Oldřich Kowalski

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THE INVARIANT CLASSIFICATION OF 3-DIMENSIONAL LINEAR SUB-SPACES OF INFINITESIMAL ISOMETRIES OF $\mathbb{E}_3$ x)

Oldřich KOWALSKI, Brno

In this paper we shall investigate the "genral" 3-dimensional linear subspaces of a Lie algebra $\mathcal{G}$ isomorphic to the Lie algebra $\mathcal{G}_*$ of the full isometry group $\mathbb{G}_*$ of $\mathbb{E}_3$ (= the 3-dimensional Euclidean space). First, we shall determine the invariants of these subspaces with respect to all automorphisms or with respect to all inner automorphisms of $\mathcal{G}$. Further, we shall show a geometrical signification of such invariants using the "finite representations" of $\mathcal{G}$ on $\mathbb{G}_*$. Finally, we shall find a classification of the preceding subspaces from the point of view of the appearing of rotational subgroups in finite representations.

1) Basic concepts. Let $\mathbb{G}$ be a Lie group isomorphic to the full isometry group of $\mathbb{E}_3$. Let us denote by $\mathbb{G}^+$ the component of unity in $\mathbb{G}$. The Lie algebra $\mathcal{G}$ of $\mathbb{G}$ has a basis $\{ X_1, X_2, X_3, X_{12}, X_{23}, X_{31} \}$ such that

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x) This article was suggested by some questions posed by A. Švec at the Seminar of Differential Geometry in Brno.

$[X_i, X_j] = 0$, $[X_{ij}, X_i] = X_j$, $[X_{ij}, X_j] = -X_i$, $[X_{ij}, X_k] = 0$

for $i, j, k = 1, 2, 3$, $i \neq j$, $j \neq k$, $k \neq i$;

$[X_{12}, X_{31}] = X_{23}$, $[X_{31}, X_{23}] = X_{12}$, $[X_{23}, X_{12}] = X_{31}$.

Each basis with this property is said to be canonical. Here, X_1, X_2, X_3 generate the largest commutative subalgebra $\mathcal{T}$ of $\mathcal{G}$. Each 3-dimensional linear subspace $\mathcal{B}$ of $\mathcal{G}$ satisfying $\mathcal{B} \cap \mathcal{T} = \{0\}$ is said to be <u>general</u> or is called a <u>3-block</u>.

<u>Proposition 1</u>. The manifold $\mathbb{B}$ of all 3-blocks possesses the natural structure of a 9-dimensional linear space.

<u>Proof</u>. Let us choose an arbitrary canonical basis. If $\mathcal{B} \in \mathbb{B}$ then $\mathcal{B} \cap \mathcal{T} = \{0\}$ yields that $\mathcal{B}$ is generated by the (uniquely determined) vectors

$$X_{12} + a_1 X_1 + a_2 X_2 + a_3 X_3 \qquad X_{23} + b_1 X_1 + b_2 X_2 + b_3 X_3$$
$$X_{31} + c_1 X_1 + c_2 X_2 + c_3 X_3$$

Denote by x_i, x_{ij} the coordinates of any vector in $\mathcal{G}$ with respect to the basis $\{X_i, X_{ij}\}$.

Then each 3-block is determined by

$$x_1 = a_1 x_{12} + b_1 x_{23} + c_1 x_{31} \qquad x_2 = a_2 x_{12} + b_2 x_{23} + c_2 x_{31}$$
$$x_3 = a_3 x_{12} + b_3 x_{23} + c_3 x_{31} .$$

In this way, to each canonical basis of $\mathcal{G}$, it corresponds a coordinate system in $\mathbb{B}$ of the form

$$\mathcal{B} \longrightarrow \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}$$

Moreover, to each change of canonical basis in $\mathcal{G}$ (or to each automorphism of $\mathcal{G}$), it corresponds a linear transformation of coordinates (a_i, b_i, c_i). Q.E.D.

Each automorphism of $\mathcal{G}$ determines a linear transformation of $\mathbb{B}$, called an <u>automorphism</u> of $\mathbb{B}$. Now, we choose a fixed Cartesian coordinate system (x, y, z) in $\mathbb{E}_3$. Denote by $\mathbb{G}_*$, $\mathbb{G}_*^+$ and $\mathcal{G}_*$ respectively the full isometry group of $\mathbb{E}_3$, its component of unity and its Lie algebra respectively. The infinitesimal transformations

$$\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}, y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}, z\frac{\partial}{\partial y} - y\frac{\partial}{\partial z}, x\frac{\partial}{\partial z} - z\frac{\partial}{\partial x}$$

form a canonical basis in $\mathcal{G}_*$. Any canonical basis $\{X_i, X_{ij}\}$ of $\mathcal{G}$ determines exactly one representation $\Gamma : \mathcal{G} \to \mathcal{G}_*$ given by

$$X_1 \to \frac{\partial}{\partial x},\ X_2 \to \frac{\partial}{\partial y},\ X_3 \to \frac{\partial}{\partial z}$$

$$X_{12} \to y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y},\ X_{23} \to z\frac{\partial}{\partial y} - y\frac{\partial}{\partial z},\ X_{31} \to x\frac{\partial}{\partial z} - z\frac{\partial}{\partial x}.$$

Let $exp_* : \mathcal{G}_* \to \mathbb{G}_*^+$ and $exp : \mathcal{G} \to \mathbb{G}^+$ be the exponential maps (see,for example,[1] pp.94-102). Then to each Lie algebra representation $\Gamma : \mathcal{G} \to \mathcal{G}_*$ there corresponds exactly one group representation $exp\,\Gamma : \mathbb{G}^+ \to \mathbb{G}_*^+$ such that the following commutative diagram holds:

(1)
$$\begin{tikzcd} \mathbb{G}^+ \arrow[r, "exp\,\Gamma"] & \mathbb{G}_*^+ \\ \mathcal{G} \arrow[u, "exp"] \arrow[r, "\Gamma"] & \mathcal{G}_* \arrow[u, "exp_*"'] \end{tikzcd}$$

The composed map $exp_* \circ \mathbb{r}$ will be called briefly a <u>finite representation</u> of $\mathcal{G}$ on $\mathbb{E}_3$. Hence, to each canonical basis of $\mathcal{G}$, it belongs exactly one finite representation of $\mathcal{G}$ on $\mathbb{E}_3$, and conversely.

2) <u>Elementary representation properties of the 3-blocks</u>. Let $\mathbb{r} : \mathcal{G} \to \mathcal{G}_*$ be a Lie algebra representation and $\mathbb{V} : exp_* \circ \mathbb{r}$ the corresponding finite representation of $\mathcal{G}$ on $\mathbb{E}_3$. Let us take $\mathcal{B} \in \mathbb{B}$. Because $\mathcal{B} \cap \mathcal{V} = \{0\}$ the set $\mathbb{V}(\mathcal{B})$ is the union of 1-dimensional groups of screw-movements (in the large sense) in $\mathbb{E}_3$.

<u>To each unit vector $\vec{n}$ of $\mathbb{E}_3$ there is exactly one 1-dimensional subgroup in $\mathbb{V}(\mathcal{B})$ such that the axis of the movement is parallel to $\vec{n}$</u>.

Really, let $\{X_i, X_{ij}\}$ be the canonical basis corresponding to $\mathbb{V}$. Let $\vec{n} = (\cos\alpha, \cos\beta, \cos\gamma)$ and let the 3-block $\mathcal{B}$ be determined by

$$\begin{aligned} Y_{12} &= X_{12} + a_1 X_1 + a_2 X_2 + a_3 X_3 \\ Y_{23} &= X_{23} + b_1 X_1 + b_2 X_2 + b_3 X_3 \\ Y_{31} &= X_{31} + c_1 X_1 + c_2 X_2 + c_3 X_3 \end{aligned} .$$

Further put

$$Y = Y_{12}\cos\gamma + Y_{23}\cos\alpha + Y_{31}\cos\beta .$$

Then

$$\mathbb{r}(Y) = (y\cos\gamma - z\cos\beta + A)\frac{\partial}{\partial x} + (z\cos\alpha - x\cos\gamma + B)\frac{\partial}{\partial y} + (x\cos\beta - y\cos\alpha + C)\frac{\partial}{\partial z}$$

where $A = a_1 \cos\gamma + b_1 \cos\alpha + c_1 \cos\beta$

$$B = a_2 \cos\gamma + b_2 \cos\alpha + c_2 \cos\beta$$

$$C = a_3 \cos\gamma + b_3 \cos\alpha + c_3 \cos\beta$$

Introducing the new variables

$$u = x\cos\alpha + y\cos\beta + z\cos\gamma$$

$$v = y\cos\gamma - z\cos\beta$$

$$w = x(\cos^2\alpha - 1) + y\cos\alpha\cos\beta + z\cos\alpha\cos\gamma,$$

we obtain

$$\Gamma(Y) = (A\cos\alpha + B\cos\beta + C\cos\gamma)\frac{\partial}{\partial u}$$
$$+ [w - (C\cos\beta - B\cos\gamma)]\frac{\partial}{\partial v}$$
$$- [v - (C\cos\alpha\cos\gamma + B\cos\alpha\cos\beta + A(\cos^2\alpha - 1)]\frac{\partial}{\partial w}$$

Thus each transformation $\psi(Yt) = \exp_* \circ \Gamma(Yt)$ is a screw-movement around the axis $u \parallel \vec{n}$. Q.E.D.

Especially, the transformation $\psi(2\pi.Y)$ is a pure translation having its direction in u. If $2\pi \cdot T(\vec{n})$ is the vector determining this translation, then

$$T(\vec{n})\cdot\vec{n} = A\cos\alpha + B\cos\beta + C\cos\gamma \ .$$

This may be rewritten as

$$(2)\quad T(\vec{n})\cdot\vec{n} = b_1\cos^2\alpha + c_2\cos^2\beta + a_3\cos^2\gamma$$
$$+ (b_2 + c_1)\cos\alpha\cos\beta + (a_1 + b_3)\cos\alpha\cos\gamma$$
$$+ (a_2 + c_3)\cos\beta\cos\gamma \ .$$

We shall place each vector $T(\vec{n})$ into a position such that the initial point shall be the origin of the coordinate system. Then the end points of $T(\vec{n})$ $(\vec{n} \in S^2 =$ = the unit sphere of E_3) generate a surface, which will be called a characteristic surface of β with respect to ψ. The equation (2) of a characteristic surface can be expressed by means of x, y, z and the obtained equation

$$(3)\quad (x^2+y^2+z^2)^3-[b_1x^2+c_2y^2+a_3z^2+(b_2+c_1)xy+ \\ +(a_1+b_3)xz+(a_2+c_3)yz]^2=0$$

shows that the considered surface is algebraic (of degree 6) and has a centre of symmetry in the origin. The characteristic surfaces will play an important rôle in our investigations.

3) <u>Automorphism invariants of</u> $\mathbb{B}$. We start to determine the automorphism group $Aut(\mathcal{G})$ of $\mathcal{G}$. All derivatives (see e.g. [1]) of the algebra $\mathcal{G}$ with respect to a canonical basis $\{X_1, X_2, \dots, X_{31}\}$ are given by the matrix

$$D=\left(\begin{array}{ccc|ccc} a & b & c & & & \\ -b & a & d & & 0 & \\ -c & -d & a & & & \\ \hline e & g & 0 & 0 & -c & -d \\ 0 & f & -e & c & 0 & b \\ -f & 0 & -g & d & -b & 0 \end{array}\right)$$

Infinitesimal variations $\delta x_1, \delta x_2, \dots, \delta x_{31}$ (where $X=x_1X_1+x_2X_2+\dots+x_{31}X_{31}\in\mathcal{G}$) by automorphisms are represented by the transpose of D. Hence all infinitesimal transformations of $Aut(\mathcal{G})$ are determined by

$$A_1 = x_{31}\frac{\partial}{\partial x_3} - x_{12}\frac{\partial}{\partial x_2}$$
$$A_2 = x_{12}\frac{\partial}{\partial x_1} - x_{23}\frac{\partial}{\partial x_3}$$
$$A_3 = x_{23}\frac{\partial}{\partial x_2} - x_{31}\frac{\partial}{\partial x_1}$$

$$A_{12} = x_1 \frac{\partial}{\partial x_2} - x_2 \frac{\partial}{\partial x_1} + x_{23} \frac{\partial}{\partial x_{31}} - x_{31} \frac{\partial}{\partial x_{23}}$$

$$A_{23} = x_2 \frac{\partial}{\partial x_3} - x_3 \frac{\partial}{\partial x_2} + x_{31} \frac{\partial}{\partial x_{12}} - x_{12} \frac{\partial}{\partial x_{31}}$$

$$A_{31} = x_3 \frac{\partial}{\partial x_1} - x_1 \frac{\partial}{\partial x_3} + x_{12} \frac{\partial}{\partial x_{23}} - x_{23} \frac{\partial}{\partial x_{12}}$$

$$A_h = x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2} + x_3 \frac{\partial}{\partial x_3}$$

Here $A_1, A_2, \dots, A_{31}$ are the infinitesimal transformations of the group $\mathrm{Int}(\mathcal{Y}) \subset \mathrm{Aut}(\mathcal{Y})$ of inner automorphisms induced by the infinitesimal transformations $X_1, X_2, \dots, X_{31}$ of $\mathcal{Y}$. With regard to the diagram (1) we may also say, that $A_1, A_2, \dots, A_{31}$ are induced by infinitesimal isometries $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \dots,$

$x \frac{\partial}{\partial z} - z \frac{\partial}{\partial x}$ of $\mathbb{E}_3$ respectively. A_h is induced in a certain sense by the infinitesimal similarity of E_3 with centre in the origin. Corresponding infinitesimal transformations of the group $\mathrm{Aut}(\mathbb{B})$ (written by means of the coordinates (a_1, b_i, c_i)) are given by

$$B_1 = \frac{\partial}{\partial c_3} - \frac{\partial}{\partial a_2}, \quad B_2 = \frac{\partial}{\partial a_1} - \frac{\partial}{\partial b_3}, \quad B_3 = \frac{\partial}{\partial b_2} - \frac{\partial}{\partial c_1}$$

$$B_{12} = - a_2 \frac{\partial}{\partial a_1} - (b_2 + c_1) \frac{\partial}{\partial b_1} + (b_1 - c_2) \frac{\partial}{\partial c_1} + a_1 \frac{\partial}{\partial a_2} + (b_1 - c_2) \frac{\partial}{\partial b_2} + (b_2 + c_1) \frac{\partial}{\partial c_2} - c_3 \frac{\partial}{\partial b_3} + b_3 \frac{\partial}{\partial c_3}$$

$$B_{23} = \sigma B_{12}, \quad B_{31} = \sigma^2 B_{12}$$

where σ denotes simultaneous cycklic permutations of letters and indices $a \to b \to c \to a$, $1 \to 2 \to 3 \to 1$. Finally,

$$B_h = \sum_{i=1}^{3} (a_i \frac{\partial}{\partial a_i} + b_i \frac{\partial}{\partial b_i} + c_i \frac{\partial}{\partial c_i})$$

The linear partial differential system

$$B_1 f = B_2 f = B_3 f = B_{12} f = B_{23} f = B_{31} f = 0$$

is involutive and it remains involutive if we adjoin the equation $B_h f = 0$ to it. Hence we see that there are exactly 3 independent point-invariants with respect to $Int(\mathbb{B})$ (called point-semi-invariants of $\mathbb{B}$) and exactly 2 independent point-invariants with respect to $Aut(\mathbb{B})$ which are homogeneous functions of degree 0 in variables a_i, b_i, c_i. P. invariants of $\mathbb{B}$ are exactly those p.semi-invariants of $\mathbb{B}$ which are homogeneous functions of degree 0 in a_i, b_i, c_i. Introducing new variables

$$\begin{array}{lll} u_1 = a_2 + c_3 & u_2 = a_1 + b_3 & u_3 = b_2 + c_1 \\ u_4 = b_1 & u_5 = c_2 & u_6 = a_3 \\ u_7 = a_1 & u_8 = b_2 & u_9 = c_3 \end{array}$$

we see that $u_1, u_2, \dots, u_6$ form a complete system of solutions of the differential system $B_1 f = B_2 f = B_3 f = 0$ and therefore the point-semi-invariants depend on these six variables only.

To determine the p.semi-invariants, it remains to solve the involutive system

$$U_{12} f = U_{23} f = U_{31} f = 0,$$

where

$$U_{12} = u_2 \frac{\partial}{\partial u_1} - u_1 \frac{\partial}{\partial u_2} + 2(u_4 - u_5) \frac{\partial}{\partial u_3} + u_3 (\frac{\partial}{\partial u_5} - \frac{\partial}{\partial u_4})$$

$$U_{23} = u_3\frac{\partial}{\partial u_2} - u_2\frac{\partial}{\partial u_3} + 2(u_5 - u_6)\frac{\partial}{\partial u_1} + u_1\left(\frac{\partial}{\partial u_6} - \frac{\partial}{\partial u_5}\right)$$

$$U_{31} = u_1\frac{\partial}{\partial u_3} - u_3\frac{\partial}{\partial u_1} + 2(u_6 - u_4)\frac{\partial}{\partial u_2} + u_2\left(\frac{\partial}{\partial u_4} - \frac{\partial}{\partial u_6}\right).$$

In order to determine the p.invariants we must still use the equation

$$\sum_{i=1}^{6} u_i \frac{\partial f}{\partial u_i} = 0$$

The system $U_{12}f = U_{23}f = U_{31}f = 0$ may be solved using the standard methods (see e.g. [2]); after a rather long computation we obtain ~~three~~ solutions, which are homogeneous polynomials of degrees 1,2 and 3 respectively in variables u_i.

This way is not very convenient and we prefer to take advantage of the characteristic surfaces. The equation of a characteristic surface (3) takes, in the new variables, the form

$$\varphi(x, y, z, u_1, u_2, \dots, u_6) = (x^2+y^2+z^2)^3 -$$
$$- [u_4x^2 + u_5y^2 + u_6z^2 + u_3xy + u_2xz + u_1yz]^2 = 0$$

It is easy to see that the function $\varphi(x, y, z, u_1, \dots, u_6)$ satisfies the linear differential system

$$y\frac{\partial\varphi}{\partial x} - x\frac{\partial\varphi}{\partial y} - U_{12}\varphi = 0$$
$$z\frac{\partial\varphi}{\partial y} - y\frac{\partial\varphi}{\partial z} - U_{23}\varphi = 0$$
$$x\frac{\partial\varphi}{\partial z} - z\frac{\partial\varphi}{\partial x} - U_{31}\varphi = 0 .$$

Passing to finite transformations we obtain the following result:

Proposition 2. Let $\mathbb{V}$ be a finite representation of the algebra $\mathcal{G}$ on $\mathbb{E}_3$ and let $\mathcal{S}$ be the characteristic surface of a 3-block $\mathcal{B} \in \mathbb{B}$ with respect to the representation $\mathbb{V}$. Denote by $\wp$ a rotation around the origin in $\mathbb{E}_3$ and by $\wp^*$ the corresponding inner automorphism of the space $\mathbb{B}$. Then the characteristic surface of the 3-block $\wp^* \mathcal{B}$ is the surface $\wp^{-1}\mathcal{S}$.

Moreover we see that u_i (and consequently the characteristic surfaces) are not changed by such inner automorphisms of $\mathbb{B}$ which correspond to the translations of $\mathbb{E}_3$. Hence

Theorem 1. The characteristic surfaces are invariant with respect to inner automorphisms of the space $\mathbb{B}$ exactly up to rotations around the origin of $\mathbb{E}_3$.

An arbitrary metric invariant of a characteristic surface is then a p.semi-invariant of $\mathbb{B}$. Instead of a characteristic surface we now consider a "characteristic λ-surface" generated by end-points of vectors $\lambda \vec{n} + T(\vec{n})$ for $\vec{n} \in S^2$, where the number λ is chosen arbitrary but such that

$$\lambda > \sup_{\vec{n} \in S^2} | T(\vec{n}) \cdot \vec{n} | .$$

Theorem 1 holds also for the characteristic λ-surfaces. A characteristic λ-surface does not cut itself and therefore the volume V_λ of the domain bounded by the surface is defined.

We obtain without difficulty

$$V_\lambda = \frac{4}{3} \pi \lambda^3 + A\lambda^2 + B\lambda + C$$

where

$$A = \frac{4}{3}\pi(\mu_4 + \mu_5 + \mu_6)$$

$$B = \frac{4}{15}\pi\{2(\mu_4^2+\mu_5^2+\mu_6^2)+(\mu_4+\mu_5+\mu_6)^2+\mu_1^2+\mu_2^2+\mu_3^2\}$$

$$C = \frac{4}{3.35}\pi\{2\mu_1\mu_2\mu_3 + (\mu_1^2\mu_4+\mu_2^2\mu_5+\mu_3^2\mu_6) + 3(\mu_1^2\mu_5+\mu_2^2\mu_6+\mu_3^2\mu_4)+3(\mu_1^2\mu_6+\mu_2^2\mu_4+\mu_3^2\mu_5) + 3(\mu_4^2\mu_5+\mu_5^2\mu_6+\mu_6^2\mu_4)+3(\mu_4^2\mu_6+\mu_5^2\mu_4+\mu_6^2\mu_5) + 5(\mu_4^3+\mu_5^3+\mu_6^3) + 2\mu_4\mu_5\mu_6\}$$

The coefficients A, B, C are independent point-semi-invariants of $\mathcal{B}$. The coefficient C denotes the measure of an "oriented characteristic family of vectors". Provided the characteristic surface does not cut itself, the absolute value $|C|$ expresses the volume of the domain bounded by the characteristic surface. The number $\frac{3}{4\pi}A =$

$= \mu_4 + \mu_5 + \mu_6$ possesses also an interesting geometrical signification: Let us again consider the parametric equation of a characteristic surface

$$T(\vec{n})\cdot\vec{n} = \mu_4\cos^2\alpha + \mu_5\cos^2\beta + \mu_6\cos^2\gamma + \mu_3\cos\alpha\cos\beta + \mu_2\cos\alpha\cos\gamma + \mu_1\cos\beta\cos\gamma$$

If we denote by $\{\vec{i}, \vec{j}, \vec{k}\}$ the basic orthonormal triplet of E_3, then

$$T(\vec{i})\cdot\vec{i} + T(\vec{j})\cdot\vec{j} + T(\vec{k})\cdot\vec{k} = \mu_4 + \mu_5 + \mu_6$$

and the same holds for the vectors $-\vec{i}, -\vec{j}, -\vec{k}$. Because $(\mu_4 + \mu_5 + \mu_6)$ is a point-semi-invariant of $\mathbb{B}$ which is not changed by rotations of the characteristic surface around the origin, we obtain, for each orthonormal triplet $\{ \vec{a}, \vec{b}, \vec{c} \}$

$$T(\vec{a}) \cdot \vec{a} + T(\vec{b}) \cdot \vec{b} + T(\vec{c}) \cdot \vec{c} = u_4 + u_5 + u_6$$

Let us construct oriented lengths which are cut by a characteristic surface on arbitrary three mutually perpendicular rays starting in the origin.

Then the sum of these lengths is constant and equal to $u_4 + u_5 + u_6$.

This number may be called the parameter of a characteristic surface.

4) The characteristic roots of a 3-block. The most simple way of computing the point-semi-invariants is that using the characteristic roots.

$T(\vec{n}) \cdot \vec{n}$ is a quadratic form defined on the unit sphere $S^2 \subset \mathbb{E}_3$.

Let us denote by $\lambda_1, \lambda_2, \lambda_3$ its characteristic roots and consider a new Cartesian coordinate system with axes given by the corresponding characteristic directions. Denoting by ξ_1, ξ_2, ξ_3 the new components of a unit vector $\vec{n}$, we obtain

$$T(\vec{n}) \cdot \vec{n} = \lambda_1 \xi_1^2 + \lambda_2 \xi_2^2 + \lambda_3 \xi_3^2 \tag{4}$$

Hence the numbers $\lambda_1, \lambda_2, \lambda_3$ expressing the oriented lengths of axes of a characteristic surface are point-semi-invariants of $\mathbb{B}$.

Consequently the characteristic determinant

$$\begin{vmatrix} u_4-\lambda & \frac{u_3}{2} & \frac{u_2}{2} \\ \frac{u_3}{2} & u_5-\lambda & \frac{u_1}{2} \\ \frac{u_2}{2} & \frac{u_1}{2} & u_6-\lambda \end{vmatrix} = -\lambda^3+\lambda^2(u_4+u_5+u_6)-\lambda(u_4u_5+ + u_5u_6+u_6u_4-\frac{u_1^2+u_2^2+u_3^2}{4})+ + u_4u_5u_6+\frac{1}{4}u_1u_2u_3-\frac{1}{4}(u_1^2u_4+u_2^2u_5+u_3^2u_6)$$

of the quadratic form (regarded as a polynomial of one variable λ) is a point-semi-invariant of $\mathbb{B}$ and the same holds for its coefficients. Evidently, applying an automorphism of $\mathfrak{G}$ (not necessarily an inner one), the characteristic roots of any 3-block will be multiplied by a positive factor at most.

Let us recall that a 1-dimensional group of screw-movements with the direction $\vec{n}$ is a rotational subgroup if and only if $T(\vec{n})\cdot\vec{n} = 0$. Hence we obtain an invariant classification of 3-blocks according to the presence of rotational subgroups in its finite representations:

1) Elliptic case. All characteristic roots of $\mathcal{B} \in \mathbb{B}$ are non-zero and they have the same sign.
The quadratic form $T(\vec{n})\cdot\vec{n}$ is positively definite or negatively definite. There are no rotations in the finite representations of $\mathcal{B}$.

2) Hyperbolic case. The characteristic roots of $\mathcal{B}$ are all non-zero and they do not have the same sign. In each finite representation $\mathcal{V}(\mathcal{B})$ of $\mathcal{B}$ there are 1-dimensional rotational subgroups; the axes of which are parallel with the generating lines of a cone.

3) Parabolic cases.

A) One of the characteristic roots is zero, the other

ones have the same sign.
In any finite representation of $\mathcal{B}$ there is exactly one 1-dimensional rotational subgroup.

B) One of the characteristic roots is zero and the other ones have opposite signs. In each finite representation of $\mathcal{B}$ there are 1-dimensional rotational subgroups; their axes are parallel to one of two mutually non-parallel planes.

C) Only one of the characteristic roots is non-zero. In each finite representation of $\mathcal{B}$ there are 1-dimensional rotational subgroups; their axes are parallel to a plane.

D) All characteristic roots are zero. Then $\mu_1 = \mu_2 = \mu_3 = \mu_4 = \mu_5 = \mu_6 = 0$. The 3-block $\mathcal{B}$ is a subalgebra, its finite representations are the isotropy groups of $\mathbb{E}_3$.

From the preceding we obtain

Corollary. If the finite representations of a 3-block $\mathcal{B}$ admit rotations only, then $\mathcal{B}$ is a subalgebra of $\mathcal{G}$.

Finally, we obtain the following theorem:

Theorem 2. Let a 3-block $\mathcal{B} \in \mathbb{B}$ be not a subalgebra. Then, after a convenient denotation, the ratia $\lambda_1 : \lambda_3$, $\lambda_2 : \lambda_3$ of oriented lengths of axes of its characteristic surface do not depend on finite representation of $\mathcal{G}$ on $\mathbb{E}_3$. Let a fixed finite representation of $\mathcal{G}$ be chosen. Let $\mathcal{B}_1$, $\mathcal{B}_2$ be two 3-blocks. Then two following conditions are equivalent:

a) There is an automorphism of $\mathcal{G}$ which sends $\mathcal{B}_1$ onto $\mathcal{B}_2$.

b) The preceding ratia are the same for $\mathcal{B}_1$ and $\mathcal{B}_2$.

R e f e r e n c e s

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[2] E. GOURSAT: Vorlesungen über die Integration der partiellen Differentialgleichungen erster Ordnung.(Teubner 1893).

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