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Commentationes Mathematicae Universitatis Carolinae, Vol. 9 (1968), No. 2, 207--222

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9,2 (1968)

ON DEMICONTINUITY AND HEMICONTINUITY OF NONLINEAR INTEGRAL OPERATORS

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1. Introduction. The notions demicontinuity and hemicontinuity of nonlinear operators have been introduced and largely studied by F.E. Browder in connection with the theory of monotone operators in series of his papers. Recently W.V. Petryshyn (cf.[7]) has discovered a two-way connection between the range and demicontinuity of nonlinear operators and T. Kato [5] has shown that every hemicontinuous monotone operator defined on an open subset of Banach space X to its dual X^* is always demicontinuous.

The purpose of this note is to give some conditions for demicontinuity and hemicontinuity of two main types of nonlinear operators in the spaces of integrable functions. The first type (Urysohn's operators) is studied in the section 2, while the second one, the operators of Nemyckij, is investigated in the section 3. These operators are discussed here without the assumption of monotonicity.

First of all we introduce some notations and recall some known facts.

The symbol E_r $(r = 1,2,\dots)$ denotes the Euclidean r-space. A function $f: [x, y] \to f(x, y)$, where x is fixed and y is variable, is denoted by $f(x, .)$.

Let G be a bounded measurable subset of E_n, g be a function of two variables defined on $G \times E_1$. Let $g(\cdot u)$ be a measurable function on G for every (fixed) $u \in E_1$, $g(t,\cdot)$ be a continuous function on E_1 for almost every (fixed) $t \in G$. Then g is called the N-function on $G \times E_1$ (see [8]).

Let G be a bounded closed subset of E_n, K be a function of three variables defined on $G \times G \times E_1$. Let $K(\cdot,t,u)$ be a measurable function on G for almost every $t \in G$ and every $u \in E_1$, $K(s,\cdot,\cdot)$ be an N-function on $G \times E_1$ for almost every $s \in G$. Then K is called the U_L-function on $G \times G \times E_1$ (see [2]).

<u>Lemma 1</u> (see [6], § 2). Let g be an N-function on $G \times E_1$ where G is a bounded measurable subset of E_n, let $p, q \geq 1$. Suppose there exist an integer n, numbers $\gamma_i \geq 0$ $(i=1,\dots,n)$, $b \geq 0$ and functions $T_i \in L_{\frac{pq}{p-q\gamma_i}}(G)$ $(i=1,\dots,n)$

such that $0 \leq q\gamma_i < p$ $(i=1,\dots,n)$ and

$$|g(t,u)| \leq \sum_{i=1}^{n} T_i(t)|u|^{\gamma_i} + b|u|^{\frac{p}{q}}$$

for almost every $t \in G$ and every $u \in E_1$. Then the operator of Nemyckij generated by the function g is a continuous bounded mapping from the space $L_p(G)$ into $L_q(G)$.

<u>Lemma 2</u> (see [1], th.39(9.2)). Let K be a U_L-function on $G \times G \times E_1$, where G is a bounded closed subset of E_n, let F be the operator of Urysohn generated by the function K, let $p, q \geq 1$. Suppose there exist an integer n,

numbers $\lambda_i \in \langle 0, p)$ $(i = 1, \dots, n)$ and functions $M_0 \in L_q(G)$, M_i $(i=1,\dots,n)$ on $G \times G$ such that

$$[\int_G |M_i(s,t)|^{\frac{p}{p-\lambda_i}} dt]^{\frac{p-\lambda_i}{p}} \in L_q(G) \quad (i=1,\dots,n)$$

and

$$|K(s,t,u)| \leq \sum_{i=1}^{n} M_i(s,t)|u|^{\lambda_i} + M_0(s)|u|^p$$

for almost every $s, t \in G$ and every $u \in E_1$. Then F is a continuous bounded mapping from $L_p(G)$ into $L_q(G)$.

2. Operators of Urysohn. Throughout this section we assume that G is a bounded closed subset of E_n, K is a U_L-function on $G \times G \times E_1$ and that F is the operator of Urysohn generated by this function K. Furthermore, we assume that p, q are arbitrary real numbers without any relation among them, $p \geq 1$, $q \geq 1$. We denote $q' = \frac{q}{q-1}$ for $q > 1$; in the case $q = 1$, we mean by q' the symbol ∞.

Theorem 1. Let $D \subset L_{q'}(G)$. Suppose there exist an integer n_φ, numbers $\lambda_i^\varphi \in \langle 0, p)$ $(i = 1, \dots, n_\varphi)$ and functions $M_0^\varphi \in L(G)$, M_i^φ on $G \times G$ $(i = 1, \dots, n_\varphi)$ for every $\varphi \in D$ such that *either*

$$(\int_G |M_i^\varphi(\cdot, t)|^{\frac{p}{p-\lambda_i^\varphi}} dt)^{\frac{p-\lambda_i^\varphi}{p}} \in L(G) \; (i = 1, \dots, n_\varphi) \tag{1}$$

or

$$\int_G M_i^\varphi(s, \cdot)\, ds \in L_{\frac{p}{p-\lambda_i^\varphi}}(G) \quad (i = 1, \dots, n_\varphi) \tag{2}$$

and

$$|K(s,t,u)\varphi(s)| \leq \sum_{i=1}^{n_\varphi} M_i^{\varphi}(s,t)|u|^{\lambda_i^{\varphi}} + M_0^{\varphi}(s)|u|^{r}$$

for almost every $s,\ t \in G$ and every $u \in E_1$. Then the following assertions are valid:

(a) If $D = L_{q'}(G)$, then F is a demicontinuous operation from $L_p(G)$ into $L_q(G)$.

(b) Let the linear hull of D be dense in $L_{q'}(G)$, let $x_0 \in L_p(G)$. Assume there exist a constant C and a neighbourhood U of the point x_0 in the space $L_p(G)$ such that

$$(3) \qquad \int_G \left| \int_G K(s,t,x(t))\,dt \right|^q ds \leq C$$

whenever $x \in U$. Then F maps U into $L_q(G)$ and is demicontinuous at the point x_0.

<u>Proof</u> Let φ be an arbitrary element of D ; we shall prove that

$$(4) \qquad \langle Fx_n, \varphi \rangle \to \langle Fx_0, \varphi \rangle$$

whenever $x_n \to x_0$ in $L_p(G)$. If $D = L_{q'}(G)$ (the case (a) of Theorem), then (4) gives demicontinuity of F at x_0 and the proof is finished. Assuming (b), according to well-known theorem [4; chapt. VIII, § 2] and (3), (4), it follows that the relation (4) holds for every $\varphi \in L_{q'}(G)$ and hence the demicontinuity of F at x_0 will be proved, too.

I. Suppose the condition (1) is fulfilled; we shall prove (4). Set

$$R_\varphi x(s) = \int_G K(s,t,x(t))\,\varphi(s)\,dt \quad (s \in G)$$

for $x \in L_p(G)$, $\varphi \in \mathcal{D}$. According to Lemma 2, R_φ is a continuous mapping from $L_p(G)$ into $L(G)$, i.e. $\|R_\varphi x_n - R_\varphi x_0\|_L \to 0$ for $x_n, x_0 \in L_p(G)$, $\|x_n - x_0\|_{L_p} \to 0$. Furthermore,

$$\|R_\varphi x_n - R_\varphi x_0\|_L = \int_G \Big| \int_G K(s,t,x_n(t))\varphi(s)dt - \int_G K(s,t,x_0(t))\varphi(s)dt \Big| ds \geq$$

$$\geq \Big|\int_G \Big[\int_G K(s,t,x_n(t))dt\,\varphi(s) - \int_G K(s,t,x_0(t))dt\,\varphi(s)\Big] ds\Big| =$$

$$= \Big|\int_G Fx_n(s)\varphi(s)ds - \int_G Fx_0(s)\varphi(s)ds\Big| =$$

$$= |\langle Fx_n, \varphi\rangle - \langle Fx_0, \varphi\rangle|$$

and hence

$$\langle Fx_n, \varphi\rangle \to \langle Fx_0, \varphi\rangle$$

for every $\varphi \in \mathcal{D}$ whenever $x_n \to x_0$ in $L_p(G)$.

II. Consider the condition (2), let $x \in L_p(G)$, $\varphi \in \mathcal{D}$. The integral $\int_G \int_G |M_i^\varphi(s,t)\,|x(t)|^{\lambda_i^\varphi}|\,dt\,ds$ exists and so

$$\int_G \int_G |M_i^\varphi(s,t)|x(t)|^{\lambda_i^\varphi}|\,dt\,ds \leq$$

$$\leq \Big(\int_G \Big[\int_G |M_i^\varphi(s,t)|ds\Big]^{\frac{p}{p-\lambda_i^\varphi}} dt\Big)^{\frac{p-\lambda_i^\varphi}{p}} \cdot \Big(\int_G |x(t)|^p dt\Big)^{\frac{\lambda_i^\varphi}{p}} < \infty$$

for $i = 1, \dots, n_\varphi$ by Theorems of Fubini and Hölder; similarly

$$\int_G \int_G |M_0^\varphi(s)|x(t)|^p|\,dt\,ds \leq \Big(\int_G |M_0^\varphi(s)|ds\Big) \cdot \Big(\int_G |x(t)|^p dt\Big) < \infty .$$

Hence,

$$\int_G \int_G |K(s,t,x(t))\varphi(s)|\,dt\,ds \le$$

$$\le \sum_{i=1}^{n_\varphi} \int_G \int_G |M_i^\varphi(s,t)|x(t)|^{\lambda_i^\varphi}|\,dt\,ds + \int_G \int_G |M_0^\varphi(s)|x(t)|^{r}|\,dt\,ds < \infty$$

which implies

$$(5) \qquad \left|\int_G \int_G K(s,t,x(t))\varphi(s)\,dt\,ds\right| < \infty \, .$$

Put

$$H_\varphi(t,u) = \int_G K(s,t,u)\varphi(s)\,ds \, ,$$

$$N_0^\varphi = \int_G M_0^\varphi(s)\,ds \, , \quad N_i^\varphi(t) = \int_G M_i^\varphi(s,t)\,ds \quad (i = 1,\dots,n_\varphi) \, .$$

Then according to Hölder's inequality

$$|H_\varphi(t,u)| \le \sum_{i=1}^{n_\varphi} N_i^\varphi(t)|u|^{\lambda_i^\varphi} + N_0^\varphi|u|^{r}$$

for almost every $t \in G$ and every $u \in E_1$ and according to (2)

$$N_0^\varphi \in E_1 \, , \quad N_i^\varphi \in L_{\frac{r}{r-\lambda_i^\varphi}}(G) \quad (i = 1,\dots,n_\varphi) \, ,$$

simultaneously. Lemma 1 implies that the operator of Nemyckij R_φ generated by the function H_φ is a continuous mapping from $L_r(G)$ into $L(G)$. Hence $\|R_\varphi x_n - R_\varphi x_0\|_L \to 0$ whenever $\|x_n - x_0\|_{L_r} \to 0$, $x_n, x_0 \in L_r(G)$ and so

$$\int_G (R_\varphi x_n)(t)\,dt \to \int_G (R_\varphi x_0)(t)\,dt \, .$$

In view of (5), using Theorem of Fubini,we obtain

$$\int_G (R_\varphi x)(t)\,dt = \int_G \left(\int_G K(s,t,x(t))\varphi(s)\,ds\right)dt =$$

$$= \int_G \left(\int_G K(s,t,x(t)) \varphi(s)\,dt \right) ds = \int_G \left(\int_G K(s,t,x(t))\,dt \right) \varphi(s)\,ds =$$

$$= \int_G (Fx)(s)\,\varphi(s)\,ds = \langle Fx, \varphi \rangle$$

for $x \in L_p(G)$. Hence

$$\langle Fx_n, \varphi \rangle \longrightarrow \langle Fx_o, \varphi \rangle$$

for every $\varphi \in D$ whenever $x_n \longrightarrow x_o$ in $L_p(G)$. The formula (4) is proved and the whole proof is concluded.

The assumptions of Theorem 1 can be made more easily verifiable by a definite choice of the set D. In this way, we can obtain a series of further theorems. Theorem 2 is one of such theorems; it is presented in the local form.

Theorem 2. Let x_o be an element of $L_p(G)$. Suppose there exist a constant C and a neighbourhood U of the point x_o in $L_p(G)$ such that

$$(6) \qquad \int_G \left| \int_G K(s,t,x(t))\,dt \right|^q ds \leq C$$

for all $x \in U$. Let there exist an integer n, numbers $\lambda_i \in \langle 0, p)$ $(i = 1,\dots,n)$ and functions $M_o \in L(G)$, M_i on $G \times G$ $(i = 1,\dots,n)$ such that either

$$\left(\int_G |M_i(\cdot,t)|^{\frac{p}{p-\lambda_i}}\,dt \right)^{\frac{p-\lambda_i}{p}} \in L(G) \quad (i = 1,\dots,n)$$

or

$$\int_G M_i(s,\cdot)\,ds \in L_{\frac{p}{p-\lambda_i}}(G) \quad (i = 1,\dots,n)$$

and that

$$|K(s,t,u)| \leq \sum_{i=1}^{n} M_i(s,t)\,|u|^{\lambda_i} + M_o(s)\,|u|^p$$

for almost every $s, t \in G$ and every $u \in E_1$. Then F is an operation from U into $L_q(G)$ demicontinuous at the point x_o.

Proof. Let $\{G_\alpha : \alpha \in A\}$ be a measurable disjoint subdivision of the set G, φ_α be the characteristic function of G_α $(\alpha \in A)$. Denote by M the linear hull of the set $\{\varphi_\alpha : \alpha \in A\}$, i.e. M is the set of all simple-functions on G. If we put $M_k = \{x \in L_{q'}(G) : |x(t)| \leq k$ for $t \in G\}$ (k is natural number), then M is dense in M_k for every k under the topology of equiconvergence [cf. 3, th.39] and hence, M is dense in every M_k even under the topology which is generated on M_k by topology of the space $L_{q'}(G)$. Hence M is dense in $\bigcup_{k=1}^{\infty} M_k$; the set $\bigcup_{k=1}^{\infty} M_k$ is dense in $L_{q'}(G)$ and so M is dense in $L_{q'}(G)$.

For every $\alpha \in A$,

$$|K(s,t,u)\varphi_\alpha(s)| \leq |K(s,t,u)|$$

for almost every $s, t \in G$ and all $u \in E_1$. Setting $D = \{\varphi_\alpha : \alpha \in A\}$ and $n_\varphi = n$, $M_0^\varphi = M_0$, $M_i^\varphi = M_i$ $(i = 1,\dots,n)$, $\lambda_i^\varphi = \lambda_i$ $(i = 1,\dots,n)$ for all $\varphi \in D$, the assertion of our theorem follows at once from (b) of Theorem 1.

Remark 1. Urysohn's operator F satisfying the conditions of Theorem 2 maps $L_p(G)$ into $L_1(G)$ and it is continuous at the point x_0. Hence this Theorem does not any new results when $q = 1$.

Remark 2. The assertion in Remark 1 is consequence of the special choice of the set D (the part (b) of Theorem 1). Under another choice of D, any similar assertion has not to be valid and so we can obtain more general theorem than Theorem 2; for example, if $G = \langle a, b \rangle$, $a, b \in E_1$, then we can set $D = \{1, s, s^2, s^3, \dots\}$ (for $s \in \langle a, b \rangle$).

Remark 3. Let H be an open subset of $L_p(G)$. Let there exist a constant C such that the formula (6) holds for all $x \in H$ and let the other assumptions of Theorem 2 be fulfilled. Then F is a demicontinuous operation from H into $L_q(G)$.

Theorem 3. Let $x_0 \in L_p(G)$. Assume F maps a neighbourhood U of the point x_0 in $L_p(G)$ onto a set $M \subset L_q(G)$, let $D \subset L_{q'}(G)$. Suppose there exist a number $\delta_{\varphi,h} > 0$ and a function $N_{\varphi,h}$ on $G \times G$ for every $\varphi \in D$, $h \in L_p(G)$ with $\|h\|_{L_p} = 1$ such that

$$(7) \qquad \int_G N_{\varphi,h}(\cdot, t)\, dt \in L(G)$$

and

$$(8) \qquad |K(s,t,x_0(t)+\tau h(t))\varphi(s)| \leq N_{\varphi,h}(s,t)$$

for almost every $s, t \in G$ and every $\tau \in (0, \delta_{\varphi,h})$. Furthermore, assume one of the following two conditions is fulfilled:

(a) $D = L_{q'}(G)$.

(b) The linear hull of D is dense in $L_{q'}(G)$ and M is bounded in $L_q(G)$. Then F is an operation from U into $L_q(G)$ hemicontinuous at the point x_0.

Proof. Let $\varphi \in D$ and $h \in L_p(G)$, $\|h\|_{L_p} = 1$, be arbitrary elements. Continuity of the function $K(s,t,\cdot)$ on G for almost every $s, t \in G$ implies

$$(9) \qquad K(s,t,x_0(t)+\tau h(t))\varphi(s) \to K(s,t,x_0(t))\varphi(s)$$

for almost every $s, t \in G$ whenever $\tau \to 0$. According to Theorem on continuous dependence of integral by parameter, the formulas (7),(8),(9) imply

$$(10) \quad \int_G K(s,t,x_0(t)+\tau h(t))\varphi(s)ds \to \int_G K(s,t,x_0(t))\varphi(s)ds$$

for almost every $t \in G$ whenever $\tau \to 0$. Furthermore, (7) and (8) give

$$\int_G |K(s,t,x_0(t)+\tau h(t))\varphi(s)|dt \le \int_G N_{\varphi,h}(s,t)dt$$

for all $\tau \in (0, \sigma_{\varphi,h})$; using the last inequality and the relations (7),(9), we have that

$$\int_G (\int_G K(s,t,x_0(t)+\tau h(t))\varphi(s)ds)dt \to$$

$$\to \int_G (\int_G K(s,t,x_0(t))\varphi(s)ds)dt$$

if $\tau \to 0$. As in the part II of the proof of Theorem 1, we can prove now (according to the theorem of Fubini) that

$$\int_G (\int_G K(s,t,x(t))\varphi(s)ds)dt = \langle Fx,\varphi\rangle \quad (x \in L_p(G), \varphi \in D)$$

and hence

$$\langle F(x_0+\tau h),\varphi\rangle \to \langle Fx_0,\varphi\rangle$$

for $\tau \to 0$, $\varphi \in D$. Under each of the conditions (a) and (b) of Theorem, this relation means hemicontinuity of F at the point x_0.

In the same way as we have derived Theorem 2 from Theorem 1, we can obtain the next theorem from Theorem 3 now.

Theorem 4. Let x_0 be an element of $L_p(G)$. Suppose there exist a neighbourhood U of the point x_0 in $L_p(G)$

and a constant C such that

$$\int_G \left| \int_G K(s,t,x(t))\,dt \right|^q ds \leq C$$

for all $x \in U$. Let there exist a number $\sigma_h > 0$ and a function N_h on $G \times G$ for every $h \in L_p(G)$ with $\|h\|_{L_p} = 1$ such that

$$\int_G N_h(\cdot, t)\,dt \in L(G)$$

and

$$|K(s,t,x_0(t) + \tau h(t))| \leq N_h(s,t)$$

for almost every $s, t \in G$ and every $\tau \in (0, \sigma_h)$. Then F is an operation from U into $L_q(G)$ hemicontinuous at the point x_0.

<u>Theorem 5</u>. Let H be an open subset of $L_p(G)$, suppose there is a constant C such that

$$\int_G \left| \int_G K(s,t,x(t))\,dt \right|^q ds \leq C$$

for all $x \in H$. Let there be such a number $\sigma_{x,h} > 0$ and a function $N_{x,h}$ on $G \times G$ for every $x \in H$ and $h \in L_p(G)$, $\|h\|_{L_p} = 1$, that

$$\int_G N_{x,h}(\cdot, t)\,dt \in L(G)$$

and that

$$|K(s,t,x(t) + \tau h(t))| \leq N_{x,h}(s,t)$$

for almost every $s, t \in G$ and every $\tau \in (0, \sigma_{x,h})$. Then F is a hemicontinuous operation from H into $L_q(G)$.

<u>Proof</u>. It is evident the operator F satisfying the conditions of this theorem fulfils the conditions of Theorem 4 for each point $x_0 \in H$. Hence F is hemicontinuous at all points of H.

3. Operators of Nemyckij. We turn our attention to demicontinuity and hemicontinuity of operators of Nemyckij now. In the following all theorems, we shall assume that G is a bounded measurable subset of E_r , g is an N-function on $G \times E_1$ and that h is the operator of Nemyckij generated by this function g . The assumptions concerning p, q and q' are the same as formerly.

Theorem 6. Let $x_0 \in L_p(G)$, let D be a subset of $L_{q'}(G)$ the linear hull of which is dense in $L_{q'}(G)$. Suppose there are an integer n_φ , numbers $\lambda_i^\varphi \in \langle 0, p)$ $(i = 1, \dots, n_\varphi)$, a constant M_0^φ and functions $M_i^\varphi \in L_{\frac{p}{p-\lambda_i^\varphi}}(G)$ $(i = 1, \dots, n_\varphi)$ for every $\varphi \in D$ such that

$$|g(t,u)\varphi(t)| \le \sum_{i=1}^{n_\varphi} M_i^\varphi(t)|u|^{\lambda_i^\varphi} + M_0^\varphi |u|^p$$

for almost every $t \in G$ and every $u \in E_1$. If there exist a constant C and a neighbourhood U of the point x_0 in $L_p(G)$ such that

$$\int_G |g(t, x(t))|^q \, dt \le C$$

whenever $x \in U$, then h is an operation from U into $L_q(G)$ demicontinuous at the point x_0 .

Proof. Let φ be an arbitrary element of D. Set

$$k_\varphi(t,u) = g(t,u)\varphi(t) ;$$

k_φ is also N-function on $G \times E_1$ and so it is possible to introduce the operator of Nemyckij generated by the function k_φ - denote it by R_φ . According to Lemma 1, it follows from the assumptions of our theorem that R_φ is a continuous operation from $L_p(G)$ into $L_q(G)$, i.e.

$$\int_G |(Rx_n)(t) - (Rx_0)(t)|dt \to 0$$

whenever $\|x_n - x_0\|_{L_p} \to 0$, $x_n, x_0 \in L_p(G)$. This relation is equivalent to

$$\int_G hx_n(t)dt \to \int_G hx_0(t)\varphi(t)dt .$$

We have proved that $\langle Fx_n, \varphi \rangle \to \langle Fx_0, \varphi \rangle$ for every $\varphi \in D$ whenever $\|x_n - x_0\|_{L_p} \to 0$; but the linear hull of D is dense in $L_{q'}(G)$, $\|Fx\| \leq C$ for $x \in U$ and so F is demicontinuous at the point x_0 [cf. 4, chapt. VIII, § 2]. The proof is complete.

Theorem 7. Let x_0 be an element of $L_p(G)$. Let there exist a constant C and a neighbourhood U of the point x_0 in $L_p(G)$ such that

$$(11) \qquad \int_G |g(t, x(t))|^q dt \leq C$$

for all $x \in U$. Assume there are an integer n, numbers $\lambda_i \in \langle 0, p)$ $(i = 1, \dots, n)$, a constant M_0 and functions

$M_i \in L_{\frac{p}{p-\lambda_i}}(G)$ $(i = 1, \dots, n)$ such that

$$|g(t,u)| \leq \sum_{i=1}^{n} M_i(t)|u|^{\lambda_i} + M_0|u|^p$$

for almost every $t \in G$ and every $u \in E_1$. Then h is an operation from U into $L_q(G)$ demicontinuous at the point x_0.

Remark 4. The operator h satisfying the conditions of Theorem 7 is a continuous mapping from U into $L_1(G)$ (see Remarks 1,2).

Remark 5. Suppose the assumptions of Theorem 7 are fulfilled, let H be an open subset of $L_p(G)$. If there is a constant C such that the inequality (11) holds for all $x \in H$, then h is a demicontinuous operation from H into the space $L_q(G)$.

Theorem 8. Let $x_0 \in L_p(G)$, $D \subset L_{q'}(G)$, let the linear hull of the set D be dense in the space $L_{q'}(G)$. Suppose h maps certain neighbourhood U of the point x_0 in $L_p(G)$ onto a set M which is bounded in $L_q(G)$. Let there exist a number $\delta_{\varphi,\xi} > 0$ and a function $N_{\varphi,\xi} \in L(G)$ for every $\varphi \in D$ and $\xi \in L_p(G)$ with $\|\xi\|_{L_p} = 1$ such that

$$(12) \qquad |g(t, x_0(t) + \tau\xi(t))\varphi(t)| \leq N_{\varphi,\xi}(t)$$

for almost every $t \in G$ and every $\tau \in (0, \delta_{\varphi,\xi})$. Then h is an operation from U into $L_q(G)$ which is hemicontinuous at the point x_0.

Proof. Let $\varphi \in D$, $\xi \in L_p(G)$, $\|\xi\|_{L_p} = 1$. It follows from continuity of the function $g(t, \cdot)$ on E_1 that

$$g(t, x_0(t) + \tau\xi(t)) \to g(t, x_0(t))$$

for almost every $t \in G$ whenever $\tau \to 0$. From (12) and according to Theorem on continuous dependence of integral by parameter, we have that

$$\int_G h(x_0 + \tau\xi)(t) \cdot \varphi(t)\,dt \to \int_G h x_0(t) \cdot \varphi(t)\,dt$$

for $\tau \to 0$. This relation means that

$$\langle h(x_0 + \tau\xi), \varphi \rangle \to \langle h x_0, \varphi \rangle$$

whenever $\tau \to 0$ for all $\xi \in L_p(G)$ with $\|\xi\|_{L_p} = 1$ and for every $\varphi \in D$. Since the linear hull of D is dense in $L_{q'}(G)$ and M is bounded, the mapping h is hemicontinuous at the point x_0 .

The following two theorems are implied by the preceding theorem and we present them without the proofs (compare Theorems 4,5).

Theorem 9. Let $x_0 \in L_r(G)$, suppose there are such a constant C and a neighbourhood U of the point x_0 in $L_r(G)$ such that

$$\int_G |g(t, x(t))|^q dt \leq C$$

for all $x \in U$. Let there exist a number $\sigma_\xi > 0$ and a function $N_\xi \in L(G)$ for every $\xi \in L_r(G)$ with $\|\xi\|_{L_r} = 1$ such that

$$|g(t, x_0(t) + \tau\xi(t))| \leq N_\xi(t)$$

for almost every $t \in G$ and every $\tau \in (0, \sigma_\xi)$. Then h is an operation from U into $L_q(G)$ hemicontinuous at the point x_0 .

Theorem 10. Let H be an open set in the space $L_r(G)$, suppose h maps H onto a bounded subset of $L_q(G)$. Let there exist a number $\sigma_{x,\xi} > 0$ and a function $N_{x,\xi} \in L(G)$ for every $x \in H$ and $\xi \in L_r(G)$ with $\|\xi\|_{L_r} = 1$ such that

$$|g(t, x(t) + \tau\xi(t))| \leq N_{x,\xi}(t)$$

for almost every $t \in G$ and all $\tau \in (0, \sigma_{x,\xi})$. Then h is a hemicontinuous operation from the set H into the space $L_q(G)$.

R e f e r e n c e s

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(Received December 22, 1967)