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APPROXIMATE SOLUTIONS OF EQUATIONS IN BANACH SPACES BY THE NEWTON ITERATIVE METHOD.

PART 1. GENERAL THEOREMS.

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§ 1.

Let X be a Banach space, $\Omega \subset X$ an open set, Φ a nonlinear operator on Ω and $\{\Phi_m\}_{m=1}^{\infty}$ a sequence of nonlinear operators which in some sense approximate Φ. The main problem studied in this paper is the following one: if there converges the Newton iterative process for the equation

(1) $$\Phi u = 0 ,$$

under which assumptions the same takes place for the equations

(2) $$\Phi_m u = 0 ,$$

m being large enough.

When $u_0 \in X$, $a > 0$, let us denote by $S(u_0, a)$ the open ball

(3) $$S(u_0, a) = \{u \,/\, u \in X,\ \|u - u_0\| < a\} .$$

$\bar{S}(u_0, a)$ denotes its closure.

The following theorem is of great importance for many considerations in this paper.

Theorem of Kantorovich ([1],pp.636-637).

Let the operator Φ map the set $\Omega = S(u_0, R) \subset X$ into X. Let there exist the first and second Fréchet derivatives of Φ on $\bar{S}(u_0, r) \subset \Omega$ and a linear bounded operator $\Gamma : X \to X$ such that the following inequalities take place:

(4) $\| \Gamma \Phi u_0 \| \leq c$,

(5) $\| \Gamma \Phi'(u_0) - I \| \leq \sigma$,

(6) $\| \Gamma \Phi''(u) \| \leq k, \quad u \in S(u_0, r)$.

If

(7) $h = \frac{ck}{(1-\sigma)^2} \leq \frac{1}{2}$, $\sigma < 1$,

(8) $r \geq r_0 = \frac{1 - \sqrt{1-2h}}{h} \quad \frac{c}{1-\sigma}$,

there exists the linear bounded operator

(9) $\Gamma_0 = [\Phi'(u_0)]^{-1}$

and the equation (1) has a solution u^* to which there converges the Newton iterative process

(10) $u_{n+1} = u_n - [\Phi'(u_n)]^{-1} \Phi u_n \qquad n = 0, 1, \ldots$

as well as the process

(11) $u_{n+1} = u_n - \Gamma_0 \Phi u_n$, $\qquad n = 0, 1, \ldots$.

Furthermore,

(12) $\| u^* - u_0 \| \leq r_0$.

If

(13) $h < \frac{1}{2}$ and $r < r_1 = \frac{1 + \sqrt{1-2h}}{h} \quad \frac{c}{1-\sigma}$

or

(14) $\quad h = \frac{1}{2}$ and $r \le r_1$,

the solution u^* is unique on $\bar{S}(u_0, r)$.

Remark 1. The inequalities (4) and (6) take place when

(15) $\quad \| \Gamma \| \le \gamma$,

(16) $\quad \| \Phi u_0 \| \le \alpha$,

(17) $\quad \| \Phi''(u) \| \le \beta , \quad u \subset \bar{S}(u_0, r)$,

(18) $\quad \gamma \alpha \le c , \quad \gamma \beta \le k$.

Remark 2. If there exists the operator Γ_0 defined by (9), the inequality (5) for $\Gamma = \Gamma_0$ takes place with $\delta = 0$ and the process (11) is the modified Newton process([1], p. 623).

Definition. We say that the operator Φ has the property Z with the point u_0 and the constants c, k, r_0 , r if Φ has the first and second Fréchet derivatives on the ball $\bar{S}(u_0, r) \subset \Omega$ and

1) there exists the linear bounded Γ_0 defined by (9),

2) there exist constants c, k, r such that

(19) $\quad \| \Gamma_0 \Phi u_0 \| \le c$,

(20) $\quad u \in \bar{S}(u_0, r) \Rightarrow \| \Gamma_0 \Phi''(u) \| \le k$,

(21) $\quad r > r_0 = \dfrac{1 - \sqrt{1 - 2h}}{h} c$,

(22) $\quad h = ck < \frac{1}{2}$.

Remark 3. When Φ has the property Z then evidently the assumptions (4) - (8) of the Theorem of Kantorovich are fulfilled for $\Gamma = \Gamma_0$, $\delta = 0$.

Remark 4. Let us denote by Q_0 the operator

(23) $Q_0 = I - \Gamma_0 \Phi$

mapping X into X . The modified process for Φ is then identical with the process of successive approximations

(24) $u_{n+1} = Q_0 u_n \qquad n = 0, 1, \dots$

for the equation

(25) $u = Q_0 u$.

Lemma 1. Let Φ have the property Z . Then

1) Q_0 maps the ball $\bar{S}(u_0, r)$ into itself,

2)

(26) $\alpha \overset{df}{=} \sup_{u \in \bar{S}(u_0, r_0)} \| Q'(u) \| < 1 - \sqrt{1 - 2h} < 1$,

3) Q_0 is a contractive operator on $\bar{S}(u_0, r_0)$ with the constant of contraction α , i.e.

(27) $u_1, u_2 \in \bar{S}(u_0, r_0) \Rightarrow \| Q_0 u_1 - Q_0 u_2 \| \le \alpha \| u_1 - u_2 \|$.

Proof.

1) Let $\| u_0 - u \| \le r_0$. Then, according to the assumptions on Φ and its derivatives,

$$\| u_0 - u + \Gamma_0 \Phi u \| \le \| \Gamma_0 \Phi u_0 \| + \| \Gamma_0 \Phi'(u_0)(u_0 - u) + \Gamma_0 \Phi u - \Gamma_0 \Phi u_0 \| \le$$

$$\le \| \Gamma_0 \Phi u_0 \| + \frac{1}{2} \sup_{0 < \vartheta < 1} \| \Gamma_0 \Phi''(u_0 + \vartheta [u - u_0]) \| \, \| u - u_0 \|^2 ,$$

and, according to (19),(20),

$$\| u_0 - u + \Gamma_0 \Phi u \| \le c + \frac{1}{2} k r_0^2 = r_0 .$$

2) There is

$$Q_0'(u) = \Gamma_0(\Phi'(u_0) - \Phi'(u))$$ and

$$u_0 \in S(u_0, r_0) \Rightarrow \| Q_0'(u) \| = \| \Gamma_0 \Phi'(u) - \Gamma_0 \Phi'(u_0) \| \leq$$

$$\leq \sup_{0<\vartheta<1} \| \Gamma_0 \Phi''(u_0 + \vartheta[u - u_0]) \| \, \| u - u_0 \| \leq k \, r_0 = 1 - \sqrt{1 - 2h} .$$

The function on the right-hand side is an increasing function of h in $\langle 0, \frac{1}{2} \rangle$ with the value 0 at h = 0 and the value 1 at $h = \frac{1}{2}$. As we assume $h < \frac{1}{2}$ the right-hand side is a number $\alpha < 1$.

3) The third assertion follows immediately from the well known theorem ([1], p.592).

Theorem 1.

A) Let Φ have the property Z . Let

B) the operators Φ_m have second Fréchet derivatives on the ball $\bar{S}(u_0, r)$, and

$$(28) \quad \lim_{m\to\infty} \| \Phi u_0 - \Phi_m u_0 \| = 0 ,$$

$$(29) \quad \lim_{m\to\infty} \| \Phi'(u_0) - \Phi_m'(u_0) \| = 0 ,$$

(30) $\lim_{m\to\infty} \| \Phi''(u) - \Phi_m''(u) \| = 0$ uniformly on $\bar{S}(u_0, r)$.

Then there exists a number $m_\varepsilon \in \mathcal{N}$ ($\mathcal{N}$ is the set of natural numbers) such that, for $m \geq m_\varepsilon$,

a) the Newton processes for the equations $\Phi_m u = 0$ with the initial approximation u_0 , are convergent, and the limits $u_*^{(m)}$ of the Newton sequences $\{ u_n^{(m)} \}$ for $n \to \infty$ are solutions of the equations $\Phi_m u = 0$;

b) u^* being the solution of the equation $\Phi u = 0$ to which there converges the Newton process with initial approximation u_0, there exists a positive number $\alpha < 1$ such that

$$(31) \quad \| u^* - u_*^{(m)} \| \leq \frac{1}{1-\alpha} \| \Gamma_0 \Phi u_*^{(m)} \| .$$

c) If, in addition to the preceding assumptions,

C)(30a) $\lim_{m \to \infty} \| \Phi u - \Phi_m u \| = 0$ uniformly on $\bar{S}(u_0, \varkappa)$,

then

$$(32) \quad \lim_{m \to \infty} \| u^* - u_*^{(m)} \| = 0 .$$

d) If furthermore

D)

$$(33) \qquad \varkappa < \varkappa_1 = \frac{1+\sqrt{1-2h}}{h} \, c$$

(the equation $\Phi u = 0$ has exactly one solution on $\bar{S}(u_0, \varkappa)$), then the equations $\Phi_m u = 0$ starting from a certain m, have unique solutions on $\bar{S}(u_0, \varkappa)$.

The proof will be presented after we have recalled one known lemma from the theory of linear operators (e.g. [3] p. 164).

<u>Lemma 2</u>. Let K, L be linear bounded operators mapping a Banach space X into itself. Let there exist, in X, the linear bounded K^{-1} and let

$$(*) \quad \| K - L \| \, \| K^{-1} \| < 1 .$$

Then there exists the linear bounded operator L^{-1}, and we have

$$(**) \quad \| L^{-1} \| \leq \frac{\| K^{-1} \|}{1 - \| K^{-1} \| \, \| K - L \|} ,$$

$$(***) \quad \| K^{-1} - L^{-1} \| \quad \frac{\| K^{-1} \|^2 \, \| K - L \|}{1 - \| K^{-1} \| \, \| K - L \|} \quad .$$

We shall also need the following

Lemma 3. Let the assumptions A, B, of Theorem 1 take place. Then, for an arbitrary ε such that

$$(34) \qquad 0 < \varepsilon < \min \left(\frac{1}{2} - h, \; \varkappa - \varkappa_0 \right) ,$$

there exists $m_\varepsilon \in \mathcal{N}$ such that, for $m \geq m_\varepsilon$, the operators Φ_m have the property Z′ with the point u_0 and with constants c_m, h_m, $\varkappa_0^{(m)}$, $\varkappa$, where

$$(35) \quad h \leq h_m \leq h + \varepsilon < \frac{1}{2} \; ; \; \varkappa_0 \leq \varkappa_0^{(m)} \leq \varkappa_0 + \varepsilon < \varkappa \quad .$$

Proof. Let us use Lemma 2 for $\Phi'(u_0)$, $\Phi_m'(u_0)$. According to (20), starting from a certain $m_1 \in \mathcal{N}$, the inequality $(*)$ takes place. Thus, for $m \geq m_1$, there exist the operators $\Gamma_0^{(m)} = [\Phi_m'(u_0)]^{-1}$ and from $(***)$ it follows that

$$(36) \qquad \lim_{m \to \infty} \| \Gamma_0^{(m)} - \Gamma_0 \| = 0 \; .$$

Furthermore,

$$\| \Gamma_0 \Phi u_0 - \Gamma_0^{(m)} \Phi_m u_0 \| \leq \| \Gamma_0^{(m)} \| \, \| \Phi u_0 - \Phi_m u_0 \| + \| \Gamma_0 - \Gamma_0^{(m)} \| \, \| \Phi u_0 \| \quad ,$$

and, according to (28), (36),

$$(37) \qquad \lim_{m \to \infty} \| \Gamma_0 \Phi u_0 - \Gamma_0^{(m)} \Phi_m u_0 \| = 0 \; .$$

Similarly, for all $u \in \overline{S}(u_0, \varkappa)$,

$$\| \Gamma_0 \Phi''(u) - \Gamma_0^{(m)} \Phi_m''(u) \| \leq \| \Gamma_0^{(m)} \| \, \| \Phi''(u) - \Phi_m''(u) \| + \| \Gamma_0 - \Gamma_0^{(m)} \| \, \| \Phi''(u) \|$$

and, according to (30), (36),

(38) $\lim_{m \to \infty} \| \Gamma_0 \Phi''(u) - \Gamma_0^{(m)} \Phi_m''(u) \| = 0$ uniformly on $\bar{S}(u_0, r)$.

It follows from (37),(19),(38) and (20) that, for each $\eta > 0$, there exists $m(\eta) \in \mathcal{N}$, $m(\eta) \geq m_1$, such that for $m \geq m(\eta)$ there not only exists $\Gamma_0^{(m)}$ but we have

$$\| \Gamma_0^{(m)} \Phi_m u_0 \| = c_m \leq c + \eta \ ,$$

$$u \in \bar{S}(u_0, r) \Longrightarrow \| \Gamma_0^{(m)} \Phi_m''(u) \| = k_m \leq k + \eta \ .$$

Let us introduce the notations

$$h(\eta) = (c + \eta)(k + \eta) \ ,$$

$$r(\eta) = \frac{1}{h(\eta)} \left(1 - \sqrt{1 - 2h(\eta)}\right)(c + \eta) \ .$$

The function r given by the last equation is defined on the interval $\langle 0, x_0 \rangle$ where $h(x_0) = \frac{1}{2}$. Both the functions $h(x)$, $r(x)$ are increasing on $\langle 0, x_0 \rangle$; $h(0) = h$, $r(0) = r_0$.

Thus, for each ε satisfying (34), there exists $m_\varepsilon \in \mathcal{N}$, $m_\varepsilon \geq m(\eta)$, such that, for $m \geq m_\varepsilon$,

$$h_m = c_m k_m = h + \varepsilon < \frac{1}{2} \ ,$$

$$r_0^{(m)} = \frac{1 - \sqrt{1 - 2h_m}}{h_m} c_m \leq r_0 + \varepsilon < r \ ,$$

and Φ_m, for $m \geq m_\varepsilon$, have the property Z with the point u_0 and constants c_m, k_m, $r_0^{(m)}$, r, and the numbers h_m, $r_0^{(m)}$ are less than fixed numbers $h + \varepsilon$, $r_0 + \varepsilon$. That completes the proof of Lemma 3.

<u>Proof of Theorem 1</u>

a) follows from Lemma 3 and Theorem of Kantorowich. In the following considerations, let us choose a fixed ε satisfying (34), and let us assume $m \geq m_\varepsilon$.

b) The operators Φ_m satisfy the assumptions of Lemma 1, and therefore each operator Q_m,

$$Q_m = I - [\Phi_m'(u_0)]^{-1}\Phi_m ,$$

maps the ball $\bar{S}(u_0, r_0^{(m)})$ into itself and is contractive, the constant of contraction being

$$\alpha_m = \sup_{u \in \bar{S}(u_0, r_0^{(m)})} \| Q_m'(u) \| \leq 1 - \sqrt{1 - 2h_m} .$$

As $h \leq h_m \leq h + \varepsilon < \frac{1}{2}$, there is, for all $m \geq m_\varepsilon$,

$$\alpha_m \leq 1 - \sqrt{1 - 2(h+\varepsilon)} < 1 .$$

We have

$$u_{n+1}^{(m)} = Q_m u_n^{(m)}, \quad u_0^{(m)} = u_0 , \qquad n = 0, 1, 2, \dots ,$$

$$\lim_{m \to \infty} u_n^{(m)} = u_*^{(m)} = Q_m u_*^{(m)} .$$

Now,

$$\| u^* - u_*^{(m)} \| = \| Q u^* - Q_m u_*^{(m)} = \| Q u^* - Q u_*^{(m)} + Q u_*^{(m)} - Q_m u_*^{(m)} \| \leq$$

$$\leq \alpha \| u^* - u_*^{(m)} \| + \| \Gamma_0 \Phi u_*^{(m)} - \Gamma_0 \Phi_m u_*^{(m)} \| ,$$

or, according to the fact that $\Phi_m u_*^{(m)} = 0$,

$$\| u^* - u_*^{(m)} \| \leq \frac{1}{1-\alpha} \| \Gamma_0 \Phi u_*^{(m)} \| .$$

c) As $u_*^{(m)} \in \bar{S}(u_0, r_0^{(m)}) \subset \bar{S}(u_0, r)$ and as $\| \Gamma_0 \Phi u_*^{(m)} \| \leq$ $\leq \| \Gamma_0 \| \, \| (\Phi - \Phi_m) u_*^{(m)} \|$, assumption C) yields immediately

$$\| u^* - u_*^{(m)} \| \xrightarrow[m \to \infty]{} 0 .$$

d) It is sufficient to show that, from a certain $m_{\varepsilon'} \geq m_\varepsilon$, the operators Φ_m have not only the property Z but also

$$\kappa_1^{(m)} \overset{df}{=} \frac{1+\sqrt{1-2h_m}}{h_m}\, c_m > \kappa \quad .$$

From the proof of Lemma 3 we see that there exists a number $\varepsilon' > 0$ such that, for $m \geq m_{\varepsilon'} \geq m_\varepsilon$, not only the assertion of Lemma 3 takes place, but also the numbers $|c_m - c|, |h_m - h|$ - and therefore also $|\kappa_1^{(m)} - \kappa_1|$ - are less than an arbitrary positive number chosen beforehand.

The proof of Theorem 1 is complete.

Remark 5. As $hc < \frac{1}{2}$, the Newton algorithm for the equation $\Phi u = 0$ converges with each initial approximation $u \in \bar{S}(u_0, \frac{1-2h}{4h})$([1],p.638). If the assumptions A, B, of the theorem are fulfilled, then an analoguous assertion takes place for the equation (2), starting from some sufficiently large $m \leq \mathcal{N}$.

§ 2.

Let again X be a Banach space. Let $\tilde{X}_m$ be an m-dimensional subspace of X , P_m a projection from X into $\tilde{X}_m$. $\bar{X}_m$ be an m-dimensional space isomorphic with $\tilde{X}_m$. The isomorphism from $\tilde{X}_m$ onto $\bar{X}_m$ be denoted by ψ_m; ψ_m^{-1} be the inverse of it mapping $\bar{X}_m$ onto $\tilde{X}_m$. Let us define an extension φ_m of ψ_m onto all of X by

(1) $\qquad \varphi_m u \overset{df}{=} \psi_m P_m u$

so that

(2) $\qquad P_m = \psi_m^{-1} \varphi_m \quad .$

Φ being an operator mapping X into itself, the finite dimensional operator $P_m \Phi$ maps X into $\tilde{X}_m$, and $\varphi_m \Phi \psi_m^{-1}$ maps $\overline{X}_m$ into $\overline{X}_m$.

The elements of $\tilde{X}_m$ will be denoted by an upper index m - $u^{(m)}$, the corresponding elements of $\overline{X}_m$ by $\overline{u}^{(m)}$. Denote by I the identity operator in X and by $\overline{I}$ the identity operator in $\overline{X}_m$. When there is no danger of confusion we will omit the index m in the notation of elements, operators and spaces.

Let us now apply the results of § 1 to the case when Φ , Φ_m are of special types such that the equations (1),(2) of § 1 are:

$$\text{I} \quad \Phi u \overset{df}{=} u - KFu - g = 0 ,$$

$$\text{II} \quad \Phi_m u^{(m)} \overset{df}{=} u^{(m)} - P_m KFu^{(m)} - P_m g = 0 ,$$

$K : X \to X$ being a linear bounded operator, $F : \Omega \to X$ a nonlinear operator, and g and element of X .
Let us furthermore consider the following equations in $\overline{X}$:

$$\text{III} \quad \overline{\Phi}_m \overline{u}^{(m)} \overset{df}{=} \overline{u}^{(m)} - \varphi_m KF\psi_m^{-1} \overline{u}^{(m)} - \varphi_m g = 0 ;$$

$$\text{IV} \quad \overline{\Psi}_m \overline{u}^{(m)} \overset{df}{=} \overline{u}^{(m)} - \varphi_m K P_m F\psi_m^{-1} \overline{u}^{(m)} - \varphi_m g = 0 .$$

<u>Lemma 1</u>. Let F be defined on the ball $\Omega = S(u_0, R)$, $u_0 \in \tilde{X}_m$, and let it have bounded second Fréchet derivative on $\overline{S}(u_0, r) \subset \Omega$. Then

1) the same takes place for the operators Φ , Φ_m , $m = 1, 2, \dots$,

2) the operators $\overline{\Phi}_m$, $\overline{\Psi}_m$ are defined on $\Omega_m = S(\psi_m u_0, \|\psi_m^{-1}\|^{-1} R)$, and they have the first two Fréchet

derivatives on $\bar{S}(\psi_m u_0, \|\psi_m^{-1}\|^{-1} r)$.

Proof is evident. For example, the last assertion follows from the implication

$$\|\psi_m^{-1}\| \, \|\bar{u}^{(m)} - \psi_m u_0\| \le r \rightarrow \|\psi_m^{-1}\bar{u}^{(m)} - u_0\| \le r .$$

Remark 1. The derivatives of Φ, Φ_m, $\bar{\Phi}_m$, $\bar{\Psi}_m$ are given by

$$\Phi'(u)h = h - KF'(u)h ,$$

$$\Phi''(u)(h,k) = -KF''(u)hk ,$$

$$\Phi_m'(u)h = h - P_m KF'(u)h ,$$

$$\Phi_m''(u)(h,k) = -P_m KF''(u)hk ,$$

$$\bar{\Phi}_m'(\bar{u})\bar{h} = \bar{h} - \varphi_m KF'(\psi_m^{-1}\bar{u})(\psi_m^{-1}\bar{h}) ,$$

$$\bar{\Phi}_m''(\bar{u})(\bar{h},\bar{k}) = -\varphi_m KF''(\psi_m^{-1}\bar{u})(\psi_m^{-1}\bar{h})(\psi_m^{-1}\bar{k}) ,$$

$$\bar{\Psi}_m'(\bar{u})\bar{h} = \bar{h} - \varphi_m KP_m F'(\psi_m^{-1}\bar{u})(\psi_m^{-1}\bar{h}) ,$$

$$\bar{\Psi}_m''(\bar{u})(\bar{h},\bar{k}) = -\varphi_m KP_m F''(\psi_m^{-1}\bar{u})(\psi_m^{-1}\bar{h})(\psi_m^{-1}\bar{k}) .$$

Remark 2. Under the assumptions of Lemma 1, the operators $F'(u)$, F are bounded on $\bar{S}(u_0, r)$. ([4],pp.30,56).

Lemma 2. Let the operator F satisfy the assumptions of Lemma 1. Let

$$(3) \qquad \lim_{m\to\infty} \|K - P_m K\| = 0 ,$$

$$(4) \qquad \lim_{m\to\infty} \|g - P_m g\| = 0 .$$

Then the assumptions B), C), of Theorem 1, § 1, take place.

Proof. With regard to Lemma 1, it suffices to show that the relations (29), (30), (30 a) of § 1 are fulfilled. There is

$$\| \Phi u - \Phi_m u \| \leq \| K - P_m K \| \| F u \| + \| g - P_m g \| ,$$

$$\| \Phi'(u_0) - \Phi'_m (u_0) \| \leq \| K - P_m K \| \| F'(u_0) \| ,$$

$$\| \Phi''(u) - \Phi''_m (u) \| \leq \| K - P_m K \| \| F''(u) \| .$$

According to (3), (4), it is sufficient to show that F, $F'(u)$, $F''(u)$ are bounded on $\bar{S}(u_0, r)$ this being true according to Lemma 1 and Remark 2.

The following lemma shows the relation between the solutions of II and III.

Lemma 3. A) If one of the equations II, III has a solution $u_*^{(m)} \in \tilde{X}_m$ or $\bar{u}_*^{(m)} \in \bar{X}_m$, respectively, then so has the second one, and

$$\bar{u}_*^{(m)} = \psi_m u_*^{(m)} . \tag{5}$$

B) If the Newton iterative process (ordinary or modified) converges for the equation II with the initial approximation $u_0^{(m)} \in \tilde{X}_m$ then it converges also for the equation III with the initial approximation

$$\bar{u}_0^{(m)} = \psi_m u_0^{(m)} , \tag{6}$$

and the same assertion takes place conversely. Furthermore, if $u_n^{(m)}$ or $\bar{u}_n^{(m)}$ resp. are the solutions of the n-th equations of the Newton processes (ordinary resp. modified), then

$$\bar{u}_n^{(m)} = \psi_m u_n^{(m)} , \quad n = 1, 2, \ldots . \tag{7}$$

Proof of A) is given by the application of the operator ψ_m on the equation II or of the operator ψ_m^{-1} on the equation III resp. taking regard to (2).

B) Let us present the proof for the more complicated case of the ordinary Newton process.

We have two sequences of equations

(8) $u_{n+1} = u_n + \eta_n, \ \eta_n - P_m K F'(u_n)\eta_n + \Phi_m u_n = 0,$

(7_0) $u_0 = u_0^{(m)} \in \tilde{X}_m, \quad n = 0, 1, 2, \dots ,$

(9) $\bar{u}_{n+1} = \bar{u}_n + \bar{\eta}_n, \ \bar{\eta}_n - \varphi K F'(\psi^{-1}\bar{u}_n)\psi^{-1}\eta_n + \bar{\Phi}_m \bar{u}_n = 0,$

(10) $\bar{u}_0 = \bar{u}_0^{(m)} = \psi_m u_0^{(m)}, \quad n = 0, 1, 2, \dots .$

For the proof being done by induction, we have to show:
a) if there exists a unique solution of one of the equations (8),(9) in $\tilde{X}_m$ or $\bar{X}_m$ respectively, for $n = 0$, then the other has also a unique solution, and we have

(7_1) $$\bar{u}_1 = \psi u_1 .$$

b) Let the equations (8),(9),for $n = 0,1,\dots, k-1$, have unique solutions in $\tilde{X}_m$ or $\bar{X}_m$ resp., and let (7) take place for $n = 0, 1, \dots, k-1, k$. Then if (8) has for $n = k$ a unique solution the same is true for (9) and conversely. Furthermore, (7) takes place for $n = k+1$.

In the part a), the assertion about the existence of the solution and the relations $\bar{\eta}_0^{(m)} = \psi_m \eta_0^{(m)}$ are given in the same way as that in the proof of A) with respect to $\psi_m \Phi_m u_0^{(m)} = \bar{\Phi}_m \bar{u}_0^{(m)}$. (7_1) then follows from the relation $\bar{\eta}_0^{(m)} = \psi_m \eta_0^{(m)}$, from (7_0) and from the definitions of $u_1^{(m)}$, $\bar{u}_1^{(m)}$. The uniqueness follows from the fact that the operator ψ_m is simple.

The proof of b) is analogous.

Remark 3. Lemma 3 A for linear operators is sometimes called lemma of Gavurin (Kis [2]).

Corollary of Lemma 3 A. If the solution $u_*^{(m)}$ is unique on the set $\mathcal{M} \subset \tilde{X}_m$ then the solution $\bar{u}_*^{(m)}$ is unique on the set $\bar{\mathcal{M}} \subset \bar{X}_m$, $\bar{\mathcal{M}} = \{\bar{u}^{(m)} \mid \bar{u}^{(m)} = \psi_m u^{(m)},\ u^{(m)} \in \mathcal{M}\}$.

Theorem 1. Let the operators in the equations I,II,III have the following properties

A) The operator $K : X \to X$ is linear bounded. The nonlinear operator $F : \Omega \to X$ ($\Omega = S(u_0, R)$, $u_0 \in \tilde{X}_p$) has a bounded second Fréchet derivative on $\bar{S}(u_0, r) \subset \Omega$.

B) Let

$$\lim_{m \to \infty} \| K - P_m K \| = 0 , \tag{11}$$

$$\lim_{m \to \infty} \| g - P_m g \| = 0 . \tag{12}$$

C) There exists a linear bounded operator $\Gamma_0 \overset{df}{=} [\Phi'(u_0)]^{-1}$ and real numbers c, k, r_0 , r such that

$$\| \Gamma_0 \Phi u_0 \| \le c , \tag{13}$$

$$u \in \bar{S}(u_0, r) \Longrightarrow \| \Gamma_0 \Phi''(u) \| \le k , \tag{14}$$

$$r > r_0 \overset{df}{=} \frac{1 - \sqrt{1 - 2h}}{h}\, c , \tag{15}$$

$$h \overset{df}{=} c k < \frac{1}{2} . \tag{16}$$

Then

1) the equations III have, from a certain $m_0 \geq p$, solutions $\bar{u}_*^{(m)}$ to which there converges the Newton iterative process (ordinary or modified) with the initial approximation $\bar{u}_0 = \psi_m u_0$.

2) There is

$$(17) \qquad \lim_{m \to \infty} \| \psi_m^{-1} \bar{u}_*^{(m)} - u^* \| = 0 ,$$

u^* being the solution of I to which there converges the Newton iterative process with the initial approximation u_0.

3) If furthermore

$$\text{D)} \quad r < r_1 = \frac{1 + \sqrt{1 - 2h}}{h} c ,$$

then there exists $m_1 \in \mathcal{N}$ such that, for $m \geq m_1$, the solutions of the equations III are unique on the ball $S(\bar{u}_0, \| \psi_m^{-1} \|^{-1} r)$.

<u>Proof</u>. The first two assumptions contain the assumptions of Lemmas 1 and 2. From them and from the other assumptions there follows that Φ and Φ_m satisfy all the assumptions of Theorem 1, § 1. Therefore the assertion of that theorem takes place for the equations I and II. Thus, there exists a solution u^* of I to which there converges the Newton iterative process with the initial approximation u_0, and, for the equation II, the assumptions of Lemma 3 are fulfilled. That means that the assertion 1) takes place and (7) is fulfilled for all m starting from a certain m_0. Thus we have, for the solutions of II and III (received by Newton processes with the initial approximations u_0 and $\psi_m u_0$), the limit relation

$$\lim_{m\to\infty} \| \bar{u}_*^{(m)} - \psi_m u_*^{(m)} \| = 0 .$$

Furthermore, Theorem 1, § 1 gives

$$\lim_{m\to\infty} \| u^* - u_*^{(m)} \| = 0 ,$$

which is just the relation (17).

When the assumptions under 3) take place, then the same is true for the assumption D) of Theorem 1, § 1. That means that, from a certain $m_1 \geq m_0$, the equations II have unique solutions on the ball $\bar{S}(u_0, r)$. This and the Corollary of Lemma 3 implies the assertion 3.

The next theorem gives an information about the relation between the solutions of III and IV.

<u>Theorem 2</u>. Let

A) the operators Φ and Φ_m in the equations I and II satisfy the assumptions A),B),C) of Theorem 1,

B) there exist the inverse operators to the operators $\bar{\Phi}_m'(\bar{u}_*^{(m)})$ for $m \geq m_0$ (where $\bar{u}_*^{(m)}$ is the solution of III, m_0 a number - both from the assertion of Theorem 1), and let us have

$$(18)\quad \| [\bar{\Phi}_m'(\bar{u}_*^{(m)})]^{-1} \| \leq c \qquad \text{for } m \geq m_0 .$$

C)

$$(19)\quad \|\varphi_m\|^r (1+\|P_m\|)^k \|P_m\|^s \|\psi_m^{-1}\|^t \|K - P_m K\| \xrightarrow[m\to\infty]{} 0 ,$$

$$(20)\quad \|\varphi_m\|^r (1+\|P_m\|)^k \|P_m\|^s \|\psi_m^{-1}\|^t \|g - P_m g\| \xrightarrow[m\to\infty]{} 0$$

for the values (r, k, s, t): $(1,1,0,0)$, $(1,1,0,1)$, $(2,1,1,2)$,

$$(21)\quad \|\varphi_m\|^r \|P_m\|^s \|\psi_m^{-1}\|^t \|(I - P_m) F u^*\| \xrightarrow[m\to\infty]{} 0$$

for the values $(r, s, t) : (1,0,0), (2,1,2), (1,0,1)$;

$$(22) \quad \|\varphi_m\| \|\psi_m^{-1}\| \|(I - P_m) F'(u^*)\| \xrightarrow[m \to \infty]{} 0 .$$

Then there exists a number $m_2 \in \mathcal{N}$, $m_2 \geq m_0$, such that the equations IV have, for $m \geq m_2$, solutions $\bar{v}_*^{(m)}$ for which

$$(23) \qquad \|\bar{u}_*^{(m)} - \bar{v}_*^{(m)}\| \xrightarrow[m \to \infty]{} 0 .$$

Proof. Let us write

$$[\bar{\Phi}_m'(\bar{u}_*^{(m)})]^{-1} = \bar{\Gamma}_m^* .$$

In the Theorem of Kantorovich (§ 1), let us put, for a fixed m,

$$\Gamma = \bar{\Gamma}_m^*, \quad \Phi = \bar{\Psi}_m, \quad u_0 = \bar{u}_*^{(m)} .$$

That gives the following corollary:

Let

$$(24) \quad A_m = \|\bar{\Gamma}_m^* \varphi_m K [F(\psi_m^{-1} \bar{u}_*^{(m)}) - P_m F(\psi_m^{-1} \bar{u}_*^{(m)})]\| \leq \eta_m ,$$

$$(25) \quad B_m = \|\bar{\Gamma}_m^* \varphi_m K [F'(\psi_m^{-1} \bar{u}_*^{(m)}) - P_m F'(\psi_m^{-1} \bar{u}_*^{(m)})]\| \|\psi_m^{-1}\| \leq \delta_m < 1,$$

$$(26) \quad C_m = \|\bar{\Gamma}_m^* \varphi_m K P_m F''(\psi_m^{-1} \bar{u}^{(m)})\| \|\psi_m^{-1}\|^2 \leq \varkappa_m$$

for all $\bar{u}^{(m)} \in \bar{S}(\bar{u}_*^{(m)}, \rho_m)$,

$$(27) \qquad h_m = \frac{\eta_m \varkappa_m}{(1 - \delta_m)^2} \leq \frac{1}{2} ,$$

$$(28) \quad \rho_m \geq \kappa_0^{(m)} = \frac{1-\sqrt{1-2h_m}}{h_m} \; \frac{\eta_m}{1-\sigma_m} \;.$$

Then the equation $\overline{\Psi}_m \, \overline{u} = 0$ has a solution $\overline{v}_*^{(m)}$ for which

$$(29) \quad \| \overline{v}_*^{(m)} - \overline{u}_*^{(m)} \| \leq \kappa_0^{(m)} \;.$$

It is thus sufficient to show:

1) $\lim\limits_{m\to\infty} A_m = 0$,

2) $\lim\limits_{m\to\infty} B_m = 0$,

3) there exists $\{\rho_m\}$ such that

α) $\lim\limits_{m\to\infty} A_m C_m = 0$ for all $\overline{u}^{(m)} \in \overline{S}(\overline{u}_*^{(m)}, \rho_m)$,

β) from a certain $m_0' \in \mathcal{N}$, $m_0' \geq m_0$, there is $\rho_m \geq \kappa_0^{(m)}$.

Indeed, in this case all the assumptions of the corollary are fulfilled, and we have

$$\lim_{m\to\infty} \kappa_0^{(m)} = 0 \;.$$

1) Let us assume $m \geq m_0$. Then, according to (18),

$$A_m \leq c \, \| K \| \, \| \varphi_m \| \, \| (I - P_m) F(\psi_m^{-1} \overline{u}_*^{(m)}) \| \;.$$

By Lemma 3 there is

$$(30) \quad \psi_m^{-1} \overline{u}_*^{(m)} = u_*^{(m)} ,$$

$u_*^{(m)}$ being the solution of II given by the Newton process from the point u_0 . According to Theorem 1 there is

$$(31) \quad u_*^{(m)} \xrightarrow[m\to\infty]{} u^* \;.$$

Thus we have the inequality following from the last one:

$$(32)\quad A_m \leq c\,\|K\|\,\|\varphi_m\|\,(1+\|P_m\|)\,\|Fu_*^{(m)} - Fu^*\| + $$
$$+\, c\,\|K\|\,\|\varphi_m\|\,\|(I-P_m)Fu^*\|\,.$$

From (31) there follows that, starting from a certain m, there is $u_*^{(m)} \in \bar{S}(u_0, r)$ and, according to the assumption about the existence of the derivative $F'(u)$ on $\bar{S}(u_0, r)$, we have

$$(33)\quad \|Fu_*^{(m)} - Fu^*\| \leq \sup_{0<\vartheta<1} \|F'(u^* + \vartheta[u_*^{(m)} - u^*])\|\,\|u^* - u_*^{(m)}\|\,.$$

According to Theorem 1, § 1, there exists a number α, $0 < \alpha < 1$, such that

$$(34)\quad \|u^* - u_*^{(m)}\| \leq \frac{\|\Gamma_0\|}{1-\alpha}\{\|K - P_m K\|\,\|Fu_*^{(m)}\| + \|g - P_m g\|\}\,.$$

$F'(u)$ is bounded on $\bar{S}(u_0, r)$ see Remark 2 . It follows from (32),(33),(34) that for having $A_m \to 0$ it is sufficient

$$\|\varphi_m\|\,(1+\|P_m\|)\,\|K - P_m K\| \xrightarrow[m\to\infty]{} 0\,,$$

$$\|\varphi_m\|\,(1+\|P_m\|)\,\|g - P_m g\| \xrightarrow[m\to\infty]{} 0\,,$$

$$\|\varphi_m\|\,\|(I - P_m)Fu^*\| \xrightarrow[m\to\infty]{} 0\,,$$

this being true according to (19),(20) for $(p, r, s, t) = (1, 1, 0, 0)$ and according to (21) for $(p, s, t) = (1, 0, 0)$.

2) Similarly, (25) yields

$$B_m \le c\|K\|\,\|\varphi_m\|\,\|(I-P_m)F'(u_*^{(m)})\|\,\|\psi_m^{-1}\| \le$$

$$\le c\|K\|\,\|\varphi_m\|(1+\|P_m\|)\|\psi_m^{-1}\|\,\|F'(u_*^{(m)})-F'(u^*)\| +$$

$$+ c\|K\|\,\|\varphi_m\|\,\|(I-P_m)F'(u^*)\|\,\|\psi_m^{-1}\| .$$

According to the assumption about the existence of $F''(u)$ in $\overline{S}(u_0, r)$, there is

$$\|F'(u_*^{(m)})-F'(u^*)\| \le \sup_{0<\vartheta<1} \|F''(u^*+\vartheta[u_*^{(m)}-u^*]\|\,\|u^*-u_*^{(m)}\| .$$

From the boundedness of $F''(u)$ on $\overline{S}(u_0, r)$ and from the two last relations there follows that for having $B_m \longrightarrow 0$ it is sufficient

$$\|\varphi_m\|(1+\|P_m\|)\|\psi_m^{-1}\|\,\|K-P_mK\| \xrightarrow[m\to\infty]{} 0 ,$$

$$\|\varphi_m\|(1+\|P_m\|)\|\psi_m^{-1}\|\,\|g-P_mg\| \xrightarrow[m\to\infty]{} 0 ,$$

$$\|\varphi_m\|\,\|(I-P_m)F'(u^*)\|\,\|\psi_m^{-1}\| \xrightarrow[m\to\infty]{} 0 .$$

this being true according to (19),(20) for $(p,r,s,t)=(1,1,0,1)$ and (22).

3) Let us choose $\rho_m = \dfrac{r-r_0}{2}\dfrac{1}{d_m}$, $d_m = \max(1, \|\psi_m^{-1}\|$.

Then, from a certain m , the following implication takes place:

(35) $\bar{u}^{(m)} \in \overline{S}(\bar{u}_*^{(m)}, \rho_m) \Rightarrow \psi_m^{-1}\bar{u}^{(m)} \in \overline{S}(u_0, r)$, i.e.

$$\|\bar{u}^{(m)}-\bar{u}_*^{(m)}\| \le \rho_m \Rightarrow \|\psi_m^{-1}\bar{u}^{(m)}-u_0\| \le r .$$

In fact, there is

$$\|\psi_m^{-1}\bar{u}^{(m)}-u_0\| \le \|\psi_m^{-1}\|\,\|\bar{u}^{(m)}-\bar{u}_*^{(m)}\| + \|\psi_m^{-1}\bar{u}_*^{(m)}-u^*\| +$$

$$+\|u^*-u_0\| \le \frac{r-r_0}{2} + \varepsilon_m + r_0 = r - \frac{r-r_0}{2} + \varepsilon_m ,$$

where $\kappa > \kappa_0$ according to (15) and $\varepsilon_m \to 0$ according to (17). Now there is, similarly as in the previous points,

$$C_m \le c \, \| K \| \, \| \varphi_m \| \, \| P_m \| \, \| \psi_m^{-1} \|^2 \, \| F''(\psi_m^{-1} \bar{u}^{(m)}) \| \; .$$

By (35) there is, for the given φ_m, $\psi_m^{-1} \bar{u}^{(m)} \in$
$\in \bar{S}(u_0, \kappa)$.

According to the assumption, $F''(u)$ is bounded on $\bar{S}(u_0, \kappa)$.
Thus, to fulfil $3\alpha)$, the following is sufficient:

$$\| \varphi_m \|^2 (1 + \| P_m \|) \| P_m \| \, \| \psi_m^{-1} \|^2 \, \| K - P_m K \| \xrightarrow[m \to \infty]{} 0 \, ,$$

$$\| \varphi_m \|^2 (1 + \| P_m \|) \| P_m \| \, \| \psi_m^{-1} \|^2 \, \| g - P_m g \| \xrightarrow[m \to \infty]{} 0 \, ,$$

$$\| \varphi_m \|^2 \, \| P_m \| \, \| \psi_m^{-1} \|^2 \, \| (I - P_m) F u^* \| \xrightarrow[m \to \infty]{} 0 \, ;$$

but this is true according to (19),(20) for $(p, \kappa, s, t) =$
$= (2, 1, 1, 2)$ and according to (21) for $(p, s, t) = (2, 1, 2)$.
The condition $3\beta)$ is equivalent to the condition

$$\frac{\kappa - \kappa_0}{2} \le \frac{1 - \sqrt{1 - 2h_m}}{h_m} \, \frac{A_m}{1 - B_m} \max (1, \| \psi_m^{-1} \|) \, .$$

If the preceding conditions are fulfilled, there is
$A_m \to 0$, $B_m \to 0$, $A_m C_m \to 0$, $h_m \to 0$, and therefore

$$\frac{1 - \sqrt{1 - 2h_m}}{h_m} \xrightarrow[m \to \infty]{} 1 \, .$$

To satisfy $3\beta)$, it suffices to have $A_m \| \psi_m^{-1} \| \to 0$,
this being true if

$$\| \varphi_m \| (1 + \| P_m \|) \| \psi_m^{-1} \| \, \| K - P_m K \| \xrightarrow[m \to \infty]{} 0 \, ,$$

$$\| \varphi_m \| (1 + \| P_m \|) \| \psi_m^{-1} \| \, \| g - P_m g \| \xrightarrow[m \to \infty]{} 0 \, ,$$

$$\| \varphi_m \| \, \| \psi_m^{-1} \| \, \| (I - P_m) F u^* \| \xrightarrow[m \to \infty]{} 0 \, ,$$

but this is fulfilled according to (19),(20) for

$(p, r, s, t) = (1, 1, 0, 1)$, and according to (21) for $(p, s, t) = (1, 0, 1)$.

Remark 4. If we assume that, for some m , there exists a solution $\bar{v}_*^{(m)}$ of the equation IV and that we know some its approximation $\bar{v}^{(m)}$ the Theorem of Kantorovich gives us the following assertion:

Let K be linear bounded and let F have bounded second Fréchet derivative on the ball $\bar{S}(\psi^{-1}\bar{v}^{(m)}, \|\psi^{-1}\| \bar{\varrho}_m)$. Let there exist the linear bounded operator

$$\bar{\Delta}_m \overset{df}{=} [\bar{\Psi}_m'(\bar{v}^{(m)})]^{-1} ,$$

Let furthermore

1) $\|\bar{\Delta}_m \bar{\Phi}_m \bar{v}^{(m)}\| \leq c_m$,

2) $\|\bar{\Delta}_m \bar{\Phi}_m'(\bar{v}^{(m)}) - \bar{I}\| = \|\bar{\Delta}_m [\bar{\Phi}_m'(\bar{v}^{(m)}) - \bar{\Psi}_m'(\bar{v}^{(m)})]\| \leq \delta_m < 1$,

3) $\|\bar{\Delta}_m \bar{\Phi}_m''(\bar{u})\| \leq k_m$ for $\bar{u} \in \bar{S}(\bar{v}^{(m)}, \bar{\varrho}_m)$,

4) $\bar{h}_m \overset{df}{=} \dfrac{c_m k_m}{(1-\delta_m)^2} \leq \dfrac{1}{2}$,

$$\bar{\varrho}_m \geq \bar{\varrho}_0 \overset{df}{=} \frac{1-\sqrt{1-2\bar{h}_m}}{\bar{h}_m} \frac{c}{1-\delta_m} .$$

Then the equation III has a solution $\bar{u}_*^{(m)}$ for which

$$\|\bar{u}_*^{(m)} - \bar{v}^{(m)}\| \leq \bar{\varrho}_0 .$$

R e f e r e n c e s

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