Svatopluk Fučík

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FIXED POINT THEOREMS FOR SUM OF NONLINEAR MAPPINGS

Svatopluk FUČÍK, Praha

I. Introduction. Let H be a real Hilbert space, K a closed bounded convex subset of H. The following theorem has been obtained by R.I. Kačurovskij, M.A. Krasnoselskij and P.P. Zabrejko [9]:

Theorem 1. Let $T(K) \subset K$ and $T = B + C$, where $\|Bx - By\| \leq q\|x-y\|$ for all $x, y \in K$. Let one of the following conditions be fulfilled:

a) $0 \leq q < 1$, C completely continuous,

b) $q = 1$, C strongly continuous.

Then T has a fixed point in K (i.e. there exists at least one point $x_0 \in K$ such that $Tx_0 = x_0$).

Using the results of F.E. Browder and G.D. de Figueiredo concerning G-operators ([4],[6],[7],[8]) we can state the last Theorem in a more general setting. The extensions of Theorem 1 that we give below follow two directions (see Theorem 5,6 and 7). In the first place, we assume special Banach spaces (not only a Hilbert space) and the second direction of generalization is the weakening of the assumption $T(K) \subset K$.

In Section V there are given some examples of operators, which are the sum of two mappings, which map the unit ball into itself and have no fixed point property.

II. Terminology, notations and definitions.

Let X be a real Banach space with the norm $\|\cdot\|$, θ its zero element; X^* denotes the adjoint (dual) space of all bounded linear functionals on X. The pairing between $x^* \in X^*$ and $x \in X$ is denoted by (x, x^*). We shall use the symbols " $\longrightarrow$ ", " $\rightharpoonup$ " to denote the strong convergence in X (or in X^*) and weak convergence in X (or in X^*), respectively.

Definition 1: Let F be a mapping with domain $D \subset X$ and values in X (or X^*). Then

(1) F is said to be strongly continuous if $x_n \rightharpoonup x_o$ in D implies $Fx_n \longrightarrow Fx_o$.

(2) F is said to be weakly continuous if $x_n \rightharpoonup x_o$ in D implies $Fx_n \rightharpoonup Fx_o$.

(3) F is said to be continuous if $x_n \longrightarrow x_o$ in D implies $Fx_n \longrightarrow Fx_o$.

(4) F is said to be completely continuous on D if for each bounded subset $M \subset D$, $F(M)$ is compact and F is continuous on D.

(5) F is said to be nonexpansive mapping on D if for every $x, y \in D$ there is $\|Fx - Fy\| \leq \|x - y\|$.

(6) F is said to be contractive mapping on D if there exists q $(0 \leq q < 1)$ such that for every $x, y \in D$ we have $\|Fx - Fy\| \leq q\,\|x - y\|$.

(7) F is said to be hemicontinuous mapping on D if F is continuous from each segment in D to weak topology in X.

Definition 2. A Banach space X is said to be strictly convex if $\|\lambda x + (1-\lambda)y\| < 1$ for all λ, $0 < \lambda < 1$, and all $x, y \in X$ with $\|x\| = \|y\| = 1$.

Definition 3 ([6],[7],[8]): A Banach space X is said to have Property (π_1) if there exists a collection of finite dimensional subspaces F, $F \in \Lambda$, such that:

(8) The collection $\{F; F \in \Lambda\}$ is directed by inclusion. That is, given any two elements F_α, $F_\beta \in \Lambda$, there exists a third one which contains both.

(9) The union of all F, $F \in \Lambda$ is dense in X.

(10) Each F, $F \in \Lambda$ is the range of a continuous linear projection P_F of norm ≤ 1.

Remark 1: Hilbert space (separable or not), Banach space with monotone Schauder basis and $C[0,1]$ have Property (π_1). (See [6],[7],[8]).

Definition 4 ([6],[7],[8]) a) A gauge function is a real-valued continuous function μ defined in the interval $\langle 0,\infty)$ such that

(11) $\mu(0) = 0$

(12) $\lim_{t\to\infty} \mu(t) = \infty$

(13) μ is strictly increasing.

b) The duality mapping in X with a gauge function μ is a mapping J from X into the set 2^{X^*} of all subsets of X^* such that

(14) $$Jx = \begin{cases} \{\theta^*\} & x = \theta \\ \{x^*, x^* \in X^*, (x, x^*) = \|x\| \cdot \|x^*\|, \|x^*\| = \mu(\|x\|)\} & x \neq \theta . \end{cases}$$

Remark 2 ([6],[8]) a) The set Jx is non-empty.

b) Let X be a Banach space with a strictly convex dual space X^*. Let J be the duality mapping in X with a gauge function μ . Then the set Jx consists of precisely one point. c) Let X be a Banach space with strictly convex dual space X^*. Let $J: X \to X^*$ be the duality mapping with a gauge function μ and $t > 0$. Then $J(tw) = \beta(t) Jw$, where β is positive function of t.

Definition 5: A Banach space X is said to have Property $(\pi_1)^*$ if

(15) X is reflexive.

(16) X^* is strictly convex.

(17) X has Property (π_1).

(18) The duality mapping J in X with gauge function μ is weakly continuous.

Remark 3 ([4],[8]): A Hilbert space, ℓ_p $(1 < p < \infty)$ have Property $(\pi_1)^*$. The Banach space $L_p[0,1]$, $1 < p < \infty$, $p \neq 2$ has not Property $(\pi_1)^*$.

Definition 6: Let K be a closed bounded convex subset of a Banach space X with Property (π_1). An operator $T: K \to X$ is said to be Galerkin approximable (or for short a G-operator) if

(19) $P_F T: K \cap F \to F$ is continuous for all but a finite number of $F \in \Lambda$.

(20) T has a fixed point in K whenever there exists $x_F \in F$ for all but a finite number of $F \in \Lambda$ such that $P_F T x_F = x_F$.

Remark 4 ([6],[7],[8]) Let K be a convex closed bounded subset of a Banach space X with Property $(\pi_1)^*$. Let $T: K \to X$ be strongly (or weakly or completely) continuous. Then T is G-operator.

III. G-operators.

Lemma 1. Let X be a Banach space with X^* strictly convex. Let J be a duality mapping with a gauge function μ. Suppose that T is a hemicontinuous mapping of an open set M of X into X. Let $u_0 \in M$ and $w_0 \in X$ be such elements of X that for each $u \in M$ there is

$$(21) \qquad (Tu - w_0, J(u - u_0)) \geq 0 .$$

Then $w_0 = Tu_0$.

Proof: (The proof is similar to that of Lemma 1 [2]). For $t > 0$, set $u_t = u_0 + t(w_0 - Tu_0)$ (t is sufficiently small). Replacing in (21) u by u_t, we obtain according to property c) of Remark 2 that $(Tu_t - w_0, J(w_0 - Tu_0)) \geq 0$, i.e. $(Tu_t - Tu_0 + Tu_0 - w_0, J(w_0 - Tu_0)) \geq 0$

$$(Tu_t - Tu_0, J(w_0 - Tu_0)) \geq (w_0 - Tu_0, J(w_0 - Tu_0)) =$$
$$= \|w_0 - Tu_0\| \cdot \mu(\|w_0 - Tu_0\|) .$$

By hemicontinuity of T the left-hand side goes to 0 as $t \to 0$. The right-hand side is independent of t. Hence $w_0 = Tu_0$. Q.E.D.

Theorem 2. Let X be a Banach space with Property $(\pi_1)^*$, K a closed bounded convex subset of X, $M \supset K$ an open subset of X, $A : K \to X$ a completely continuous (resp. strongly continuous) operator and $B : M \to X$ a contractive (resp. nonexpansive) mapping.

Then $T = A + B$ is G-operator.

Proof: Condition (19) is clear.

a) Let A be completely continuous, B contractive mapping and $P_F Tx_F = P_F Ax_F + P_F Bx_F = x_F$ for all $F \in \Lambda$. For each $x, y \in M$ and $F \in \Lambda$ there is

$$((I-P_F B)x-(I-P_F B)y, J(x-y)) \geq (1-q)\|x-y\|\mu(\|x-y\|) .$$

Let be $y \in M$ arbitrary but fixed. Using a standard argument (see the proof of Proposition 1 [7] or Theorem IV.3 [8]) we can prove that there exists the sequence $\{x_n ; x_n \in F_n\} \subset \subset \{x_F ; F \in \Lambda\}$ such that

$$x_n \to x_o, \; Ax_n \to u, \; P_{F_n} Ax_n \to u, \; P_{F_n} By \to By .$$

For all natural number n and for each $x \in M$ we have

$$(22) \quad ((I-P_{F_n} B)x-(I-P_{F_n} B)y, \; J(x-y)) \geq$$
$$\geq (1-q)\|x-y\|\mu(\|x-y\|) .$$

Replacing in (22) x by x_n, we obtain $(P_{F_n} - P_n)$

$$((I-P_n B)x_n-(I-P_n B)y, \; J(x_n-y)) \geq 0 .$$

Because $(I-P_n B)x_n \to u$, $(I-P_n B)y \to (I-B)y$ and $J(x_n-y) \to J(x_o-y)$ we obtain $(u-(I-B)y, J(x_o-y)) \geq 0$ for each $y \in M$.

Using Lemma 1 we have, that

$$(23) \qquad (I-B)x_o = u .$$

From (22) it follows that

$$0 \leq \limsup_{n\to\infty} (1-q)\|x_n-x_o\|\mu(\|x_n-x_o\|) \leq$$
$$\leq \lim_{n\to\infty} ((I-P_n B)x_n-(I-P_n B)x_o, \; J(x_n-x_o)) = 0 ,$$

i.e., $x_n \to x_o$. Since A is continuous and (23) holds, we have that $(I-B)x_o = Ax_o$.

This completes the proof of a).

b) Let A be strongly continuous and B nonexpansive mapping.

The proof of this part is analogous to that of a).

By strongly continuity of A we have $u = Ax_o$ and by (23) we obtain $(I-B)x_o = Ax_o$. Q.E.D.

IV. Fixed Point Theorems

The following two theorems are due to D.G. de Figueirero [6],[7],[8]:

Theorem 3: Let K be a closed bounded convex subset of a Banach space X with Property (π_1). Let $T: K \to X$ be a G-operator defined in K. Assume that

(24) θ belongs to the interior of $K \cap F$, for all but a finite number of $F \in \Lambda$.

(25) For all but a finite number of $F \in \Lambda$ we have $P_F(Tx - \lambda x) \neq \theta$, for all $\lambda > 1$ and all $x \in \partial K \cap F$ (∂K is boundary of K).

Then T has a fixed point.

Theorem 4: Let K be a closed bounded convex subset of a Banach space X with Property (π_1). Let $T: K \to X$ be a G-operator defined in K. Assume that (24) is fulfilled and

(26) $(Tx, Jx) \leq \|x\| \mu(\|x\|)$, for all $x \in \partial K$.

Then T has a fixed point in K.

Applying Theorem 3 and 4 to our results (Section III) we obtain immediately the following fixed-point theorems.

Theorem 5: Let X be a Banach space with Property $(\pi_1)^*$, K a closed bounded convex subset of X, $M \supset K$ an open subset of X, $A: K \to X$ a completely continuous (resp. strongly continuous) operator and $B: M \to X$ a contractive (resp. nonexpansive) mapping. Set $T = A + B$. Assume that (24) and (25) (or (24) and (26)) are fulfilled. Then T has a fixed point in K.

Lemma 2 ([3]): Let K be a closed convex bounded subset of a Hilbert space H. Then there exists an operator $L: H \to K$ such that

(27) $Lx = x$ for each $x \in K$.

(28) $\| Lx - Ly \| \leq \| x - y \|$ for each $x, y \in H$.

Theorem 6: Let K be a closed bounded convex subset of a Hilbert space H , $A : K \to H$ a completely continuous (resp. strongly continuous) operator and $B : K \to H$ a contractive (resp. nonexpansive) mapping.
Set $T = A + B$. Assume that (24) and (25)(or (24) and (26)) are fulfilled with $J = I$ (I is identity operator).
Then T has a fixed point in K .

Proof: For $x \in H$, set $\tilde{B}x = BLx$. Using Theorem 5 we have $M = H$ and $\tilde{B}$ is contractive (resp. nonexpansive) mapping. Theorem 5 and Lemma 2 proved Theorem 6.

Theorem 7: Let A , B and T have the same properties as in Theorem 6. Assume that $T(\partial K) \subset K$. Suppose that (24) holds.
Then T has a fixed point in K .

Another fixed-point theorem for sum of operators has been proved by W.V. Petryshyn [12]:

Theorem 8: Let H be a complex Hilbert space, F a hemicontinuous mapping from H to H such that

(29) $|(Fx - Fy ; x - y)| \geq \beta \| x - y \|^2$

holds for every x and $y \in H$ and some constant $\beta > 0$.
Let S be a completely continuous mapping such that $(1/\beta)\{Sx - F\theta\}$ maps the ball $B_r = \{x ; x \in H, \|x\| \leq r\}$ into B_r . Set $T = I - F + S$.

Then T has a **fixed** point in B_r .

Remark 5 : M. Altman [1] has proved the following

Theorem 9: Let H be a separable Hilbert space, F weakly closed (i.e. if $x_n \rightharpoonup x_0$, $Fx_n \rightharpoonup y$ then $y = Fx_0$) and maps unit ball into bounded subset of H . If

$$(Fx, x) \leq (x, x) \quad \text{for each } x \text{ with } \|x\| = 1,$$

then F has a fixed point.

It was shown [11], that weakly compact and weakly closed mapping is weakly continuous. Hence the assumptions of Altman's Theorem say that F is weakly continuous.

V. Examples

Let H be a Hilbert space, K unit ball, $T: K \to K$ such that $T = A + B$.

The author investigated the question concerning the fixed point of T , when A and B are from the class of mappings which contains strongly continuous, completely continuous, weakly continuous, nonexpansive and contractive operators. From next examples it follows that T has fixed point property only if A is completely continuous (resp. strongly continuous) and B is contractive (resp. nonexpansive).

Example 1 (see [5],[10]):

Let H be a separable Hilbert space, $\{y_n; n = 0, \pm 1, \pm 2, \dots\}$ be an orthonormal basis for H and define the transformations A and B as follows:

$$x = \sum_{n=-\infty}^{+\infty} a_n y_n, \quad Bx = \sum_{n=-\infty}^{+\infty} a_n y_{n+1},$$

$$Ax = (1 - \|x\|) y_0 .$$

Set $T = A + B$. $TK \subset K$ and T has no fixed point in K . A is nonexpansive, completely continuous and B is weakly continuous and nonexpansive.

Example 2: Set $Ax = \frac{1}{2}(1 - \|x\|)y_0$ and B as in Example 1. Then $T = A + B$ transforms K into K and has no fixed point in K . A is completely continuous and contractive and B is weakly continuous and nonexpansive.

Example 3: Set $A_1 x = \frac{1}{3}(1 - \|x\|)y_0 + \frac{1}{2}Bx$ (B is from Example 1) and $B_1 x = \frac{1}{2}Bx$.

This example shows that A_1 is contraction and B_1 is contraction and weakly continuous, $T = A_1 + B_1$ maps K into K and has no fixed point in K .

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