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STRONG EMBEDDING OF CATEGORY OF ALL GRUPOIDS INTO CATEGORY OF SEMIGROUPS

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In [3] a strong embedding of concrete categories x) is introduced as follows: A strong embedding of $\langle \mathcal{K}, \Box \rangle$ into $\langle \mathcal{K}', \Box' \rangle$ is a full embedding $\Phi : \mathcal{K} \rightarrow \mathcal{K}'$ such that for some set functor $F : \mathcal{S} \rightarrow \mathcal{S}$ there is

$$\Box' \circ \Phi = F \circ \Box .$$

Thus, the images of the objects of $\mathcal{K}$ with the same underlying set have also the same underlying set and analogously for morphisms of $\mathcal{K}$ and their underlying mappings. The functor F will be called the underlying functor of the strong embedding Φ .

In [3] the following result is proved: Every category

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x) A concrete category is, as usual, a couple $\langle \mathcal{K}, \Box \rangle$, where $\mathcal{K}$ is a category, $\Box$ is a faithfull functor of $\mathcal{K}$ into the category $\mathcal{S}$ of all sets and all their mappings. If μ is an object or a morphism of $\mathcal{K}$ then $\Box(\mu)$ is called its underlying set or its underlying mapping, respectively.

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of all universal algebras of a given type (and all their homomorphisms) can be strongly embedded into every category of all universal algebras with at least two unary or at least one at least binary operation.

In [2] the full embedding of the category $\mathcal{R}$ of all graphs and all their graph-homomorphisms into the category Sgp of all semigroups is constructed. This full embedding is not a strong embedding, $\mathcal{R}$ cannot be strongly embedded into any category of algebras, [3].

The aim of the present note is a construction of a strong embedding of the category A(2) of all universal algebras with one binary operation into the category Sgp x).

I am indebted to A. Pultr and J. Sichler for valuable discussions, suggestion of the problem and turning my attention to the paper [1].

<u>Theorem</u>: The category A(2) of all universal algebras with the binary operation can be strongly embedded into the category Sgp of all semigroups.

Proof of the Theorem:

I. Let D be a semigroup considered in [1], i.e. the semigroup with two generators a,b and one relation $ab^2 = baba$. This semigroup is rigid, i.e. it has no endomorphism other than the identity automorphism, [1].

Moreover D is right cancellative i.e. if $vu = wu$ then

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x) A strong embedding of A(1,1) into Sgp was independently constructed by J. Sichler.

$v = w$ (this is not explicitly given in [1] but it follows from facts about D proved there, namely: every element $u \in D$ can be uniquely written in the form $u = b^t a^{s_1} b a^{s_2} \dots b a^{s_n}$ where $t \geq 0$, $n \geq 1$, $s_n \geq 0$ and $s_i \geq 1$ for $1 \leq i < n$. If this normal form of u is given then the formulas for the normal forms of u b are listed in [1]. From these it follows easily: if v, $w \in D$, $vb = wb$, then $v = w$. Thus if $v, \neq w$ then $vb \neq wb$ and we can repeat this step, i.e. then $vb^t \neq wb^t$. If $u = b^t a^{s_1} b a^{s_2} \dots b a^{s_n}$ is given and $v \neq w$, then $vb^t \neq wb^t$ and, considering the normal form of vb^t and wb^t, and the fact that $s_1 \neq 0$, we obtain $vu \neq wu$.

II. Let $M = \{a, ab, ba, aba, bab, ab^2 = baba\}$, $N = D - M$. Let F be the set functor $F(X) = (X \times M) \vee N$ (where by $\vee$ is denoted a disjoint union), $F(f) = (f \times id_M) \vee \vee\, id_N$, where by id_M, id_N are denoted the identity mappings. Denote by $\pi_X : F(X) \longrightarrow D$ the mapping for which $\pi_X(\langle x, u \rangle) = u$ whenever $\langle x, u \rangle \in \in X \times M$, $\pi_X(v) = v$ whenever $v \in N$.

III. We shall define an embedding $\Phi : A(2) \longrightarrow Sgp$ with F as an underlying functor. Let $\langle X, \cdot \rangle$ be an object of $A(2)$ (i.e. $\cdot$ is a binary operation on a set X). We define $\Phi(\langle X, \cdot \rangle) = \langle F(X), \oplus \rangle$, where $\oplus$ is the following binary operation on $F(X)$:

a) if $r, s \in F(X)$, $\pi_X(r) \cdot \pi_X(s) \in N$, then

$r \oplus s = \pi_X(r) \cdot \pi_X(s)$;

b) $v \oplus \langle x, u \rangle = \langle x, vu \rangle$ whenever $x \in X$, $v \in \{b, b^2\}$, $vu \in M$; $\langle x, u \rangle \oplus v = \langle x, uv \rangle$ whenever $x \in X$, $v \in \{b, b^2\}$, $uv \in M$;

c) $\langle x, u \rangle \oplus \langle y, v \rangle = \langle x \cdot y, uv \rangle$ whenever x, $y \in X$, $u, v \in M$, $uv \in \{aba, baba\}$.

Evidently $\pi_X : \langle F(X), \oplus \rangle \longrightarrow D$ is a homomorphism.

IV. It is easy to see that $\oplus$ is associative and if a mapping $f : \langle X, \cdot \rangle \to \langle Y, \cdot \rangle$ is a homomorphism, then $F(f) : \langle F(X), \oplus \rangle \longrightarrow \langle F(Y, \oplus \rangle$ is also a homomorphism. Thus $\Phi : A(2) \to Sgr$ is really an embedding with F as an underlying functor. We must prove that it is a full embedding.

V. Let $g : \langle F(X), \oplus \rangle \longrightarrow \langle F(Y), \oplus \rangle$ be a homomorphism. We shall prove that $g = F(f)$, where $f : \langle X, \cdot \rangle \to \langle Y, \cdot \rangle$ is a homomorphism:

1) First we prove: if $x, y \in X$, then $\pi_Y g(\langle x, baba \rangle) = \pi_Y g(\langle y, baba \rangle)$. For, $\langle x, baba \rangle \oplus b = babab = \langle y, baba \rangle \oplus b$ and therefore $\pi_Y g(\langle x, baba \rangle) \cdot \pi_Y g(b) = \pi_Y (g(\langle x, baba \rangle) \oplus g(b)) = \pi_Y g(babab) = \pi_Y g(\langle y, baba \rangle) \cdot \pi_Y g(b)$.

Now use cancellation in D.

2) Choose $x \in X$. Let $\varphi, \psi : D \to F(X)$ be mappings identical on N and

$\varphi(u) = \psi(u) = \langle x, u \rangle$ whenever $u \in \{a, ba, ab, bab\}$;
$\varphi(aba) = \psi(aba) = \langle x \cdot x, aba \rangle$;
$\varphi(baba) = \langle x \cdot x, baba \rangle$, $\psi(baba) = \langle x, baba \rangle$.

Using 1) we obtain $\pi_Y g \varphi = \pi_Y g \psi$.

3) Consequently, $\pi_Y g \varphi : D \to D$ is a homomorphism and since D is rigid, $\pi_Y g \varphi$ is the identity. Thus g is identical on N and $g(\langle x, a \rangle) = \langle \bar{x}, a \rangle$ for some $\bar{x} \in Y$.

4) Define $f : X \to Y$ such that $f(x) = \bar{x}$. Then $g(\langle x, ab \rangle) = g(\langle x, a \rangle \oplus b) = \langle f(x), a \rangle \oplus b = \langle f(x), ab \rangle$; analogously $g(\langle x, ba \rangle) = \langle f(x), ba \rangle$, $g(\langle x, bab \rangle) = \langle f(x), bab \rangle$. Moreover $g(\langle x, baba \rangle) = g(\langle x, ab^2 \rangle) = g(\langle x, a \rangle \oplus b^2) = \langle f(x), a \rangle \oplus b^2 = \langle f(x), baba \rangle$ and

$$b \oplus g(\langle x, aba \rangle) = g(b) \oplus g(\langle x, aba \rangle) = g(\langle x, baba \rangle) = \langle f(x), baba \rangle = b \oplus \langle f(x), aba \rangle,$$

consequently $g(\langle x, aba \rangle) = \langle f(x), aba \rangle$. Thus $g = F(f)$.

5) Since $\langle f(x) \cdot f(y), aba \rangle = \langle f(x), a \rangle \oplus \langle f(y), ba \rangle = g(\langle x, a \rangle) \oplus g(\langle y, ba \rangle) = g(\langle x \cdot y, aba \rangle) = \langle f(x \cdot y), aba \rangle$

for every $x, y \in X$, then $f : \langle X, \cdot \rangle \to \langle Y, \cdot \rangle$ is a homomorphism.

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