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HOMOGENEITY PROBLEMS FOR EXTREMALLY DISCONNECTED SPACES

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Denote by CH(m) the statement that there is no cardinal between m and 2^m. The main result is the following

Theorem 1. If either CH($\aleph_0$) holds or CH(exp $\aleph_0$) does not hold then no infinite compact space embeddable in an extremally disconnected space is homogeneous.

We say that a space P is homogeneous if for each x and y in P there exists a homeomorphism h of P onto P such that h x = y . The proof is based on Theorems 2 and 3 below. For the proof of Theorem 3 two results from [F_2] are needed.

Extremally disconnected spaces will be often called ED-spaces. The closure of a set X is a space P is denoted by $c\ell_P X$, or, simply $c\ell X$. The symbol X^* stands for $c\ell X - X$.

Lemma 1. Assume that a space P admits an embedding in an extremally disconnected space Q , and let X and Y be discrete countable sets in P . The set

$$Z = (X \cap Y) \cup (X^* \cap Y) \cup (X \cap Y^*)$$

is discrete and normally embedded in P , and

$$c\ell Z = c\ell X \cap c\ell Y , \quad Z^* = X^* \cap Y^* .$$

Proof. Evidently the set Z is discrete, and the inclusions $\subset$ hold. Any countable discrete set in an ED-space is normally embedded, and hence Z is normally embedded in Q (and so in P), and also the set

$$Z_0 = (X - c\ell Y) \cup (Y - c\ell X)$$

is normally embedded. It follows that

$$Z_1 = c\ell(X - c\ell Y) \cap c\ell(Y - c\ell X) = \emptyset.$$

On the other hand, clearly

$$c\ell X \cap c\ell Y \subset c\ell Z \cup Z_1, \text{ and}$$

$$X^* \cap Y^* \subset Z^* \cap Z_1 .$$

Thus

$$c\ell X \cap c\ell Y \subset c\ell Z, \text{ and } X^* \cap Y^* \subset Z^*.$$

The proof is complete.

Now we are going to introduce a partial order in the set of the germs of countable normally embedded discrete sets. This is not necessary, however, it will simplify the description of some simple reasonings.

<u>Definition 1</u>. Let x be a point in a space P . Two sets X_1 and X_2 define the same germ at x if $U \cap X_1 = U \cap X_2$ for some neighborhood U of x . Denote by $\mathcal{N}_P(x)$, or simply $\mathcal{N}(x)$, the set of all non-trivial germs at x of normally embedded discrete countable sets in P . Of course the trivial germs are those with representatives (x) and $\emptyset$. Thus two discrete normally embedded countable sets X_1 and X_2 define the same non-trivial germ if and only if

$$x \in c\ell(X \cap Y) - (X \cup Y) .$$

A partial order $<$ on $\mathcal{N}(x)$ is defined as follows:

$$n_1 > n_2 \text{ iff } X_2 \subset c\ell X_1 - X_1 \text{ for some } X_i \in n_i .$$

Theorem 2. If P is embeddable in an extremally disconnected space then $\mathcal{N}(x)$ is linearly ordered for each x in P.

Proof. Immediate corollary to Lemma 1.

In what follows let T be the set of all types as introduced in $[F_1]$. Roughly speaking, the types are equivalence classes of the class of all pairs $\langle X, \xi \rangle$, where ξ is a free ultrafilter on a countable set X, and two pairs $\langle X_1, \xi_1 \rangle$, $\langle X_2, \xi_2 \rangle$ are equivalent iff there exists a one-to-one mapping f of X_1 onto X_2 such that $f[\xi_1] = \xi_2$. The set T is ordered by the relation Φ which is read "produces", see $[F_1$, Definition 1.4].

Definition 2. Let x be a point in a space P. If X is a normally embedded discrete countable set in P with $x \in cl\,X - X$, then the intersections of X with the neighborhoods of x form an ultrafilter on X, and the type of this ultrafilter is called the type of x wrt X, and denoted by $\tau(x,X,P)$. If $n \in \mathcal{N}(x)$, and $X_1, X_2 \in n$, then clearly

$$\tau(x, X_1, P) = \tau(x, X_2, P),$$

and the common value of all $\tau(x,X,P)$, $X \in n$, is called the type of n, and denoted by $\tau(n,P)$ or simply τn. The set of all τn, $n \in \mathcal{N}(x)$, is denoted by $T(x,P)$ or simply Tx. Finally, denote by τ the relation consisting of all pairs $\langle n, \tau(n,P) \rangle$. Evidently, the mapping $\tau : \mathcal{N}(x) \to T$ is order-preserving for each x and P.

Theorem 3. Let P be a space embedded in an extremally disconnected space. For each x in P the mapping

$$\tau : \mathcal{N}(x) \to T$$

is one-to-one, and the cardinal of any section $\{t \mid t \in Tx, t < t_0\}$ of Tx is at most $\exp \aleph_0$. If P is compact then Tx is a section-like set in T, i.e. if $t_0 \in Tx$ then $\{t \mid t < t_0\} \subset Tx$.

<u>Proof</u>. Assume that $n_i \in \mathcal{N}(x)$, $n_1 \neq n_2$. We may, and shall, assume that $n_1 < n_2$. There exist normally embedded discrete countable sets $X_i \in n_i$ such that $X_1 \subset \subset cl X_2 - X_2$. Consider the subspace $R = cl X_2$ of P, and the Čech-Stone compactification βR of R. Clearly βR is a free compact separable space (i.e. a copy of βN); by Theorem C in [F_2] (the proof follows from Theorem B in [F_1]) the type of x wrt X_2, and the type of x wrt X_1 are distinct. This proves that $\tau : \mathcal{N}(x) \to T$ is one-to-one. To prove the second statement consider a $t_0 = \tau n_0$, and choose a normally embedded discrete set X_1 in n_1. By definition, for each $t < t_0$ there exists a normally embedded discrete countable set $X_t \subset cl X_0 - X_0$ in $n_t = \tau^{-1} t$. Theorem C in [F_1] applies to $\beta cl X_0$ and gives the estimate for $\{t \mid t \in Tx, t < t_0\}$. If P is compact then $cl X_0$ is compact, and hence $\beta cl X_0 = cl X_0$; the last statement follows by definition of the definition of types.

Now we are prepared to prove Theorem 1. For convenience, we state the following evident lemmas.

<u>Lemma 2</u>. Let h be a homeomorphism of P onto itself. For any x in P, h induces an isomorphism of $\mathcal{N}x$ onto $\mathcal{N}hx$, and Tx onto Thx.

<u>Lemma 3</u>. If a space P contains a copy of βN (N denotes the discrete set of natural members) then $T(P) = T$, where $T(P) = \bigcup\{Tx \mid x \in P\}$.

Proof of Theorem 1. Assume that an infinite compact homogeneous space P admits an embedding into an ED-space. Then P contains a copy of βN by Lemma 1, and hence, by Lemmas 2 and 3 $Tx = T$ for each x in P. By Theorem 3 there exists no cardinal between $\exp \aleph_0$ and $\exp \exp \aleph_0$. Thus the second condition in Theorem 1 is sufficient. To prove that the first condition is sufficient we shall verify the following, may be a little more general, proposition.

Theorem 4. Assume that P is an infinite compact space embeddable into an extremally disconnected space. Then P is not homogeneous provided anyone of the following conditions is fulfilled:

1. $CH(\aleph_0)$.

2. There exist two distinct types of P-points.

3. There exist two distinct incomparable types, i.e. T is not linearly ordered.

4. $Tx \neq T$ for each compact space K embeddable in an ED-space and each x in K.

5. $Tx \neq T$ for each x in any compact ED-space.

6. $Tx \neq T$ for each x in any free compact ED-space.

Proof. Condition 1 implies Condition 2 by W.Rudin [R]. The type of any P-point is produced by no type (because a P-point is the cluster point of no countable set), and therefore 2 implies 3. By Theorem 3 Condition 3 implies Condition 4. Evidently 4 implies 5, and 5 implies 6. Condition 6 implies Condition 4 because any space in 4 is a subspace of some space in 6 . Thus Conditions 4,5 and 6 are equivalent. By Lemmas 2 and 3, it follows from Condition 4 that there exists no homogeneous infinite compact space embeddable in an ED-space.

<u>Remark</u>. Without any assumption on the set theory (except for the axiom of choice) an infinite compact space K is not homogeneous provided that one of the following conditions is fulfilled:

a. K is embeddable in an ED-space, and $Tx \neq T$ for some x in K .

b. A type of a point of K lives in an ED-space $E \supset K$ outside of K. [A particular case of a.]

c. K is a subspace of a compact ED-space E such that E - K is not countably compact. [A particular case of b.]

d. K is a subspace of βN (or equivalently, of a separable ED-space x)). [A particular case of b.]

e. There exists an extremally disconnected space P such that K is embeddable in P and contains a copy of P .[F_3].

f. There exists a compact ED-space P such that K is a nowhere dense subspace of P , and contains a copy of P . [F_4].

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x) A.V.Archangelskij has observed in Dokl.Akad.N.SSSR,175,pp. 451-4, that the condition is equivalent to the statement that the total character of K is at most c .

R e f e r è n c e s

[Č] E. ČECH: Topological spaces. Academia, Praha 1965.

[F_1] Z. FROLÍK: Sums of ultrafilters. Bull. Amer. Math. Soc. 73 (1967), 87-91.

[F_2] " : Types of ultrafilters on countable sets. Proc. Second Prague Symposium on Topology, pp. 142-3, Academia, Praha 1967.

[F_3] " : Fixed points of maps of β M. To appear in Bull. Amer. Math. Soc.

[F_4] " : Fixed points of maps of extremally disconnected spaces and Boolean Algebras. Submitted to Bull. Acad. Polon.

[R] W. Rudin: Homogeneity problems in the theory of Čech compactifications, Duke Math. J. 23 (1956), pp. 409-19, 633.

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