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REDUCED DIMENSION OF PRIMITIVE CLASSES OF UNIVERSAL ALGEBRAS

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This paper is a continuation of my paper [1].

Let us define the reduced dimension of a primitive class $\mathcal{C}$ of algebras of (an infinitary) type τ as the least regular number ϑ^* such that $\mathcal{C}$ is equivalent to a primitive class of algebras of dimension ϑ^*. In this paper we shall find a necessary and sufficient condition for a primitive class to be of a reduced dimension $\leq \vartheta^*$ where ϑ^* is a given regular number; see Theorem 1 below. If $\vartheta^* = \aleph_0$, then this result can be strengthened; see Theorem 2.

Theorem 2 follows easily from Theorem 1 and "Hauptsatz über algebraische Hüllensysteme" (J. Schmidt [2],p.25). However, we shall give an independent proof of Theorem 2, not requiring any of the two theorems.

<u>Lemma</u>. Let $\mathbb{A}$ be an algebra of type τ (dimension ϑ) with an independent set of generators X of cardinality $\geq \vartheta$. Let $\mathbb{A}^*$ be an algebra of type τ^* (dimension ϑ^*) such that $A = A^*$. Let each fundamental operation of $\mathbb{A}^*$ be algebraic in $\mathbb{A}$ and

(1) $\quad C_{\mathbb{A}}(M) = C_{\mathbb{A}^*}(M) \qquad$ for all $M \subseteq X$.

Then the algebras $\mathbb{A}$, $\mathbb{A}^*$ are equivalent.

Proof. It is sufficient to prove that each fundamental operation of $\mathbb{A}$ is algebraic in $\mathbb{A}^*$ (see [1], theorem 3). Let $i \in I$. There exists an injection $\mathfrak{a}$ of K_i into X. Put $a = f_i(\mathfrak{a})$. By (1) we get $a \in C_{\mathbb{A}^*}(W(\mathfrak{a}))$. By Corollary 1 of Theorem 5 of [3] there exists an algebraic operation $h \in H^{K_i}(\mathbb{A}^*)$ such that $a = h(\mathfrak{a})$. By our assumption, $h \in H^{K_i}(\mathbb{A})$. Hence, both f_i and h are algebraic in $\mathbb{A}$ and $f_i(\mathfrak{a}) = h(\mathfrak{a})$; as the set $W(\mathfrak{a})$ is independent in $\mathbb{A}$, we get $f_i = h$ by Corollary 1 of Theorem 11 of [3]. As h is algebraic, f_i is algebraic in $\mathbb{A}^*$, too.

Let λ be an infinite cardinal number. A set M of sets is called λ-directed if for all $N \subseteq M$ such that $\mathit{Card}\ N < \lambda$ there exists an $A \in M$ with $B \subseteq A$ for all $B \in N$. (Every λ-directed set is evidently non-empty.) A set is called directed if it is $\aleph_0$-directed. Every non-empty chain of sets is directed.

Theorem 1. Let $\mathcal{O}\!\iota$ be a non-trivial primitive class of algebras of type τ (dimension ϑ). Let ϑ^* be a regular number. Let X be a set of cardinality $\geq \max(\vartheta, \vartheta^*)$ and $\mathbb{C}$ an $\mathcal{O}\!\iota$-free algebra with $\mathcal{O}\!\iota$-basis X. The following conditions are equivalent:

(i) $\mathcal{O}\!\iota$ is equivalent to a primitive class of algebras of dimension ϑ^*.

(ii) If $\mathbb{A} \in \mathcal{O}\!\iota$, then the union of any ϑ^*-directed set of sets closed in $\mathbb{A}$ is also closed in $\mathbb{A}$.

(iii) The union of any ϑ^*-directed set of sets closed in $\mathbb{C}$ is also closed in $\mathbb{C}$.

Proof. (i) $\Longrightarrow$ (ii): well-known and easy. (ii) $\Longrightarrow$ (iii): evident. (iii) $\Longrightarrow$ (i): Let us define a type τ^* in this way: its domain I^* is the set of all ordered pairs $\langle M, c \rangle$ such that $M \subset X$, $\mathit{Card}\ M < \vartheta^*$ and $c \in C_{\mathfrak{C}}(M)$; if $i = \langle M, c \rangle \in I^*$, then put $K_i^* = M$. Evidently, ϑ^* is the dimension of τ^*. Let us define an algebra $\mathfrak{C}^*$ of type τ^* with $C^* = C$ in this way: if $i = \langle M, c \rangle \in I^*$, then there exists (by [3], Corollary 1 of Theorem 5 and Corollary 1 of Theorem 11) exactly one algebraic operation $h \in H^{K_i^*}(\mathfrak{C}) = H^M(\mathfrak{C})$ such that $h(id_M) = c$ (where id_M denotes the identical mapping of M onto itself); put $h_i^* = h$ (the i-th fundamental operation of $\mathfrak{C}^*$).

Hence, each fundamental operation of $\mathfrak{C}^*$ is algebraic in $\mathfrak{C}$.

Let $M \subseteq X$. Put

(2) $$\mathcal{D} = \{ N ;\ N \subseteq M \ \&\ \mathit{Card}\ N < \vartheta^* \}$$

and

(3) $$E = \{ C_{\mathfrak{C}}(N) ;\ N \in \mathcal{D} \} .$$

If $N \in \mathcal{D}$, then it follows easily from the independence of X that $X \cap C_{\mathfrak{C}}(N) = N$. Hence, the mapping φ defined by $\varphi(N) = C_{\mathfrak{C}}(N)$ is a one-to-one mapping of $\mathcal{D}$ onto E and it is an order-isomorphism if we consider $\mathcal{D}$ and E as partially ordered by the set-theoretic inclusion. The set $\mathcal{D}$ is ϑ^*-directed because ϑ^* is regular; hence, also the set E is ϑ^*-directed. By our assumption (iii) we get that the union $\bigcup_{N \in \mathcal{D}} C_{\mathfrak{C}}(N)$ is closed in $\mathfrak{C}$. As M is evidently contained in this union, we get

(4) $$C_{\mathfrak{C}}(M) = \bigcup_{N \in \mathcal{D}} C_{\mathfrak{C}}(N) .$$

Let us prove

(5) $\qquad C_{\mathfrak{C}}(M) = C_{\mathfrak{C}^*}(M)$.

The inclusion " $\supseteq$ " is trivial. Let $a \in C_{\mathfrak{C}}(M)$. By (4) there exists an $N \in \mathcal{D}$ such that $a \in C_{\mathfrak{C}}(N)$. Put $i = \langle N, a \rangle$. As $N \subset X$ and $\mathit{Card}\ N < \vartheta^*$, we get $i \in I^*$. By the construction of h_i^* we get $a = h_i^*(id_N)$. Hence, $a \in C_{\mathfrak{C}^*}(N) \subseteq C_{\mathfrak{C}^*}(M)$.
We have proved (5).

Conditions of the lemma are thus satisfied and we infer that the algebras $\mathfrak{C}$, $\mathfrak{C}^*$ are equivalent. Hence, X is also an independent set of generators of $\mathfrak{C}^*$. There exists exactly one primitive class $\mathcal{L}$ such that $\mathfrak{C}^*$ is $\mathcal{L}$ -free with $\mathcal{L}$ -basis X . By Theorem 6 of [1] the classes $\mathcal{O}\!\mathcal{L}$, $\mathcal{L}$ are equivalent.

<u>Theorem 2</u>. Let $\mathcal{O}\!\mathcal{L}$ be a non-trivial primitive class of algebras of type τ (dimension ϑ). Let X be a set of cardinality $\geq \vartheta$ and $\mathfrak{C}$ an $\mathcal{O}\!\mathcal{L}$ -free algebra with $\mathcal{O}\!\mathcal{L}$ -basis X . The following conditions are equivalent:

(i) $\mathcal{O}\!\mathcal{L}$ is equivalent to a primitive class of finitary algebras.

(ii) If $\mathfrak{A} \in \mathcal{O}\!\mathcal{L}$, then the union of any non-empty well-ordered chain of sets closed in $\mathfrak{A}$ is also closed in $\mathfrak{A}$.

(iii) The union of any non-empty well-ordered chain of sets closed in $\mathfrak{C}$ is also closed in $\mathfrak{C}$.

<u>Proof</u>. (i) $\Longrightarrow$ (ii) and (ii) $\Longrightarrow$ (iii) is easy.
(iii) $\Longrightarrow$ (ii): Construct τ^* and $\mathfrak{C}^*$ as in the proof of Theorem 1. Let us prove by transfinite induction that

for each cardinal number α the following holds:

(6) If $M \subseteq X$ and $Card\ M = \alpha$, then $C_{\mathfrak{C}}(M) = C_{\mathfrak{C}^*}(M)$.

If α is finite, we can repeat the proof of (5) if we put there $N = M$. Let α be infinite and let (6) hold for all cardinal numbers less than α. As $Card\ M = \alpha$, there exists a one-to-one mapping η of α onto M (recall that α is the set of all ordinal numbers less than α). Evidently,

$$(7) \qquad C_{\mathfrak{C}}(M) = C_{\mathfrak{C}}\left(\bigcup_{\gamma<\alpha} \eta''\gamma\right)$$

(where $\eta''\gamma$ denotes the range of $\eta \restriction \gamma$). The set of all $C_{\mathfrak{C}}(\eta''\gamma)$ for $\gamma < \alpha$ is evidently a non-empty well-ordered chain of sets closed in $\mathfrak{C}$; hence, its union is closed in $\mathfrak{C}$ and thus evidently

$$(8) \qquad C_{\mathfrak{C}}\left(\bigcup_{\gamma<\alpha} \eta''\gamma\right) = \bigcup_{\gamma<\alpha} C_{\mathfrak{C}}(\eta''\gamma) .$$

If $\gamma < \alpha$, then $Card(\eta''\gamma) = Card\ \gamma < \alpha$ because α is a cardinal number; by the inductional assumption we have $C_{\mathfrak{C}}(\eta''\gamma) = C_{\mathfrak{C}^*}(\eta''\gamma)$. Hence,

$$(9) \qquad \bigcup_{\gamma<\alpha} C_{\mathfrak{C}}(\eta''\gamma) = \bigcup_{\gamma<\alpha} C_{\mathfrak{C}^*}(\eta''\gamma) .$$

As $\mathfrak{C}^*$ is finitary, we get

$$(10) \qquad \bigcup_{\gamma<\alpha} C_{\mathfrak{C}^*}(\eta''\gamma) = C_{\mathfrak{C}^*}\left(\bigcup_{\gamma<\alpha} \eta''\gamma\right) = C_{\mathfrak{C}^*}(M) .$$

By (7),(8),(9) and (10) we get (6). The proof of (iii) $\Longrightarrow$ $\Longrightarrow$ (i) can be finished similarly as in Theorem 1.

References

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