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NON-HOMOGENEITY OF $\beta P - P$

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This is to answer a question I have been asked repeatedly by my colleagues since my proofs (independent of the continuum hypothesis) of non-homogeneity of $\beta N - N$ were preprinted. It turns out that both methods, that one depending on the author's estimate of the cardinal of the set of all relative types of a given point of $\beta N - N$ (see [1]), and also that one depending on the author's non-fixed point theorem (see [2] and [3] or [4]), work in general situation. What is needed in addition is not too much, however, perhaps a little bit surprising, see Lemma and its Corollary 2 below. Non-homogeneity problems for extremally disconnected spaces will be treated in a forthcoming paper.

The main results are two proofs of the following theorem, without any assumption concerning the continuum hypothesis.

<u>Theorem 1</u>. If P is not pseudocompact then $\beta P - P$ is not homogeneous.

Under an additional assumption that P is locally compact Theorem 1 was stated and proved by W. Rudin [5]; his proof heavily depends on the Continuum Hypothesis (the existence of P-points in $\beta N - N$). Using the same method (in particular, using the continuum hypothesis) T. Isiwata [6] formulated and proved Theorem 1 above. Our proof depends on Corollary 2 of the following lemma, and the theory of types of ultrafilters.

Lemma. Let X be a completely normally embedded countable subset of a completely regular space P. Let $Y \subset \beta P$ be a countable set which is semi-separated to X (i.e. $(X \cap cl\, Y) \cup (Y \cap cl\, X) = \emptyset$). Then X and Y are functionally separated in βP, i.e.

$$cl\, Y \cap cl\, X = \emptyset .$$

Proof. The sets X and Y are countable (hence Lindelöf) and semi-separated, and hence separated, that means there exist two disjoint open sets U and V in βP such that $U \supset X$, $V \supset Y$. Put $F = P - U$. Since X is countable, there exists a zero set Z in P which is disjoint to X and contains F. Since X is completely normally embedded, X and Z are functionally separated in P. Hence X and F are functionally separated in P, which implies that $cl\, X \cap cl\, F = \emptyset$ (because of the characteristic property of Čech-Stone compactifications). Now observe that $cl\, Y \subset cl\, F$.

Corollary 1. Let X be a countable infinite completely normally embedded set in a space P. If $x \in cl\, X - X$ (in βP) is in the closure of a countable set Y with $cl\, Y \subset \beta P - P$, then

$$x \in cl(Y \cap cl\, X)$$

Corollary 2. Let X be a countably infinite completely normally embedded set in a space P, and let $x \in cl\, X - X$. If Y is a countable discrete set which is with its closure contained in $\beta P - P$, and if $x \in cl\, Y - Y$, then

$x \in cl Z - Z$ for some $Z \subset Y \cap cl X$.

The proof of Corollary 1 is easy, and Corollary 2 is a particular case of Corollary 1.

Now we are going to derive immediate consequences of Corollary 2 for types of ultrafilters. It is not necessary to introduce any notation and definitions concerning types, however, it seems to be convenient to do that, and also, the theory of types might be of some interest in itself. For the definition of types see [1;1.1]. Roughly speaking, two ultrafilters on countable sets, say $\mathfrak{X}$ on X and $\mathcal{Y}$ on Y, are defined to be equivalent, if there exists a bijective mapping f between X and Y such that $f[\mathfrak{X}] = \mathcal{Y}$; now, the types would be the equivalence classes if there were no set-theoretical troubles.

<u>Definition</u>. Let Q be a space, and let $x \in Q$. Consider the collection $\mathfrak{M}_x$ of all countable normally embedded discrete sets M such that $x \in cl M - M$. Thus the intersections of the neighborhoods of x with any M in $\mathfrak{M}_x$ form an ultrafilter $\alpha_x M$ on M, and the type of this ultrafilter is called the type of x with respect to M. If $S \subset Q$, then the types with respect to subsets M lying with their closure in S are called the types in S; the types in $\beta P - P$ are called the ideal types.

<u>Theorem 2</u>. Let X be a countable infinite discrete completely normally embedded set in a space P and let $x \in cl X - X$ (in $Q = \beta P$). Then

A. The cardinal of the set of all ideal types of x is at most $\exp \aleph_0$.

B. The type of x with respect to X is distinct from any ideal type of x.

Proof. Let t be any ideal type of x, say with respect to Y. By Corollary 2 there exists a $Z \subset Y \cap \cap \operatorname{cl} X$ such that $x \in \operatorname{cl} Z$. Clearly the types with respect to Y and Z coincide. Thus every ideal type of x is equal to the ideal type of x in $\operatorname{cl} X - - X$. Since $\operatorname{cl} X$ is a Čech-Stone compactification of X, both statements follow from the corresponding statements for the particular case $P = N$; for A see Theorem C in [1], for B see Theorem in [2] or Theorem B in [3], or Proposition 2 in [4].

Proof of Theorem 1. Assume that P is not pseudocompact. It follows that there exists a countably infinite discrete completely normally embedded X in P. For each x in $\beta P - P$ let T_x denote the set of all types of x in $\beta P - P$. If h is any homeomorphism of $\beta P - P$ onto itself, and if $hx = y$ then $T_x = T_y$. Now let $x \in \operatorname{cl} X - X$. We want to find a y in $\operatorname{cl} X - X$ such that $T_x \neq T_y$. Now if we want to apply the assertion A in Theorem 2, we pick a type t which is not in T_x (the cardinal of the set of all types is $\exp \exp \aleph_0$), and then we select a y in $\operatorname{cl} X - X$ such that $t \in T_y$. If we want to apply the assertion B, we pick an y in $\operatorname{cl} X - X$ such that the type of x with respect to X belongs to T_y. Since $t \notin T_x$ by B, necessarily $T_x \neq T_y$.

R e f e r e n c e s

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