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A NOTE ON THE MINIMAX PRINCIPLE FOR K-POSITIVE OPERATORS

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In this note definitions and notation of the paper [4] will be used. Instead of the assumption (β) in [4] the normality of the cone K will be requested (see [2]).

The purpose of this note is to show that some assumptions of the papers [3 - 6] can be either weakened or omitted.

Let $T \in [Y]$ be a K-positive operator which satisfies at least one of the following two conditions:

(a) T is a semi-nonsupport operator (see [7]).

(b) T is a u_0-positive operator (see [1],p.60).

Let $H' \subset K'$ be a K-total set. Then we put for $x \in K$, $x \neq \sigma$,

$$r_x = \inf_{\substack{x' \in H' \\ \langle x, x' \rangle \varkappa(x') \neq 0}} \frac{\langle Tx, x' \rangle}{\langle x, x' \rangle},$$

$$r^x = \sup_{\substack{x' \in H' \\ \langle x, x' \rangle \varkappa(x') \neq 0}} \frac{\langle Tx, x' \rangle}{\langle x, x' \rangle},$$

where $\varkappa(x') = 1$ in the case (a) and $\varkappa(x') = \langle u_0, x' \rangle$ in the case (b).

<u>Definition</u>. The operator $T \in [\mathfrak{X}]$, where $\mathfrak{X}$ denotes a complexification of Y, is said to have property (S), if the relations $\lambda \in \sigma(T)$, $|\lambda| = \rho(T)$, where $\rho(T)$ is the spectral radius of T, imply that λ is a pole of

$\mathcal{R}(\lambda, T) = (\lambda I - T)^{-1}$.

Theorem 1. Assume that

(i) $K \subset Y = K - K$ is a normal cone.

(ii) $H' \subset K' \subset Y'$ is a K-total set.

(iii) $T \in [Y]$ has property (S).

(iv) At least one of the conditions (a) and (b) is fulfilled.

Then it holds:

1. $\rho(T) = \min\limits_{\substack{x \in K \\ x \neq o}} r^x = \max\limits_{\substack{x \in K \\ x \neq o}} r_x$.

2. There are a proper vector $x_o \in K$ and a proper linear form $x_o' \in K'$ of the operator T which have the following properties. The vector x_o is a nonsupport element of the cone K (see [7]), the linear form x_o is strictly positive (see [2]). Moreover, the relations $y = \nu T y$, $y \in K$, $y' = \mu T' y'$, $y' \in K'$ imply that $y = c x_o$ and $y' = c' x_o'$ with some constants c and c' .

3. Every extremal element z with respect to the operator T (i.e. either $r^z = \rho(T)$ or $r_z = \rho(T)$) has the form $z = c x_o$.

Note. This theorem shows that the assumptions

(c) T is a strict nonsupport operator (see [7]),

(d) T is a uniformly u_o-positive operator (see [6]),

can be omitted in the main theorem of the paper [5].

The assumption (c) in [3] can be replaced by assumption (a) and the assumption (a) in [4, 6] by assumption (b). Simultaneously with these alterations some other assertions hold under the corresponding replacements of the assumptions. A typical example is the generalized Stein-Rosenberg theorem (see [3, 6]) which can be formulated as follows:

Theorem 2. If we assume that

(α) the operator $T \in [Y]$ has property (S),

(β) in the expression $B = L + \mathcal{U}$, the operators L and $\mathcal{U} \in [Y]$ are K-positive and $\mathcal{U} \neq \theta$,

(γ) the operator $H = (I - L)^{-1}\mathcal{U}$ has in K a proper vector which corresponds to the spectral radius $\rho(H)$,

(δ) the operator $(I - B)^{-1}\mathcal{U}$ has property (S),

(ε) the operator B satisfies at least one of the conditions (a) and (b),

then one of the three following conditions holds:

$$0 < \rho(H) < \rho(B) < 1,$$

$$\rho(B) = \rho(H) = 1,$$

$$\rho(H) > \rho(B) > 1.$$

Proof of theorem 1. Only the assertion 3 is to be proved.

Assume that z is an extremal element with respect to the operator T. Let $r_z = \rho(T)$. The case $r^z = \rho(T)$ can be investigated analogously. Let $v = Tz - \rho z + \sigma$, where $\rho = \rho(T)$. Then $v \in K$ and $\mathcal{P}v \neq \sigma$, where

$$\mathcal{P} = \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} [\rho(T)]^{-k} T^k$$

(see [5]). If $x \in K$, $x \neq \sigma$, then $\mathcal{P}x$ is a proper vector of T corresponding to $\rho(T)$: $T\mathcal{P}x = \rho(T)\mathcal{P}x$ (see [5]). Let $x_0' = \frac{1}{\rho(T)} T' x_0'$, $x_0' \neq \sigma$. The functional x_0' is strictly positive (see [7] and [5]). Consequently, we have

$$0 < \mathcal{P}v, x'_0 > = < (T - \rho I)\mathcal{P}z, x'_0 > = 0$$

and this contradicts the relation $v \neq o$. The proof is completed.

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