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NETS AND GROUPOIDS, II x)

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In the sequel we shall introduce and analyse the notion of a general net which has been suggested by the final remarks in [2].

Definition 1. A (general) net is defined here as a quadruplet $(\mathbb{P}, \mathcal{X}, \mathcal{Y}, \mathcal{Z})$ where $\mathbb{P}$ is a set and $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$ are partitions on $\mathbb{P}$. We shall restrict ourselves to nets $(\mathbb{P}, \mathcal{X}, \mathcal{Y}, \mathcal{Z})$ such that $card\ \mathcal{X} = card\ \mathcal{Y} \geq card\ \mathcal{Z}$ and $card(X \cap Y \cap Z) \leq 1$ for all $X \in \mathcal{X}$, $Y \in \mathcal{Y}$, $Z \in \mathcal{Z}$. ($\mathbb{P}$: the set of the points, $\mathcal{X} \cup \mathcal{Y} \cup \mathcal{Z}$: the set of the lines, $\mathcal{X}$: the set of the $\mathcal{X}$-lines, $\mathcal{Y}$: the set of the $\mathcal{Y}$-lines, $\mathcal{Z}$: the set of the $\mathcal{Z}$-lines.)

Definition 1a. Two nets $(\mathbb{P}^{(i)}, \mathcal{X}^{(i)}, \mathcal{Y}^{(i)}, \mathcal{Z}^{(i)})$, $i = 1, 2$, are said to be isomorphic if there is a bijection $\sigma: \mathbb{P}^{(1)} \to \mathbb{P}^{(2)}$ such that $X \in \mathcal{X}^{(1)} \Longrightarrow \sigma X \in \mathcal{X}^{(2)}$, $Y \in \mathcal{Y}^{(1)} \Longrightarrow \sigma Y \in \mathcal{Y}^{(2)}$, $Z \in \mathcal{Z}^{(1)} \Longrightarrow \sigma Z \in \mathcal{Z}^{(2)}$.

Definition 2. A multigroupoid is defined as a couple (S, μ) where S is a non-void set and μ a map of $S \times S$ into $\mathfrak{P}(S)$. We shall restrict ourselves to multigroupoids

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x) Part I in CMUC 8,3(1967),pp.435-451.

(S, μ) such that to every $a \in S$ there exist $a', a'' \in S$ satisfying $\mu(a', a) \neq \emptyset \neq \mu(a, a'')$.

<u>Definition 2a</u>. Two multigroupoids $(S^{(i)}, \mu^{(i)})$, $i = 1, 2$, are said to be <u>isotopic</u> if there exist bijections

$$\alpha : S^{(1)} \to S^{(2)}, \ \beta : S^{(1)} \to S^{(2)}, \ \gamma : \bigcup_{(x,y) \in S^{(1)} \times S^{(1)}} \mu^{(1)}(x, y) \to \bigcup_{(x,y) \in S^{(2)} \times S^{(2)}} \mu^{(2)}(x, y)$$

such that $\mu^{(2)}(\alpha a, \beta b) = \gamma \mu^{(1)}(a, b)$ for all $a, b \in S^{(1)}$.

<u>Construction 1</u>. Let $\mathcal{N} = (\mathbb{P}, \mathcal{X}, \mathcal{Y}, \mathcal{Z})$ be a net. Choose maps $\xi : \mathcal{X} \to S$ (a bijection), $\eta : \mathcal{Y} \to S$ (a bijection) and $\zeta : \mathcal{Z} \to S$ (an injection) where S is a set with $\mathrm{card}\, S = \mathrm{card}\, \mathcal{X} = \mathrm{card}\, \mathcal{Y}$. Now define the map $\mu : S \times S \to \mathcal{P}(S)$ in such a way that $\mu(a, b) := \{c \in S \mid \xi^{-1}(b) \cap \eta^{-1}(a) \zeta^{-1}(c) \neq \emptyset\}$ for all $a, b \in S$. Then (S, μ) is a multigroupoid in our sense.
$((S, \mu) =: \mathcal{G}_{\xi, \eta, \zeta}(\mathcal{N}))$

<u>Construction 2</u>. Let $G = (S, \mu)$ be a multigroupoid. Start from $S \times S$ and substitute each $(a, b) \in S \times S$ by the set $S_{a,b}$ where (i) there exists a bijection $\varphi_{a,b} : \mu(a, b) \to S_{a,b}$ for all $a, b \in S$ and (ii) $S_{a_1,b_1} \cap S_{a_2,b_2} = \emptyset$ for any distinct couples (a_1, b_1), $(a_2, b_2) \in S \times S$. Now define $\mathcal{X} := \{\bigcup_{a \in S} S_{a,b} \mid b \in S\}$, $\mathcal{Y} := \{\bigcup_{b \in S} S_{a,b} \mid a \in S\}$, $\mathcal{Z} := \{\bigcup_{\substack{\forall (a,b) \in S \times S \\ \text{with } c \in \mu(a,b)}} \{\varphi_{a,b}(c)\} \mid c \in \mu(S \times S)\}$.
By assumptions about G, the sets $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$ must be partitions on $\mathbb{P} := \bigcup_{(a,b) \in S \times S} S_{a,b}$, $\mathrm{card}\, \mathcal{X} = \mathrm{card}\, \mathcal{Y} \geq \mathrm{card}\, \mathcal{Z}$ and $\mathrm{card}\,(X \cap Y \cap Z) \leq 1$ for all $(X, Y, Z) \in \mathcal{X} \times \mathcal{Y} \times \mathcal{Z}$. Thus $(\mathbb{P}, \mathcal{X}, \mathcal{Y}, \mathcal{Z})$ is a net in our sense.
$((\mathbb{P}, \mathcal{X}, \mathcal{Y}, \mathcal{Z}) =: \mathcal{N}_{\{\varphi_{a,b}\}_{(a,b) \in S \times S}}(G))$

Theorem 1. If $\mathcal{N}=(\mathbb{P},\mathcal{X},\mathcal{Y},\mathcal{Z})$ is a net then each $\mathcal{N}_{\{\varphi_{a,b}\}(a,b)\in S\times S}(\mathcal{G}_{\xi,\eta,\zeta}(\mathcal{N}))$ is isomorphic to $\mathcal{N}$. If $G=(S,\mu)$ is a multigroupoid then $\mathcal{G}_{\xi,\eta,\zeta}(\mathcal{N}_{\{\varphi_{a,b}\}(a,b)\in S\times S}(G))$ is isotopic to G.

Proof. Let $\mathcal{N}=(\mathbb{P},\mathcal{X},\mathcal{Y},\mathcal{Z})$ be a net. Construct $G:=\mathcal{G}_{\xi,\eta,\zeta}(\mathcal{N})=(S,\mu)$ and $\mathcal{N}':=\mathcal{N}_{\{\varphi_{a,b}\}(a,b)\in S\times S}(G)=(\mathbb{P}',\mathcal{X}',\mathcal{Y}',\mathcal{Z}')$. Now let us define the map $\sigma:\mathbb{P}\to\mathbb{P}'$ in such a way that to each $p\in\mathbb{P}$ we associate $p'=\varphi_{\xi X,\eta Y}(\zeta Z)$ where X, Y, Z are determined by $p\in X\in\mathcal{X}$, $p\in Y\in\mathcal{Y}$, $p\in Z\in\mathcal{Z}$. We easily see that σ realizes an isomorphism between $\mathcal{N}$ and $\mathcal{N}'$.

Secondly, let $G=(S,\mu)$ be a multigroupoid. Construct $\mathcal{N}:=\mathcal{N}_{\{\varphi_{a,b}\}(a,b)\in S\times S}(G)=(\mathbb{P},\mathcal{X},\mathcal{Y},\mathcal{Z})$ and $G':=\mathcal{G}_{\xi,\eta,\zeta}(\mathcal{N})=(S',\mu')$. Now define maps $\alpha:S\to S'$, $\beta:S\to S'$ and $\gamma:\bigcup_{(x,y)\in S\times S}\mu(x,y)\to\bigcup_{(x',y')\in S'\times S'}\mu(x',y')$ in such a way that $\alpha a:\eta(\bigcup_{b\in S}S_{a,b})$ for all $a\in S$, $\beta b:=\xi(\bigcup_{a\in S}S_{a,b})$ for all $b\in S$ and $\gamma c:=$

$$=\zeta\Big(\bigcup_{\substack{(a,b)\in S\times S\\ \text{with } c\in\mu(a,b)}}\{\varphi_{a,b}(c)\}\Big)\text{ for all } c\in\bigcup_{(x,y)\in S\times S}\mu(x,y).$$

As

$$c\in\mu(a,b)\iff\gamma c\in\mu'(\alpha a,\beta b),\qquad(\alpha,\beta,\gamma)$$

represents an isotopy between G and G'. Q.E.D.

Theorem 2. Let $\mathcal{N}^{(i)}=(\mathbb{P}^{(i)},\mathcal{X}^{(i)},\mathcal{Y}^{(i)},\mathcal{Z}^{(i)})$, $i=1,2$, be nets. Then $\mathcal{N}^{(i)}$, $i=1,2$, are isomorphic iff $\mathcal{G}_{\xi^{(i)},\eta^{(i)},\zeta^{(i)}}(\mathcal{N}^{(i)})$, $i=1,2$, are isotopic (for some, and consequently for all choices of $\xi^{(i)},\eta^{(i)},\zeta^{(i)}$).

Proof. Let $\sigma:\mathbb{P}^{(1)}\to\mathbb{P}^{(2)}$ be a map which mediates an isomorphism between $\mathcal{N}^{(1)}$ and $\mathcal{N}^{(2)}$. Then $(\xi^{(2)}\circ\sigma\circ\xi^{(1)^{-1}}$, $\eta^{(2)}\circ\sigma\circ\eta^{(1)^{-1}}$, $\zeta^{(2)}\circ\sigma\circ\zeta^{(1)^{-1}})$ represents the required isotopy between $(S^{(1)},\mu^{(1)})$ and $(S^{(2)},\mu^{(2)})$. Conversely, let

(α, β, γ) represent an isotopy between $(S^{(1)}, \mu^{(1)})$ and $(S^{(2)}, \mu^{(2)})$. Define the map $\sigma : \mathbb{P}^{(1)} \to \mathbb{P}^{(2)}$ in such a way that for each $p \in \mathbb{P}^{(1)}$ it holds $\{\sigma p\} = (\xi^{(2)^{-1}} \circ \beta \circ \xi^{(1)}) A \cap \cap (\eta^{(2)^{-1}} \circ \alpha \circ \eta^{(1)}) B \cap (\zeta^{(2)^{-1}} \circ \gamma \circ \zeta^{(1)}) C$ where $(A, B, C) \in \in \mathcal{X} \times \mathcal{Y} \times \mathcal{Z}$ is determined by $p \in A \cap B \cap C$. We easily verify that σ mediates an isomorphism between $\mathcal{N}^{(1)}$ and $\mathcal{N}^{(2)}$. Q.E.D.

<u>Theorem 3</u>. Let $\mathcal{N} = (\mathbb{P}, \mathcal{X}, \mathcal{Y}, \mathcal{Z})$ be a net. Then, for each $\mathcal{G}_{\xi,\eta,\zeta}(\mathcal{N}) = (S, \mu)$, the conditions

(1) $X \cap Y \neq \emptyset \quad \forall (X, Y) \in \mathcal{X} \times \mathcal{Y}$,

(2) $card(X \cap Y) = 1$ ——— " ———,

(3) $X \cap Z \neq \emptyset \quad \forall (X, Z) \in \mathcal{X} \times \mathcal{Z}$,

(4) $card(X \cap Z) = 1$ ——— " ———

are equivalent to

(1°) $\mu(a, b) \neq \emptyset \quad \forall (a, b) \in S \times S$,

(2°) $card\, \mu(a, b) = 1$ ——— " ———,

(3°) for each $(b, c) \in S \times \mu(S \times S)$ there exists an $a \in S$ such that $c \in \mu(a, b)$,

(4°) for each $(b, c) \in S \times \mu(S \times S)$ there exists exactly one $a \in S$ such that $c \in \mu(a, b)$, respectively.

The proof is obvious and may be omitted. - Denote by (3′) and (4′) respectively the analogon of (3) and (4) respectively for $\mathcal{Y}, \mathcal{Z}$ instead of $\mathcal{X}, \mathcal{Z}$. If (2) holds then (S, μ) is actually a groupoid, whereas if (2),(4) and (4′) are valid then (S, μ) is actually a quasigroup.

<u>Definition 3</u>. Let $\mathcal{N} = (\mathbb{P}, \mathcal{X}, \mathcal{Y}, \mathcal{Z})$ be a net. Let $a \mathcal{X} b$ denote that the points a, b lie on the same $\mathcal{X}$-line, and similarly for $\mathcal{Y}$ or $\mathcal{Z}$ instead of $\mathcal{X}$. By a <u>rectangle</u> in $\mathcal{N}$ we shall mean any quadruple a, b, c, d

of points such that $a \mathcal{Y} b$, $c \mathcal{Y} d$, $b \mathcal{X} c$, $a \mathcal{X} d$; denotation: $\mathcal{R}(a\,b\,c\,d)$. Further we introduce the following closure conditions:

(T) $\mathcal{R}(a\,b\,c\,d), \mathcal{R}(a'b'c'd'), b=b', a'\mathcal{Z}c, a\mathcal{Z}c' \Longrightarrow d\mathcal{Z}d'$,

(R) $\mathcal{R}(a\,b\,c\,d), \mathcal{R}(a'b'c'd'), a\mathcal{Z}a', b\mathcal{Z}b', c\mathcal{Z}c' \Longrightarrow d\mathcal{Z}d'$,

(B_1) $\mathcal{R}(a\,b\,c\,d), \mathcal{R}(a'b'c'd'), a'\mathcal{Y}c, a\mathcal{Z}a', b\mathcal{Z}b', c\mathcal{Z}c' \Rightarrow d\mathcal{Z}d'$,

(B_2) $\mathcal{R}(a\,b\,c\,d), \mathcal{R}(a'b'c'd'), b\mathcal{X}d', a\mathcal{Z}a', b\mathcal{Z}b', c\mathcal{Z}c' \Rightarrow d\mathcal{Z}d'$,

(B_3) $\mathcal{R}(a\,b\,c\,d), \mathcal{R}(a'b'c'd'), a\mathcal{Z}c, a\mathcal{Z}a', b\mathcal{Z}b', c\mathcal{Z}c' \rightarrow d\mathcal{Z}d'$,

and

(H) $\mathcal{R}(a\,b\,c\,d), \mathcal{R}(a'b'c'd'), a'=c, a\mathcal{Z}a', b\mathcal{Z}b', c\mathcal{Z}c' \Rightarrow d\mathcal{Z}d'$.

($\hat{T}$) $\mathcal{R}(a\,b\,c\,d), \mathcal{R}(a'b'c'd'), c=c', a\mathcal{Z}a', b\mathcal{Z}b' \Longrightarrow d\mathcal{Z}d'$.

Theorem 4. Let $\mathcal{N} = (\mathbb{P}, \mathcal{X}, \mathcal{Y}, \mathcal{Z})$ be a net satisfying (2),(3) and (3'). Then $(\hat{T}) \Rightarrow (R) \Rightarrow ((B_1) \& (B_2)) \Rightarrow (B_i) \Rightarrow (H)$ for $i = 1, 2, 3$.

The proof can be given similarly to that presented in [2], pp.397–402.

Theorem 5. Let $\mathcal{N} = (\mathbb{P}, \mathcal{X}, \mathcal{Y}, \mathcal{Z})$ be a net such that there is a $G := \mathcal{G}_{\xi,\eta,\zeta}(\mathcal{N}) = (S, \mu)$ with the following properties: G is a groupoid and there exist elements $x_0, y_0 \in S$ such that $\mu(x_0, x) = \mu(x, y_0)$ for all $x \in S$. Then $(T)\Big|_{\substack{a,a' \in \xi^{-1}(x_0) \\ c,c' \in \eta^{-1}(y_0)}}$ implies the commutivity of μ. If especially x_0 is a left unity for G and if $(T)\Big|_{a,a' \in \xi^{-1}(x_0)}$ holds then the associativity of μ follows.

Proof. Let us identify $\mathcal{N}$ with $\mathcal{N}(G)$ xx) in a

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xx) The subscript by $\mathcal{N}$ can be omitted.

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natural way. Further apply the first restriction of (T) introduced above for $a=(x_0,y)$, $a'=(x_0,x)$, $c(x,y_0)$, $c'=(y,y_0)$. Such an application is possible because of $a\,\mathcal{Z}\,c'$ and $a'\,\mathcal{Z}\,c$. It follows $(x,y)\,\mathcal{Z}\,(y,x)$, i.e., $\mu(x,y)=$ $=\mu(y,x)$. Now investigate the second part and apply the second restriction of (T) for $a=(x_0,\bar{y})$, $a'=(x_0,\bar{\bar{y}})$, $c=(x,y)$, $c'=(z,y)$ where $a\,\mathcal{Z}\,c'$ and $a'\,\mathcal{Z}\,c$ is assumed. Then $(x,\bar{y})\,\mathcal{Z}\,(z,\bar{\bar{y}})$, i.e. $\mu(x,\bar{y})=$ $=\mu(z,\bar{\bar{y}})$. But by our assumptions, $a\,\mathcal{Z}\,c'$, and this is equivalent with $\mu(x,\bar{y})=\mu(z,y)\Longleftrightarrow\bar{y}=\mu(z,y)$. Similarly $a'\,\mathcal{Z}\,c\Longleftrightarrow\mu(x_0,\bar{\bar{y}})=\mu(x,y)\Longleftrightarrow\bar{\bar{y}}=\mu(x,y)$. Thus $\mu(x,\mu(z,y))=\mu(z,\mu(x,y))$ and therefore $\mu(x,\mu(y,z))=\mu(\mu(x,y),z)$, by the commutativity of μ, valid by the first part of this proof. Q.E.D.

<u>Theorem 6</u>. Let $\mathcal{N}=(\mathbb{P},\mathcal{X},\mathcal{Y},\mathcal{Z})$ be a net such that some $\mathcal{G}_{\xi,\eta,\zeta}(\mathcal{N})=(S,\mu)$ is a groupoid having a unity 0. Then $(B_1)\big|_{a',c\in\eta^{-1}(0)}$ and $(B_1)\big|_{b',c'\in\xi^{-1}(0)}$ imply $\mu(y,\mu(z,\mu(y,x)))=\mu(\mu(y,\mu(z,y)),x)$, $\forall x,y,z\in S$.

The proof is similar to the proof in [2], pp.413-415.

<u>Theorem 7</u>. Let $\mathcal{N}=(\mathbb{P},\mathcal{X},\mathcal{Y},\mathcal{Z})$ be a net satisfying (2). Then the following two conditions are equivalent:

(5) $\exists(X_0,Y_0)\in\mathcal{X}\times\mathcal{Y}$ such that $card(X_0\cap Z)=card(Y_0\cap Z)$ for all $Z\in\mathcal{Z}$,

(6) $\exists\,\xi,\eta,\zeta$ such that there is an $e\in S$ commuting with all elements of S with regard to

$$\mathcal{G}_{\xi,\eta,\zeta}(\mathcal{N})=(S,\mu).$$

__Proof__. Let (5) hold. Choose ξ, η, ζ in such a manner that $\xi X_0 = \eta Y_0 \; (=: e)$ and that

$$\xi(\{X \in \mathfrak{X} \mid X \cap Y_0 \cap Z \neq \emptyset\}) = \eta(\{Y \in \mathcal{Y} \mid X_0 \cap Y \cap Z \neq \emptyset\})$$

for all $Z \in \mathcal{Z}$. Then the corresponding μ satisfies $\mu(e, x) = \mu(x, e) \;\; \forall x \in S$. The other implication follows by reversing the preceding investigation. Q.E.D.

__Remark__. The interesting relation between multigroupoids and their representing groupoids in the sense of [4],pp.41-42 may be utilized to obtain a new meaning of closure conditions in nets over special groupoids. This will be considered in another publication.

R e f e r e n c e s

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