

Theorem 3. A space P is a B-space if and only if there exists a B-structure on P.

Remark. It follows from Theorem 2 that the elements of the coverings of any B-structure are analytic. It can be proved that a space P is Borelian if and only if there exists a complete sequence of countable coverings such that the elements of the coverings are Baire sets in P.

The external characterization of Borelian spaces described in section 2, is proved by using the first Separation Theorem. For B-spaces we need a much more complicated separation theorem which will be described in the next section.

4. Separation. Given a collection $\mathcal{M}$ of sets, two sets X and Y are said to be $\mathcal{M}$-separated, or separated in $\mathcal{M}$, if there exist X_1 and Y_1 in $\mathcal{M}$ with $X \subset X_1$,

$Y \subset Y_1$, and $X_1 \cap Y_1 = \emptyset$.

Lemma 1. Let $\mathcal{M}$ be a finitely additive and finitely multiplicative collection of subsets of a set P. Assume that $\mathcal{F}$ is a finite collection of sets in P such that any two disjoint elements of

$$[\mathcal{F}] \cap [\mathrm{compl}\,(\mathcal{M})]$$

are $\mathcal{M}$-separated. Then for any $M \supset \cap \mathcal{F}$, $M \in \mathcal{M}$, there exists a family $\mathcal{M}_M = \{M_F \mid F \in \mathcal{F}\}$ ranging in $\mathcal{M}$ such that $F \subset M_F$ for each F in $\mathcal{F}$, and

$$\cap \mathcal{M}_M \subset M .$$

E.g., if F is a finite collection of compact sets in a separated space, and U is an open neighborhood of $\cap \mathcal{F}$, then there exist open neighborhoods U_F of F, $F \in \mathcal{F}$, with $\cap \{U_F \mid F \in \mathcal{F}\} \subset U$.

Lemma 2. Let $\mathcal{M}$ be a finitely additive and σ-multiplicative collection of subsets of a set P, and let $P \in \mathcal{M}$. Assume that $\mathcal{A}$ is a countable collection of sets in P such that any two disjoint elements of $[\mathcal{A}] \cap [\mathrm{compl}\,(\mathcal{M})]$ are $\mathcal{M}$-separated. If $\{M_{\mathcal{F}} \mid \mathcal{F} \subset \mathcal{A}$, $\mathcal{F}$ finite$\}$ is a family ranging in $\mathcal{M}$ such that $\cap \mathcal{F} \subset M_{\mathcal{F}}$ for each $\mathcal{F}$, then there exists a family $\{K_A \mid A \in \mathcal{A}\}$ ranging in $\mathcal{M}$ such that $K_A \supset A$ for all A in $\mathcal{A}$, and

$$\cap \{K_A \mid A \in \mathcal{F}\} \subset M_{\mathcal{F}}$$

for each finite $\mathcal{F} \subset \mathcal{A}$.

If P is a space, and if $\mathcal{A}$ is a countable collection of analytic sets in P, and if $\mathcal{M}$ is the set of all Baire sets in P, then the assumptions of Lemma 2 are fulfilled by the first separation theorem ([3] or [5]), and we get the following important

<u>Theorem 4</u>. Let $\mathcal{A}$ be a countable collection of analytic sets in a space P , and let

$$\{ B_{\mathcal{F}} \mid \mathcal{F} \subset \mathcal{A} \, , \; \mathcal{F} \text{ finite} \}$$

be a family of Baire sets in P such that $B_{\mathcal{F}} \supset \bigcap \mathcal{F}$ for each $\mathcal{F}$. Then there exists a family $\{ Z_A \mid A \in \mathcal{A} \}$ of Baire sets such that $Z_A \supset A$, and $\bigcap \{ Z_A \mid A \in \mathcal{F} \} \subset B_{\mathcal{F}}$ for each $\mathcal{F}$.

5. <u>The main results</u>

<u>Definition</u>. A space R is called quasi-classical if there exists an usco-compact correspondence of a separable metrizable space onto R . A space P is said to be quasi-classical at infinity if K - P is quasi-classical for some, and then any, compactification of P .

The class of all quasi-classical spaces is closed under usco-compact correspondences, and the class of all spaces quasi-classical at infinity is closed under proper mappings in both directions. For a metrizable space P it is equivalent: 1) P is separable; 2) P is quasi-classical; and 3) P is quasi-classical at infinity.

<u>Theorem 5</u>. If P is a B-space, and if $P \subset Q$ such that Q - P is quasi-classical then

$$(*) \qquad P \in ([\text{closed } (Q)] \cap [\text{Baire } (Q)])_{\sigma\delta} \, .$$

If P is a B-space that is quasi-classical at infinity, then (*) is true for any $Q \supset P$.

The proof is not easy, and requires Theorem 4.

<u>Theorem 6</u>. A metrizable space P is a separable absolute Borel set if and only if P is a B-space.

For the classical descriptive theory we get the following non-trivial results (easy corollaries):

<u>Theorem 7</u>. If f is a busco-compact correspondence of Σ onto a metrizable space P, then P is a separable absolute Borel sets.

<u>Theorem 8</u>. Each of the following conditions is necessary and sufficient for a metrizable space P to be a separable absolute Borel set:

(a) There exists a complete sequence of countable coverings of P such that the elements of the coverings are analytic subspaces of P.

(b) Condition (a) with disjoint coverings.

(c) There exists a B-structure on P.

(d) There exists a Borelian structure on P.

R e f e r e n c e s

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