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THE STRUCTURE OF TORSION-FREE RINGS

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(Preliminary communication)

Throughout this brief communication, R denotes an associative ring, not necessarily with unity and M a left R-module. Let us recall that an R-module N of M is called <u>essential</u> in M if

$N \cap X \neq 0$ for every non-zero R-submodule X of M,

and that M is called <u>uniform</u>, if every R-submodule $N \subseteq M$ is essential in M. By an <u>essential extension</u> of a ring R, we shall understand here any ring $\overline{R}$ containing the ring R as an essential R-submodule; R is then called the essential subring of $\overline{R}$. In accordance with [2] (or [3]), an R-module M is said to be <u>torsion-free</u> if there is no non-zero element of M whose order (annihilator) is essential in R; thus, a (left) torsion-free ring is just a ring with zero singular ideal in the terminology of R.E.JOHNSON [9]. Finally, a ring R containing a direct sum of its uniform (left) ideals which is essential in R will be called (left) <u>tidy</u>; alternatively, R is tidy if and only if each of its (left) ideals contains a uniform (left) ideal of R.

In its simplified form, the main result relating injective hulls of torsion-free tidy rings and matrix rings can be presented as follows (cf.[10]).

Theorem S. A ring R is torsion-free and tidy if and only if its (up-to-an-isomorphism unique) maximal essential extension is a direct product of full rings $\mathfrak{M}^{\circ}(c, D)$ of finite-row $c \times c$ matrices over division rings D .

A straightforward simple proof of Theorem S is offered in [6]; the basic idea of the proof goes back to [1]. The following routine lemma provides the intermediary step towards the matrix representation.

Lemma. Let $M = \bigoplus_{\omega \in \Omega} M_{\omega}$ be a direct decomposition of a unital R_1-module M . Then the endomorphism ring $\mathcal{E}_M = Hom_{R_1}(M, M)$ is isomorphic to the ring of all $\Omega \times \Omega$ matrices $(\varphi_{\omega', \omega''})$ such that, for every $\omega', \omega'' \in \Omega$,

(i) $\varphi_{\omega' \omega''} \in Hom_{R_1}(M_{\omega'}, M_{\omega''})$

and

(ii) $\{m \in M_{\omega} \mid m \in Ker\, \varphi_{\omega' \omega}$ for all but a finite number of ω's $\} = M_{\omega'}$.

For, a torsion-free ring R can be essentially embedded in a ring R_1 with unity. Furthermore, the injective hull H(R) of the unital R_1-module R is an essential extension of R which is (canonically) isomorphic to the endomorphism ring $\mathcal{E}_{H(R)} = Hom_{R_1}(H(R), H(R))$. Since R is tidy, there is a direct sum $\bigoplus_{\omega} U_{\omega}$ of uniform left ideals which is essential in R and thus, denoting by $H(U_{\omega})$ an injective hull of U_{ω} in H(R), $G = \bigoplus_{\omega} H(U_{\omega})$ is essential in H(R). It can be shown that $\mathcal{E}_{H(R)} \cong \mathcal{E}_G = Hom_{R_1}(G, G)$ and an application of Lemma provides the desired conclusion.

Now, we can improve the theorem showing that the cardinalities c and the division rings D are determined uniquely by the ring R (for a full account we refer to [5]); the cardinalities c are, in fact, the respective π-ranks of R introduced in [3] (or [4]) corresponding to the "generalized primes" π of R. The latter concept can be extended to a general ring in the following manner.

Definition. Two uniform R-modules U_1 and U_2 are said to be $\sim$-equivalent if they contain non-zero R-submodules $V_1 \subseteq U_1$ and $V_2 \subseteq U_2$ which are R-isomorphic. Let U be a uniform R-module; the set $\mathcal{P}_U$ of all orders of elements x belonging to uniform R-modules X $\sim$-equivalent to U is said to be a generalized prime of R (associated with U). A generalized prime $\mathcal{P}$ of R is said to be relevant if $\mathcal{P}$ contains left annihilators of non-zero elements of R.

Denoting by Π_0^R the set of all relevant generalized primes of a torsion-free ring R, one can show - applying the methods of [3] or [4] - that the $\mathcal{P}$-rank $r_{\mathcal{P}}(R)$ of R is an invariant of R for every $\mathcal{P} \in \Pi_0^R$ and that, similarly to the unital case, a division ring is associated with every $\mathcal{P} \in \Pi_0^R$, viz. $D_{\mathcal{P}} = Hom_R(U, U)$, $U \in \mathcal{P}$. Notice that if R is tidy, evidently $\sum_{\mathcal{P} \in \Pi_0^R} r_{\mathcal{P}}(R) = r(R)$.

The refined form of Theorem S then reads:

Theorem R. Let R be a torsion-free tidy ring; for every $\mathcal{P} \in \Pi_0^R$, let $c_{\mathcal{P}} = r_{\mathcal{P}}(R)$ and $D_{\mathcal{P}}$ be the division ring associated with $\mathcal{P}$. Then the (up-to-an-isomorphism unique) maximal essential extension $\overline{R}$ of R is the direct product

$$\overline{R} = \prod_{\mathcal{P} \in \Pi_0^R} \mathfrak{M}^{\circ}(c_{\mathcal{P}}, D_{\mathcal{P}})$$

of the full $\mathcal{C}_{\mathcal{P}} \times \mathcal{C}_{\mathcal{P}}$ finite-row matrix rings $\mathcal{M}^{o}(\mathcal{C}_{\mathcal{P}}, D_{\mathcal{P}})$ over $D_{\mathcal{P}}$.

On the other hand, every essential subring R of the direct product

$$\prod_{\omega \in \Omega} \mathcal{M}^{o}(\mathcal{C}_{\omega}, D_{\omega})$$

of the (non-zero) full $\mathcal{C}_{\omega} \times \mathcal{C}_{\omega}$ finite-row matrix rings $\mathcal{M}^{o}(\mathcal{C}_{\omega}, D_{\omega})$ over a division ring D_{ω} is a torsion-free tidy ring R such that the set Π_{o}^{R} of all relevant generalized primes of R can be indexed by Ω and $\mathcal{C}_{\omega} = \varkappa_{\mathcal{P}_{\omega}}(R)$ for $\omega \in \Omega$; moreover, D_{ω} is the division ring associated with $\mathcal{P}_{\omega}$.

In conclusion, let us point out that if R is a direct sum $R = \bigoplus_{\omega} U_{\omega}$ of uniform torsion-free left ideals, then the embedding of R into the direct product $\overline{R}$ of matrix rings described in the theorems can be chosen in such a way that the ideals U lie in the respective (uniform) "column" ideals of $\overline{R}$. This may help substantially when applying the theorems to derive results of the type of WEDDERBURN-ARTIN and GOLDIE Theorems (cf.[5]). And the theorems can be favourably applied to obtain such results; this is a consequence of the fact that they provide a good, viz. essential, embedding into a direct product of full matrix rings over division rings of every ring which can possibly be "closely" embedded into such matrix rings. This is the fact on which the particular value of the theorems rests. To illustrate this feature, let us formulate the following version of GOLDIE's results ([7], [8]) which can be derived from our Theorem R quite simply (using the fact that a full matrix ring $\mathcal{M}(k, D)$ of finite

$k \times k$ matrices over a division ring D is a (classical) left quotient ring for an essential subring R of $\mathfrak{M}(k, D)$ if and only if R is prime (or semiprime); see [5]).

Corollary. Let R be a torsion-free tidy semiprime (or prime) ring such that the ranks $r_{\mathcal{P}}(R)$ are finite for all $\mathcal{P} \in \Pi_0^R$. Then, $\prod_{\mathcal{P} \in \Pi_0^R} \mathfrak{M}(\kappa_{\mathcal{P}}(R), D_{\mathcal{P}})$ is the classical left quotient ring for R (or, $\Pi_0^R = \{\mathcal{P}\}, \kappa_{\mathcal{P}}(R) = \kappa(R)$, $D_{\mathcal{P}} = D_R$ and $\mathfrak{M}(\kappa(R), D_R)$ is the classical left quotient ring for R, respectively).

On the other hand, if $\prod_{\omega \in \Omega} \mathfrak{M}(k_\omega, D_\omega)$ with integers $k_\omega \geq 1$, is the classical left quotient ring for a ring R, then R is a torsion-free tidy semiprime ring such that the elements of Π_0^R can be indexed by Ω and $r_{\mathcal{P}_\omega}(R) = k_\omega$ for every $\omega \in \Omega$; if, in particular, Ω consists of a single element, then R is prime.

R e f e r e n c e s

[1] V. DLAB: Die Endomorphismenringe Abelscher Gruppen und die Darstellung von Ringen durch Matrizenringe, Czech. Math.J.7(1957),485-523.

[2] V. DLAB: Distinguished families of ideals of a ring, to appear.

[3] V. DLAB: The concept of rank and some related questions in the theory of modules, Comment.Math.Univ.Carolinae 8(1967),39-47.

[4] V. DLAB: Rank theory of modules, to appear.

[5] V. DLAB: Matrix representation of torsion-free rings, to appear.

[6] V. DLAB: Matrix rings as injective hulls of torsion-free tidy rings, to appear.

[7] A.W. GOLDIE: The structure of prime rings under ascending chain conditions,Proc.London Math.Soc. 8(1958),589-608.

[8] A.W. GOLDIE: Semi-prime rings with maximum condition, Proc.London Math.Soc.10(1960),201-220.

[9] R.E. JOHNSON: The extended centralizer of a ring over a module,Proc.Amer.Math.Soc.2(1951),891-895.

[10]R.E. JOHNSON: Quotient rings of rings with zero singular ideal,Pacific J.Math.11(1961),1385-1392.

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