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GRADIENT MAPS AND BOUNDEDNESS OF GÂTEAUX DIFFERENTIALS

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Introduction. There is a number of papers devoted to the study of properties of gradient mappings. The following result is essentially due to E.S. Citlanadze [1],[2](see also [3,§ 7]) : Suppose X is a reflexive Banach space with base, f a continuous functional on X which is weakly continuous in the open ball $\|x\| < R+\alpha$ $(R>0,\ \alpha>0)$ and such that f possesses the Fréchet derivative $f'(x)$ on the ball $D(\|x\| < R)$. Assume that the remainder $\omega(x,h)$ of $f'(x)$ (i.e. $\omega(x,h) = f(x+h) - f(x) - f'(x)h$) is uniform on $D(\|x\| < R)$. Then the gradient map $F(x) = f'(x)$ is compact on $D(\|x\| < R)$. In [1,2] there are also established the sufficient conditions under which a gradient mapping is strongly continuous on $D(\|x\| < R)$. These results have been extended by M.I. Kadec [4] to separable reflexive spaces X without assuming of the existence of the base of X and by V.J. Anosov [5] to nonreflexive spaces which satisfy a certain restrictive condition. Another results in these topics have been obtained by E.H. Rothe [6],[7],[8]. According to Rotho [8] a Banach space X is said to have the property (P) if there exists a sequence $\{\psi_i^*\}$ of linearly independent elements ψ_i^* of X^* (X^* is dual of X) and a number $M > 0$ with the following property: closed linear span of $\{\psi_i^*\}$ is X^* and for each positive n there exists a linear projection of norm at most M on the intersection $\bigcap_{i=1}^{n} N_i$, where $N_i = \{x \in X : \psi_i^*(x) = 0\}$. The main result of [7],[8] is as follows: Let X be a Banach space with property (P), f a functional defined on a convex subset $V \subset X$. Assume f possesses a continuous Fréchet derivative $f'(x)$ in V. Then the following condition is necessary and sufficient that a gradient map $F(x) = f'(x)$ be completely continuous in V: For each $\eta > 0$ there exist functionals $e_i^* \in X^*$, $i = 1,2,\dots,N$, such that

$$|f(x+h)| - f(x)| < \eta \|h\|, \quad x \in V,\ x+h \in V$$

for all $h \in X$ which satisfy the inequalities

$$|e_i^*(h)| < \frac{\eta}{2} \|h\|, \quad (i = 1, 2, \dots, N).$$

T. Ando [9] has established the sufficient conditions for the compactness of gradient map in Banach spaces X without the assumption of P-property of X. Recently J.W. Daniel [10] has established the result of E.H. Rothe [6],[7] to collectively compact sets of gradient maps.

The purpose of this note is twofold. In § 1 we shall establish sufficient conditions for the strong continuity of gradient map $F(x) = f'(x)$ where a potential f is a convex subadditive functional with $f(0) = 0$, meanwhile § 2 deals with the boundedness of Gâteaux differential $Vf(x_0, h)$ where f is a continuous functional on the space X of the second category (in particular on complete spaces). Moreover, the boundedness of "homogeneous" maps is also considered.

Notations and definitions. Let X,Y be real linear normed spaces, X^* a dual of X , E_1 a set of all real numbers, $f: X \to E_1$ a functional of X into E_1 . We shall use the symbols " $\longrightarrow$ ", " $\xrightarrow{w}$ " to denote the strong and weak convergence in X . A functional f is said to be

(a) convex on a convex subset $M \subset X$ if for each x, y ∈ ∈ M and $\lambda \in \langle 0, 1 \rangle$

$$f(\lambda x + (1-\lambda)y) \leqq \lambda f(x) + (1-\lambda) f(y) \quad ,$$

(b) subadditive on X if for every $x, y \in X$,

$$f(x+y) \leqq f(x) + f(y) \quad ,$$

(c) weakly continuous at $x_0 \in X$ if $x_n \xrightarrow{w} x_0$ implies $f(x_n) \to f(x_0)$.

A mapping $F: X \to Y$ of X into Y is said to be

(d) compact on $M \subset X$ if for each bounded subset $N \subset M$ F(N) is compact in Y (i.e. each sequence $\{y_n\} \in F(N)$ contains a subsequence $\{y_{n_k}\}$ which is convergent in Y).

(e) strongly continuous at $x_0 \in X$ if $x_n \in X$ $x_n \xrightarrow{w} x_0$ implies $F(x_n) \to F(x_0)$.

(f) completely continuous on $V \subset X$ if F is compact and continuous on V .

(g) bounded (a functional $f: X \longrightarrow E_1$ is called upper-bounded) in X if for each bounded set $M \subset X$, F(M) is bounded in Y (f(M) is upper-bounded).

For Gâteaux, Fréchet differentials and derivatives we use the notions and notations given in Vajnberg's book [3, chapt.I]. Let F be a mapping of X into Y. A Fréchet derivative $F'(x)$ (or Fréchet differential $dF(x,h)$) is said to have an uniform remainder $\omega(x,h)$ on $M \subset X$ if for any $\varepsilon > 0$ there exists $\delta > 0$ such that

$$0 < \|h\| < \delta \Rightarrow \|\omega(x,h)\| < \varepsilon \|h\|$$

for each $x \in M$, where $\omega(x,h) = F(x+h) - F(x) - F'(x)h$ (or $\omega(x,h) = F(x+h) - F(x) - dF(x,h)$).

Assume that a functional $f: X \to E_1$ has the Fréchet derivative $f'(x)$ on $M \subset X$. By gradient mapping $F: M \to X^*$ there is meant a map defined by $F(x) = f'(x)$, $x \in M$. We denote by D_R a closed ball centred about origin with radius $R > 0$. Throughout this paper we consider the <u>finite</u> functionals only.

§ 1. <u>Gradient mappings</u>. We shall prove the following

<u>Theorem 1</u>. Let X be a reflexive Banach space, D_R a closed ball in X, G an open convex subset of X containing D_R. Suppose $f: G \to E_1$ is a convex subadditive functional on G with $f(0) = 0$ and that f is upper-bounded on some convex open subset $M \neq \emptyset$ of G. Assume f possesses the Fréchet differential $df(x,h)$ on D_R and that the remainder $\omega(x,h)$ of $df(x,h)$ is uniform on D_R. Then the gradient map $F(x) = f'(x)$ where $f'(x)$ denotes the Fréchet derivative, is strongly continuous, compact and uniformly continuous on D_R and f is weakly continuous on D_R.

<u>Proof</u>. First of all f is continuous on G by Theorem

2 [11,II,§ 5]. Let x be an arbitrary (but fixed) element of D_R, $h_n \to 0$, $h_n \in X$. Then

$$|df(x, h_n)| \leqq |f(x+h_n) - f(x)| + |\omega(x, h_n)| .$$

The first term on the right side tends to 0 as $n \to \infty$ by continuity of f on G while $\omega(x, h_n) \to \infty$ as $n \to \infty$ by our assumption and in view of $h_n \to 0$. Being $df(x, h)$ linear in $h \in X$, continuity of $df(x, h)$ at h = 0 implies $df(x, h) = f'(x)h$ for each $x \in D_R$. Assume $\{x_n\} \in D_R$, $x_o \in D_R$, $x_n \xrightarrow{w} x_o$. Suppose on the contrary that f'(x) is not strongly continuous in x_o. Then there exist $\varepsilon_o > 0$ and the subsequence $\{x_{n_k}\}$ such that

(1) $$\| f'(x_{n_k}) - f'(x_o) \| > \varepsilon_o .$$

Let h be an arbitrary element of X with $\|h\| \leqq 1$. Then for $t > 0$

$$f(x_{n_k} + th) - f(x_{n_k}) = f'(x_{n_k})th + \omega(x_{n_k}, th),$$

$$f(x_o + th) - f(x_o) = f'(x_o)th + \omega(x_o, th).$$

Hence

(2) $$f'(x_{n_k})th - f'(x_o)th = f(x_{n_k} + th) - f(x_{n_k}) - \omega(x_{n_k}, th) + f(x_o) - f(x_o + th) + \omega(x_o, th).$$

For sufficiently small $t > 0$ we have that $x_{n_k} \pm th \in G$, $x_o \pm th \in G$, $th \in G$. Since f is subadditive on G,

(3) $$f(x_{n_k} + th) - f(x_{n_k}) \leqq f(th).$$

Employing convexity of f we have that

$$f(x_0) - f(x_0 + th) \leqq f(x_0 - th) - f(x_0)$$

and by subadditivity of f

$$f(x_0) - f(x_0 + th) \leqq f(-th) . \tag{4}$$

By our hypothesis

$$\begin{aligned} f(th) &= f'(0)th + \omega(0, th) , \\ f(-th) &= -f'(0)th + \omega(0, t(-h)) . \end{aligned} \tag{5}$$

Since f is convex and possesses the Fréchet derivative $f'(x)$ on D_R , there exists a number $t_1 > 0$ such that for each $t \in (0, t_1)$ we have

$$\begin{aligned} 0 &\leqq \omega(0, th) < \frac{1}{4} \varepsilon_0 t \| h \| , \\ 0 &\leqq \omega(0, t(-h)) < \frac{1}{4} \varepsilon_0 t \| h \| , \\ 0 &\leqq \omega(x_0, th) < \frac{1}{4} \varepsilon_0 t \| h \| . \end{aligned} \tag{6}$$

By our hypothesis the remainder $\omega(x, h)$ of $f'(x)$ is uniform on D_R . Hence there exists a positive number t_0 such that $0 < t_0 < t_1$ implies

$$0 \leqq \omega(x_{n_k}, t_0 h) < \frac{\varepsilon_0}{4} t_0 \| h \| \tag{7}$$

for each k $(k = 1,2,\dots)$. The relations (1) - (7) imply that

$$f'(x_{n_k}) t_0 h - f'(x_0) t_0 h < \varepsilon_0 t_0 \| h \| .$$

Hence

$$f'(x_{n_k}) h - f'(x_0) h < \varepsilon_0 \| h \| . \tag{8}$$

On the other hand, using the following inequalities

$$f(x_{n_k}+th)-f(x_{n_k}) \geqq f(x_{n_k})-f(x_{n_k}-th) \geqq -f(-th),$$

$$f(x_0)-f(x_0+th) \geqq -f(th),$$

employing (5) with changes $-f(th)$ for $f(th)$ and $-f(-th)$ for $f(-th)$ and (6),(7) with change of sign to minus, we obtain as above that

$$f'(x_{n_k})h - f'(x_0)h > -\varepsilon_0 \|h\| .$$

This inequality together with (8) imply

$$|f'(x_{n_k})h - f'(x_0)h| < \varepsilon_0 \|h\| .$$

Hence

$$\|f'(x_{n_k})-f'(x_0)\| = \sup_{\|h\| \leqq 1} |f'(x_{n_k})h - f'(x_0)h| \leqq \varepsilon_0 .$$

But this is a contradiction with (1). Hence $F(x) = f'(x)$ is strongly continuous on D_R. By Theorem 1.4 [3] $F(x)$ is compact and uniformly continuous on D_R (see also Th.1.3 [3]). According to Theorem 8.2 [3] f is weakly continuous on D_R. This completes the proof.

Remark 1. It is easy to see that the first assertion of Theorem 1 remains valid if D_R is replaced by an open convex neighbourhood $V(o)$ of 0 which is contained in G.

Theorem 2 [12] is valid if an open convex neighbourhood $V(o)$ of 0 is replaced by closed ball D_R and f is a convex subadditive functional on an open set G which contains D_R. Thus we have the following

Corollary 1. Let X be a reflexive Banach space, D_R a closed ball in X, G an open convex subset of X containing D_R . Suppose $f: G \to E_1$ is a convex subadditive functional on G with f(o) = 0 and that f is upper-bounded on some convex open subset $M \neq \emptyset$ of G . Assume f possesses the Fréchet differential $df(0, h)$ at 0 and the Gâteaux differential $Vf(x, h)$ for each $x \in D_R$, $x \neq 0$ and that the remainder $\omega(x, h)$ of the Fréchet derivative $f'(x)$ (which exists on D_R according to Th.2 [12]) is uniform on D_R . Then the gradient map $F(x) = f'(x)$ is strongly continuous, compact and uniformly continuous on D_R and f is weakly continuous on D_R .

<u>Remark 2</u>. The remainder $\omega(x, h)$ of $f'(x)$ is uniform on D_R if $F(x) = f'(x)$ is uniformly continuous on D_R (see [3,§ 4]). If X is a linear normed space, $f: X \to E_1$, a convex uniformly continuous functional on the open ball $B_{R+\alpha}$ ($\|x\| < R + \alpha$), then f has an uniformly continuous Fréchet derivative $f'(x)$ on B_R ($\|x\| < R$) $\Longleftrightarrow$ f is uniformly smooth on B_R (see [13,Theorem 8]). This assertion gives necessary and sufficient conditions that a gradient mapping $F(x) = f'(x)$ exists and be uniformly continuous on B_R (see also Th.7 [13]).

<u>Corollary 2</u>. Suppose X is a linear normed space, G an open convex subset of X containing D_R . Assume f satisfies the assumptions of Theorem 1. Then the gradient map $F(x) = f'(x)$ is strongly continuous on D_R .

§ 2. <u>Boundedness of Gâteaux differentials and maps.</u>

First of all we recall some well-known notions and result . We shall say that $f: X \to E_1$ is a function of the first Baire class if f is a point-limit of the sequence of continuous functions on X. A function $f: X \to E_1$ is said to have Baire property if there exists a subset $A \subset X$ of the 1.category in X such that $f/_{X-A}$ is continuous. We shall use the following

Lemma 1 [14, Theorem 14.3.1]. $f: X \to E_1$ is a function of the first Baire class $\iff$ for every $c \in E_1$ $\{x \in X : f(x) > c\}$, $\{x \in X : f(x) < c\}$ are F_σ-sets in X.

Theorem 2. Let X be a linear normed space of the 2. category in itself, $f: X \to E_1$ a continuous functional on X. Suppose f possesses the Gâteaux differential $Vf(x_0, h)$ at $x_0 \in X$ and that there exists a constant $M > 0$ such that for every $h_1, h_2 \in X$

$$(1)\ |Vf(x_0, h_1 + h_2)| \leqq M \max(|Vf(x_0, h_1)|, |Vf(x_0, h_2)|).$$

Then $Vf(x_0, h)$ is bounded in X.

Proof. Define a sequence $\{f_n(h)\}$ of functionals $f_n(h)$ by

$$f_n(h) = (f(x_0 + n^{-1}h) - f(x_0)) \cdot n$$

for every $h \in X$. Then $\{f_n(h)\}$ is a sequence of continuous functionals on X. By our hypothesis $\lim_{n \to \infty} f_n(h) = Vf(x_0, h)$ for every $h \in X$. Hence $Vf(x_0, h)$ is a function of the first Baire class and according to lemma 1 for every n $(n = 1, 2, \dots)$

$$A_n = \{h \in X : Vf(x_0, h) < n\},$$

$$B_n = \{h \in X : Vf(x_0, h) > -n\}$$

are F_σ sets. Since the intersection of two F_σ sets is again a F_σ set, $G_n = A_n \cap B_n$ is a F_σ-set for every n (n = 1,2,...) . Hence $G_n = \bigcup_{n=1}^{\infty} F_{nm}$, where F_{nm} are closed sets in X . Since

$$G_n = \{h \in X : |Vf(x_0, h)| < n\}$$

for every n (n = 1,2,...) , $X = \bigcup_{n=1}^{\infty} G_n$ and therefore $X = \bigcup_{n,m=1}^{\infty} F_{nm}$. By Baire category theorem at least one of F_{nm} $(n, m = 1, 2, \dots)$, say $F_{n_0 m_0}$, must contain a closed ball. Therefore there exist $r > 0$ and $h_0 \in X$ such that $\| h - h_0 \| \leqq r \Longrightarrow h \in F_{n_0 m_0}$ and for such h we have that $|Vf(x_0, h)| < n_0$ (for $F_{n_0 m_0} \subset G_{n_0}$). Set y = h - h_0 , then $\|y\| \leqq r$ and

$$|Vf(x_0, y)| \leqq M \max(|Vf(x_0, h)|, |Vf(x_0 - h_0)|) <$$

$$< M \max(n_0, c_0) ,$$

where $c_0 = |Vf(x_0, -h_0)|$. Hence $Vf(x_0, h)$ is bounded on the closed ball $\|h\| \leqq r$ and by homogeneity of $Vf(x_0, h)$ in h we see that $Vf(x_0, h)$ is bounded on each bounded subset of X . This completes the proof.

Corollary 3. Let X be a linear normed space of the 2. category in itself, $f: X \to E_1$ a continuous functional on X . Suppose f possesses the Gâteaux differential $Vf(x_0, h)$ at $x_0 \in X$ and that there exists a constant $M > 0$ such that for every $h_1, h_2 \in X$

$Vf(x_0, h_1 + h_2) \leqq M(Vf(x_0, h_1) + Vf(x_0, h_2))$.
Then $Vf(x_0, h)$ is upper-bounded in X .

Theorem 3. Let X,Y be linear normed spaces, X of the second category in itself, $F: X \to Y$ a mapping of X into Y such that

(a) $\| F(\lambda u)\| = |\lambda| \| F(u)\|$ for every $\lambda \in E_1$ and $u \in X$.

(b) $\| F(u+v)\| \leqq M \max(\| F(u)\|, \| F(v)\|)$ for every $u, v \in X$, where M is a positive constant.

(c) $u_n \in X$, $u \in X$, $u_n \to u \Rightarrow$

$$\Rightarrow \| F(u)\| \leqq \lim_{n\to\infty} \sup \| F(u_n)\| .$$

Then F is bounded in X .

Proof. Set $F_n = \{x \in X : \| F(x)\| \leqq n\}$.
Then $X = \bigcup_{n=1}^{\infty} F_n$. If $x_k \in F_n$, $x \in X$ $x_k \to x$, then $\| F(x)\| \leqq \lim_{n\to\infty} \sup \| F(x_k)\| \leqq n$.
Hence F_n $(n = 1, 2, \dots)$ are closed in X and thus at least one of them contains a closed ball. Now we proceed as in the proof of Theorem 2.

Theorem 4. Let X,Y be linear normed spaces, X of the 2.category in itself, $F: X \to Y$, $U: X \to Y$ mappings of X into Y . Suppose U possesses the Baire property, F satisfies the conditions (a),(b) of Theorem 3 and that for every $x \in X$ there is $\| F(x)\| \leqq \| U(x)\|$.
Then F is bounded map in X .

Proof. Use the arguments of Banach's proof [15,Theorem 1,p.78] and the ones of the second part of the proof of Th.2.

Remark 3. The conditions (c) of Theorem 3 and $\| F(x)\| \leqq \| U(x)\|$, $x \in X$ of Theorem 4 are sufficient that an additive map F be continuous and hence homogeneous on X . Both are due to Banach [15,p.78-79].

Remark 4. Theorems 3,4 can be used for investigations of the boundedness of the Gâteaux differentials $VF(x_0, h)$. Some other results concerning the boundedness of such differentials can be found in [3,§ 3], [13].

Remark 5. Theorem 2 can be derived at once from Theorem 4: A functional f is continuous on X , hence $Vf(x_0, h)$ possesses the Baire property and thus the condition of Theorem 4 is satisfied with $U(h) = Vf(x_0, h)$. We have the proof of Theorem 2 because it is somewhat different from the Banach's proof [15,Th.1,p.78].

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