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THE FOURTH MOMENT OF A GENERAL LINEAR STATISTIC

Marie HUŠKOVÁ, Praha

The purpose of this paper is to derive the fourth moment of the general linear rank statistic. Let $R_1, R_2, \dots, R_N$ be ranks of independent random variables $X_1, X_2, \dots, X_n$. Let

$$S = \sum_{i=1}^{N} a(i, R_i)$$

be a general linear rank statistic. For computation it is suitable to put

$$d(i,j) = a(i,j) - a(i,\cdot) - a(\cdot,j) + \bar{a} \quad ,$$

where

$$a(i,\cdot) = \frac{1}{N}\sum_{j=1}^{N} a(i,j) \quad ,$$

$$a(\cdot,j) = \frac{1}{N}\sum_{i=1}^{N} a(i,j) \quad ,$$

$$\bar{a} = \frac{1}{N}\sum_{j=1}^{N} a(\cdot,j) \quad .$$

Evidently

$$(1) \qquad \sum_{i=1}^{N} d(i,j) = \sum_{j=1}^{N} d(i,j) = 0 \; .$$

We shall formulate our result in the following theorem:

<u>Theorem</u>: If the distributions of $X_1, X_2, \dots, X_N$ are continuous and equal one another, then

$$E(S-ES)^4 = \frac{3(N^2-3N+1)}{N(N-1)(N-2)(N-3)} \left(\sum_{i=1}^{N}\sum_{j=1}^{N} d^2(i,j)\right)^2 -$$

$$- \frac{3}{(N-2)(N-3)} \sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k=1}^{N} (d^2(i,j)d^2(i,k)+d^2(i,j)d^2(k,j)) +$$

$$+ \frac{N(N+1)}{(N-1)(N-2)(N-3)} \sum_{i=1}^{N}\sum_{j=1}^{N} d^4(i,j) +$$

$$+ \frac{6}{N(N-1)(N-2)(N-3)} \sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k=1}^{N}\sum_{h=1}^{N} d(i,k)d(i,h)d(j,k)d(j,h).$$

<u>Proof</u>: From the definition of the fourth moment we receive

$$E(S-ES)^4 = E\left(\sum_{i=1}^{N} d(i,R_i)\right)^4 = \sum_{i=1}^{N} Ed^4(i,R_i) + 4\sum_{i=1}^{N}\sum_{\substack{j=1\\ j\neq 1}}^{N} Ed(j,R_j)d^3(i,R_i) + \tag{2}$$

$$+ 3\sum_{i=1}^{N}\sum_{\substack{j=1\\ j\neq i}}^{N} Ed^2(i,R_i)d^2(j,R_j) + 6\sum_{i=1}^{N}\sum_{\substack{j=1\\ j\neq i}}^{N}\sum_{\substack{k=1\\ k\neq i\\ k\neq j}}^{N} Ed^2(i,R_i)d(j,R_j)d(k,R_k) +$$

$$+ \sum_{i=1}^{N}\sum_{\substack{j=1\\ j\neq i}}^{N}\sum_{\substack{k=1\\ k\neq j\\ k\neq i}}^{N}\sum_{\substack{h=1\\ h\neq j\\ h\neq i\\ h\neq k}}^{N} Ed(i,R_i)d(j,R_j)d(k,R_k)d(h,R_h) .$$

Now we shall compute each member of the right side separately. The first member can be arranged as follows

$$\sum_{i=1}^{N} Ed^4(i,R_i) = \frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N} d^4(i,j) .$$

If we use (1) and relations

$$P(R_i=k, R_j=h) = \frac{1}{N(N-1)} \quad \text{if } k\neq h,\ i\neq j,$$
$$= 0 \quad \text{if } k = h,\ i\neq j,$$

then the second member gets the form

$$\sum_{i=1}^{N}\sum_{\substack{j=1\\ j\neq i}}^{N} Ed^3(i,R_i)d(j,R_j) = \frac{1}{N(N-1)}\sum_{i=1}^{N}\sum_{k=1}^{N} d^4(i,k) .$$

Analogously we obtain the following formulas for the remaining members

$$\sum_{i=1}^{N}\sum_{\substack{j=1\\ j\neq i}}^{N} E d^2(i,R_i)d^2(j,R_j) = \frac{1}{N(N-1)}\left(\left(\sum_{i=1}^{N}\sum_{k=1}^{N} d^2(i,k)\right)^2 - \right.$$

$$\left. - \sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k=1}^{N} d^2(i,k)d^2(j,k) - \sum_{i=1}^{N}\sum_{k=1}^{N}\sum_{h=1}^{N} d^2(i,k)d^2(i,h) + \sum_{i=1}^{N}\sum_{k=1}^{N} d^4(i,k)\right),$$

$$\sum_{i=1}^{N}\sum_{\substack{j=1\\ j\neq i}}^{N}\sum_{\substack{k=1\\ k\neq i\\ k\neq j}}^{N} E d^2(i,R_i)d(j,R_j)d(k,R_k) = \frac{1}{N(N-1)(N-2)}\left(\left(\sum_{i=1}^{N}\sum_{h=1}^{N} d^2(i,h)\right)^2 - \right.$$

$$\left. - 2\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{h=1}^{N} d^2(i,h)d^2(j,h) - 2\sum_{i=1}^{N}\sum_{h=1}^{N}\sum_{k=1}^{N} d^2(i,h)d^2(i,k) + 4\sum_{i=1}^{N}\sum_{h=1}^{N} d^4(i,h)\right),$$

$$\sum_{i=1}^{N}\sum_{\substack{j=1\\ j\neq i}}^{N}\sum_{\substack{k=1\\ k\neq i\\ k\neq j}}^{N}\sum_{\substack{h=1\\ h\neq i\\ h\neq j\\ h\neq k}}^{N} E d(i,R_i)d(j,R_j)d(k,R_k)d(h,R_h) =$$

$$= \frac{1}{N(N-1)(N-2)(N-3)}\left(6\sum_{i=1}^{N}\sum_{k=1}^{N}\sum_{j=1}^{N}\sum_{h=1}^{N} d(i,k)d(i,h)d(j,k)d(j,h) + \right.$$

$$+ 3\left(\sum_{i=1}^{N}\sum_{k=1}^{N} d^2(i,k)\right)^2 - 18\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k=1}^{N} d^2(i,k)d^2(j,k) -$$

$$\left. - 18\sum_{i=1}^{N}\sum_{k=1}^{N}\sum_{h=1}^{N} d^2(i,k)d^2(i,h) + 36\sum_{i=1}^{N}\sum_{k=1}^{N} d^4(i,k)\right).$$

Now it is sufficient to put these results into (2) and we shall get the requested formula.

<u>Remark</u>: If $a(i,j) = a_i b_j$ then our result coincides with that which appeared in the paper The correlation coefficient test, Suppl. JRSS 4(1937), 225-232, written by E.J.G. Pitman.

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