Aleš Pultr

On selecting of morphisms among all mappings between underlying sets of objects in concrete categories and realisations of these

Commentationes Mathematicae Universitatis Carolinae, Vol. 8 (1967), No. 1, 53--83

Persistent URL: `http://dml.cz/dmlcz/105093`

Commentationes Mathematicae Universitatis Carolinae
8,1 (1967)

ON SELECTING OF MORPHISMS AMONG ALL MAPPINGS BETWEEN UNDERLYING SETS OF OBJECTS IN CONCRETE CATEGORIES AND REALISATIONS OF THESE

Aleš PULTR, Praha

Introduction. Let us begin with a simple example: If $\leqslant$ is a partial ordering on a set X, define a topology $t(\leqslant)$ on X as follows:

U is open iff for every $x \in U$ $y \leqslant x$ implies $y \in U$.

Let us take notice of the fact that if $(X, \leqslant)$, $(Y, \preccurlyeq)$ are two partially ordered sets, then, among all the mappings of X into Y, the continuous mappings with respect to $t(\leqslant)$, $t(\preccurlyeq)$ are exactly the isotone mappings with respect to $\leqslant$, $\preccurlyeq$. Thus, the structure of topology is, in certain sense, richer than the structure of partial ordering, namely, if a system of mappings of X into Y may be described as a system of all isotone mappings with respect to partial orderings, it may be described as a system of all continuous mappings with respect to suitable topologies.

For a moment, understand under a structure anything taking part in selection of "suitable" mappings. We shall deal with replacing of structures by other ones, richer in the sense mentioned above (i.e., able to describe at least all the systems of mappings which may be described by the former ones).

We shall now reformulate this in the category language. Usually, the term "concrete category" is used for a category $\mathcal{K}$ such that there exists a faithfull functor from $\mathcal{K}$ into the category of sets (the so called forgetful functor). Let us agree to understand here under a concrete category a category together with a given forgetful functor (for the former notion we may use the atribut concretisable). Roughly speaking, objects of concrete categories are sets (endowed by structures) and morphisms are some mappings between these sets. We say that a concrete category $(\mathcal{K}, \square)$ is realisable in $(\mathcal{K}', \square')$ ($\square$, $\square'$ are the forgetful functors), if there is a <u>full</u> embedding $\Phi : \mathcal{K} \Rightarrow \mathcal{K}'$ preserving the underlying sets and the actual form of morphisms (see definition 1.1 in § 1). The example above may be now formulated as follows: The category of partially ordered sets and their isotone mappings is realisable in the category of topological spaces and their continuous mappings.

Another example: Among the results of [1] is the (otherwise formulated) fact that a number of topologylike categories is realisable in the category of merotopic spaces.

Let us recall some definitions from [2]. If r, s are α-nary relations on X, Y, we say that a mapping $f : X \to Y$ is rs-compatible, if, for every $\{x_\iota \mid \iota < \alpha\} \in r$, $\{f(x_\iota) \mid \iota < \alpha\} \in s$. (We remark that a unary relation is a subset of X ; hence, if r, s are unary relations on X, Y respectively, $f : X \to Y$ is rs-compatible iff $f(r) \subset s$). A type $\Delta = \{\alpha_\beta \mid \beta < \gamma\}$ is a sequence of ordinals indexed by ordinals. $r = \{r_\beta\}$ is said to be a relational system of the type Δ on X

if, for every $\beta < \gamma$, r_β is an α_β-nary relation on X; if r, s are relational systems of a type $\Delta = \{\alpha_\beta \mid \beta < \gamma\}$ on X, Y respectively, we say that $f : X \to Y$ is rs-compatible, iff it is $r_\beta s_\beta$-compatible for every $\beta < \gamma$. Let $F_1, \dots, F_n$ be functors from the category of sets into itself, $\Delta_1, \dots, \Delta_n$ types. $S((F_1, \Delta_1), \dots, (F_n, \Delta_n))$ is defined as follows: The objects are systems $(X, r_1, \dots, r_n)$ where r_i is a relational system of the type Δ_i on $F_i(X)$; morphisms from $(X, r_1, \dots, r_n)$ into $(Y, s_1, \dots, s_n)$ are mappings $f : X \to Y$ such that $F_i(f)$ are $r_i s_i$-compatible for covariant F_i, $s_i r_i$-compatible for contravariant F_i. (More exactly, the morphisms are triples $((X, r_1, \dots, r_n), f, (Y, s_1, \dots, s_n)))$.

Many important concrete categories may be considered as full subcategories of categories $S((F_1, \Delta_1), \dots, (F_n, \Delta_n))$. Some examples:

1) The category $\mathfrak{A}(\Delta)$ of all algebras of the type $\Delta = \{\alpha_\beta \mid \beta < \gamma\}$ is a full subcategory of $S((I, \bar{\Delta}))$, where I is the identity functor and $\bar{\Delta} = \{\alpha_\beta + 1 \mid \beta < \gamma\}$.

2) The category of topological spaces and their continuous mappings is a full subcategory of $S((P^-, \{1\}))$ where P^- is the functor associating with every set its power set, and, with every $f : X \to Y$, $P^-(f) : P^-(Y) \to P^-(X)$ defined by $P^-(f)(A) = f^{-1}(A)$.

3) Similarly, the category of uniform spaces and their uniformly continuous mappings is a full subcategory of $S((P^- \circ Q, \{1\}))$ where $Q(X) = X \times X$, $Q(f)(x, y) = (f(x), f(y))$.

4) The category of topological groups and their continuous homomorphisms is a full subcategory of $S((P^-, \{1\}), (I, \{3\}))$.

5) The category of merotopic spaces (see [1]) is a full subcategory of $S((P^+ \circ P^+, \{1\}))$, where $P^+(X) = \{A \mid A \subset X\}$, $P^+(f)(A) = f(A)$ (the image of the set A).

6) The category of proximity spaces may be considered as a full subcategory of $S((P^+, \{2\}))$ (if the proximities are defined by the relation "to be near") or of $S((P^-, \{2\}))$ (if the proximities are defined as the relations "to be far").

7) The category of differentiable manifolds and their differentiable mappings is a full subcategory of $S((P_{E_1}, \{1\}))$, where $P_{E_1}(X) = E_1^X$, $P_{E_1}(f)(\varphi) = \varphi \circ f$.

8) The category of topological spaces and their open continuous mappings is a full subcategory of $S((P^-, \{1\}), (P^+, \{1\}))$.

In the examples we met some particular set functors (I, P^-, P^+, Q, P_{E_1}). Q is a special case of Q_A defined by $Q_A(X) = X^A$, $Q_A(f)(\varphi) = f \circ \varphi$, P_{E_1} is a special case of P_A defined by $P_A(X) = A^X$, $P_A(f)(\varphi) = \varphi \circ f$. Denote by K_A the functor defined by $K_A(X) = X \times A$, $K_A(f) = f \times id_A$, by V_A the functor defined by $V_A(X) = X \times \{0\} \cup A \times \{1\}$, $V_A(f)(x, 0) = (f(x), 0)$ for $x \in X$, $V_A(f)(a, 1) = (a, 1)$ for $a \in A$.

In the present paper we shall deal with representations of $S((F_1, \Delta_1), \dots, (F_n, \Delta_n))$ such that the functors F_i are obtained from the mentioned ones by operations of composition, cartesian product ($\times$, see § 2), join ($\vee$, see § 2) and a further operation defined in § 2. The main result is that such $S((F_1, \Delta_1), \dots, (F_n, \Delta_n))$ is always realisable in $S(((P^-)^k \circ V_A, \{1\}))$ with a sufficiently large natural number k and a set A. This is stated in Theorem 6.5 in a somewhat more general form.

§ 1 contains some definitions and a particular case of realisation following from [2]. In § 2 the mentioned operations with set functors are described. § 3 contains an auxiliary notion and some statements concerning this. In §§ 4 and 5 the functors obtained from I, V_A, K_A, Q_A, P_A, P^-, P^+ are discussed and a canonical majorisation of these is found. In § 6 is proved the main theorem; as an easy consequence we obtain a theorem on boundability (i.e. full embeddability into categories of algebras, see [2],[3]). § 7 contains some remarks, in particular two examples of realisations (namely, of $S(Q_A)$ in $S(P_B)$ with sufficiently large B and of $S(P^-)$ in $S((P^+)^3)$) not following from the previous theory.

§ 1. Some definitions and notation.

As stated above, in the present paper a concrete category $(\mathfrak{K}, \square)$ is a category together with a fixed forgetful functor.

1.1. Definition. Let $(\mathfrak{K}, \square)$, $(\mathfrak{K}', \square')$ be concrete categories. A full embedding (i.e. a one-to-one covariant functor onto a full subcategory) $\Phi : \mathfrak{K} \to \mathfrak{K}'$ is said to be a realisation of $(\mathfrak{K}, \square)$ in $(\mathfrak{K}', \square')$ if

$$\square' \circ \Phi = \square \quad .$$

We write then $\Phi : (\mathfrak{K}, \square) \Rightarrow (\mathfrak{K}', \square')$. To indicate the realisability of $(\mathfrak{K}, \square)$ in $(\mathfrak{K}', \square')$, i.e. the existence of such Φ, we write simply $(\mathfrak{K}, \square) \rightrightarrows (\mathfrak{K}', \square')$.

1.2. Remark: Obviously $(\mathfrak{K}, \square) \rightrightarrows (\mathfrak{K}', \square')$ and $(\mathfrak{K}', \square') \rightrightarrows (\mathfrak{K}'', \square'')$ imply $(\mathfrak{K}, \square) \rightrightarrows (\mathfrak{K}'', \square'')$.

1.3. Conventions: We write simply $S(F_1, \dots, F_n)$ instead of $S((F_1, \{1\}), \dots, (F_n, \{1\}))$ (see Introduction). Thus, if e.g. F is a covariant functor, the objects of $S(F)$ are couples (X, κ), where $\kappa \subset F(X)$, and $f : X \to Y$ is a morphism from (X, κ) into $(Y, \varkappa)$ iff $F(f)(\kappa) \subset \varkappa$.

The category $S((F_1, \Delta_1), \dots, (F_n, \Delta_n))$ is always considered to be endowed by the forgetful functor associating X with $(X, \kappa_1, \dots, \kappa_n)$ and f with $((X, \kappa_1, \dots, \kappa_n), f, (Y, \varkappa_1, \dots, \varkappa_n))$ (as a rule, we write simply f instead of $((X, \kappa_1, \dots, \kappa_n), f, (Y, \varkappa_1, \dots, \varkappa_n))$).

1.4. __Definition__: A set functor is a functor from the category of sets into itself.

1.5. __Theorem__: Let G_ι, $\iota \in J$, be set functors, Δ_ι types. Put $F_\iota = G_\iota$ for covariant G_ι, $F_\iota = P^- \circ G_\iota$ for contravariant G_ι. Then

$$S(\{(G_\iota, \Delta_\iota) \mid \iota \in J\}) \rightrightarrows S(\{(F_\iota, \Delta_\iota) \mid \iota \in J\}).$$

__Proof__: This follows easily from the following statement: If κ is an α-nary relation on $G(X)$, define an α-nary relation $\tilde{\kappa}$ on $P^- \circ G(X)$ by: $\{A_\varkappa \mid \varkappa < \alpha\} \in \tilde{\kappa}$ iff ($x_\varkappa \in A_\varkappa$ for every $\varkappa$) implies $\{x_\varkappa \mid \varkappa < \alpha\} \notin \kappa$. Let κ, $\varkappa$ be α-nary relations on $G(X), G(Y)$ respectively. Then, for any $f : X \longrightarrow Y$, $G(f)(\varkappa) \subset \kappa$ iff $P^- \circ G(f)(\tilde{\kappa}) \subset \tilde{\varkappa}$. Let $G(f)(\varkappa) \subset \kappa$, $\{A_\varkappa\} \in \tilde{\kappa}$. We have to prove that $\{G(f)^{-1}(A_\varkappa)\} \in \tilde{\varkappa}$. If $y_\varkappa \in G(f)^{-1}(A_\varkappa)$, we have $G(f)(y_\varkappa) \in A_\iota$. Thus, $\{y_\varkappa\} \in \varkappa$ would imply $\{G(f)(y_\varkappa)\} \in \kappa$ which is impossible. On the other hand, let $P^- \circ G(f)(\tilde{\kappa}) \subset \tilde{\varkappa}$ and $\{y_\varkappa\} \in \varkappa$; if $\{G(f)(y_\varkappa)\} \notin \kappa$, we have $\{\{G(f)(y_\varkappa)\}\} \in \tilde{\kappa}$ and hence $\{G(f^{-1})\{G(f)(y_\varkappa)\}\} \in \tilde{\varkappa}$. This is a contradiction, as $y_\varkappa \in G(f)^{-1}\{G(f)(y_\varkappa)\}$.

§ 2. Operations with set functors.

We shall use the following notation: If X, Y are sets, we denote by $X \vee Y$ the set $X \times \{0\} \cup Y \times \{1\}$ (a "disjoint union" of X and Y). If $f: X \to Y$, $g: U \to V$ are mappings, we denote by $f \times g$ the mapping of $X \times U$ into $Y \times V$ defined by $(f \times g)(x, u) = (f(x), g(u))$ and by $f \vee g$ the mapping of $X \vee U$ into $Y \vee V$ defined by $(f \vee g)(x, 0) = (f(x), 0)$ and $(f \vee g)(u, 1) = (g(u), 1)$. If X, Y are sets, $\langle X, Y \rangle$ is the set of all mappings of X into Y (i.e., the set Y^X, in the more usual notation). For $f: X \to Y, g: U \to V$ define $\langle f, g \rangle : \langle Y, U \rangle \to \langle X, V \rangle$ by $\langle f, g \rangle (\alpha) = g \circ \alpha \circ f$.

2.1. Lemma: I. $(f_1 \times g_1) \circ (f_2 \times g_2) = f_1 \circ f_2 \times g_1 \circ g_2$, $id \times id = id$; if f, g are one-to-one (onto, resp.), $f \times g$ is one-to-one (onto, resp.).

II. $(f_1 \vee g_1) \circ (f_2 \vee g_2) = f_1 \circ f_2 \vee g_1 \circ g_2$, $id \vee id = id$; if f, g are one-to-one (onto, resp.), $f \vee g$ is one-to-one (onto, resp.).

III. $\langle f_1, g_1 \rangle \circ \langle f_2, g_2 \rangle = \langle f_2 \circ f_1, g_1 \circ g_2 \rangle$, $\langle id, id \rangle = id$; if f is onto and g one-to-one, then $\langle f, g \rangle$ is one-to-one; if f is one-to-one and g onto, then $\langle f, g \rangle$ is onto.

2.2. Definition: Let F, G be set functors of the same variance. We define set functors $F \times G$ and $F \vee G$ by

$$(F \times G)(X) = F(X) \times G(X), \quad (F \times G)(f) = F(f) \times G(f);$$

$$(F \vee G)(X) = F(X) \vee G(X), \quad (F \vee G)(f) = F(f) \vee G(f).$$

Let the variances of F, G be opposite. The set functor $\langle F, G \rangle$ is defined by

$\langle F, G\rangle(X) = \langle F(X), G(X)\rangle$, $\langle F, G\rangle(f) = \langle F(f), G(f)\rangle$.

<u>Theorem</u>: If F, G are covariant (contravariant), so are $F \times G$ and $F \vee G$. If F is covariant and G contravariant, $\langle F, G\rangle$ is contravariant; if F is contravariant and G covariant, $\langle F, G\rangle$ is covariant.

<u>Proof</u> is trivial.

2.3. <u>Remark</u>: The operations $F \times G$, $F \vee G$ are, up to the natural equivalence, associative. We shall write $\overset{n}{\underset{i=1}{\mathsf{X}}} F_i = F_1 \times \dots \times F_n$, $\overset{n}{\underset{i=1}{\bigvee}} F_i = F_1 \vee \dots \vee F_n$. Let $f_\iota : X_\iota \to Y_\iota$ $(\iota \in J)$ be mappings. Define $\mathsf{X}\{f_\iota\} : \mathsf{X}\{X_\iota \mid \iota \in J\} \to \mathsf{X}\{Y_\iota \mid \iota \in J\}$ ($\mathsf{X}\{X_\iota \mid \iota \in J\}$ etc. is the usual cartesian product of the system) by $\mathsf{X}\{f_\iota\}(\{x_\iota\}) = \{f_\iota(x_\iota)\}$, $\bigvee\{f_\iota\} : \bigvee\{X_\iota \mid \iota \in J\} \to \bigvee\{Y_\iota \mid \iota \in J\}$ ($\bigvee\{X_\iota\} = \bigcup\{X_\iota \times \{\iota\} \mid \iota \in J\}$) by $\bigvee\{f_\iota\}(x, \iota) = (f_\iota(x), \iota)$. Now, if for every $\iota \in J$ a covariant (contravariant) functor F_ι is given, we may define $\mathsf{X} F_\iota$, $\bigvee F_\iota$ by $(\mathsf{X} F_\iota)(X) = \mathsf{X}\{F_\iota(X)\}$, $(\mathsf{X} F_\iota)(f) = \mathsf{X}\{F_\iota(f)\}$, $(\bigvee F_\iota)(X) = \bigvee\{F_\iota(X)\}$, $(\bigvee F_\iota)(f) = \bigvee\{F_\iota(f)\}$. This is, up to the natural equivalence, in accordance with the notation above.

2.4. Composition of set functors F, G is, of course, denoted by $F \circ G$.

§ 3. <u>Majorisation of set functors</u>.

If $T : F \to G$ is a natural transformation of functors such that $T_X : F(X) \to G(X)$ is one-to-one (onto) for every X, we say that T is a monotransformation (epitransformation).

3.1. <u>Definition</u>: Let F, G be set functors. We write

$F \dashv G$ if either there is a monotransformation $T: F \to G$ or if there is an epitransformation $T: G \to F$. We say that F is majorised by G (and write $F < G$), if there are functors $F_1, \dots, F_n$ such that $F = F_1$, $G = F_n$ and $F_i \dashv F_{i+1}$ for $i = 1, \dots, n-1$.

3.2. Remarks: 1) Obviously $F < G$ and $G < H$ imply $F < H$. 2) If F, G are naturally equivalent, then $F < G$ and $G < F$.

3.3. Metalemma: Let ω be a binary operation on set functors such that

1) for any functor G with a property $\mathcal{A}$ $F_1 \dashv F_2$ implies $G \omega F_1 \dashv G \omega F_2$,

2) for any functor G with a property $\mathcal{B}$ $F_1 \dashv F_2$ implies $F_1 \omega G \dashv F_2 \omega G$.

Let $F_1 < F_2$, $G_1 < G_2$ and let either F_2 have the property $\mathcal{A}$ and G_1 have the property $\mathcal{B}$, or F_1 have the property $\mathcal{A}$ and G_2 have the property $\mathcal{B}$. Then $F_1 \omega G_1 < F_2 \omega G_2$.

Proof is easy.

3.4. Lemma: Let F_1, F_2, G be set functors. If $F_1 \dashv F_2$, then $F_1 \circ G \dashv F_2 \circ G$.

Proof: Let $T: F_1 \longrightarrow F_2$ be a monotransformation. Define $T': F_1 \circ G \to F_2 \circ G$ by $T'_X = T_{G(X)}$. Thus, T'_X is always one-to-one. T' is a transformation: Let us prove it e.g. for covariant F_1, F_2 and contravariant G, the other cases are similar. Let $f: X \to Y$ be a mapping. Then $G(f): G(Y) \to G(X)$ and we have

$$(F_2 \circ G)(f) \circ T'_Y = F_2(G(f)) \circ T_{G(Y)} = T_{G(X)} \circ F_1(G(f)) = T'_X \circ (F_1 \circ G)(f).$$

Let $T: F_2 \to F_1$ be an epitransformation. Again, define $T': F_2 \circ G \to F_1 \circ G$ by $T'_X = T_{G(X)}$. Thus, T'_X is always onto. This time, we prove the transformation property e.g. for F_1, F_2, G contravariant. For $f: X \to Y$ we have $G(f): G(Y) \to G(X)$ and we obtain

$$(F_1 \circ G)(f) \circ T'_X = F_1(G(f)) \circ T_{G(X)} = T_{G(Y)} \circ F_2(G(f)) = T'_Y \circ (F_2 \circ G)(f).$$

3.5. __Definition__: Covariant set functors are said to be nice, if they associate one-to-one and onto mappings with one-to-one and onto mappings, respectively. Contravariant nice set functors associate one-to-one mapping with onto ones and onto mappings with one-to-one ones.

__Remark__: Evidently, any composition of nice functors is nice.

3.6. __Lemma__: Let F_1, F_2, G be set functors, G nice. Let $F_1 \dashv F_2$. Then $G \circ F_1 \dashv G \circ F_2$.

__Proof__: Let $T: F_1 \to F_2$ be a monotransformation. Define T' by $T'_X = G(T_X)$. We see easily that this is really a transformation. If G is covariant, $T'_X = G(T_X)$ is one-to-one for every X, and T' transforms $G \circ F_1$ into $G \circ F_2$. If G is contravariant, T' transforms $G \circ F_2$ into $G \circ F_1$, and $T'_X = G(T_X)$ is always onto.

Analogously for an epitransformation $T: F_2 \to F_1$.

3.7. From lemmas 3.4 and 3.6 and from the metalemma there follows

__Theorem__: Let $F_1 < G_1$, $F_2 < G_2$; let F_1 or G_1 be nice. Then $F_1 \circ F_2 < G_1 \circ G_2$.

__Corrollary__: Let $F_i < G_i$ $(i = 1, \dots, n)$. Let $G_2, \dots, G_n$ be nice. Then $F_n \circ F_{n-1} \circ \dots \circ F_2 \circ F_1 < G_n \circ G_{n-1} \circ \dots \circ G_2 \circ G_1$.

3.8. __Theorem__: Let $F_1 < F_2$ and $G_1 < G_2$. Then $F_1 \times G_1 < F_2 \times G_2$.

__Proof__: It suffices to show that $F_1 \dashv F_2$ implies

$F_1 \times G \dashv F_2 \times G$ and $G \times F_1 \dashv G \times F_2$ for any G. We shall prove that $F_1 \times G \dashv F_2 \times G$, the second statement follows from this according to natural equivalences. Let $T: F_1 \to F_2$ be a monotransformation. Define $T': F_1 \times G \to F_2 \times G$ by $T'_X = T_X \times id_{G(X)}$. By 2.1, T'_X are one-to-one mappings. If F_1, F_2 and G are covariant, we have

$$(F_2 \times G)(f) \cdot T'_X = (F_2(f) \times (G(f)) \cdot (T_X \times id_{G(X)}) = (F_2(f) \cdot T_X) \times G(f) = (T_Y \cdot F_1(f)) \times (id_{G(Y)} \cdot G(f)) = T'_Y \cdot (F_1 \times G)(f).$$

If F_1, F_2 and G are contravariant, we have

$$(F_2 \times G)(f) \cdot T'_Y = (F_2(f) \cdot T_Y) \times G(f) = T_X \cdot F_1(f) \times id_{G(X)} \cdot G(f) = T'_X \cdot (F_1 \times G)(f).$$

Similarly for an epitransformation $T: F_2 \to F_1$.

3.9. <u>Theorem</u>: Let $F_1 < F_2$ and $G_1 < G_2$. Then $F_1 \vee G_1 < F_2 \vee G_2$.

<u>Proof</u>: replace the symbols $\times$ by $\vee$ in the previous proof.

3.10. <u>Theorem</u>. Let $F_1 < F_2$ and $G_1 < G_2$. Then $\langle F_1, G_1 \rangle < \langle F_2, G_2 \rangle$.

<u>Proof</u>: Again, it suffices to show that for $F_1 \dashv F_2$ always 1) $\langle F_1, G \rangle \dashv \langle F_2, G \rangle$ and 2) $\langle G, F_1 \rangle \dashv \langle G, F_2 \rangle$.

1) Let $T: F_1 \to F_2$ be a monotransformation. Define T' by $T'_X = \langle T_X, id_{G(X)} \rangle$. By 2.1, T'_X is a mapping of $\langle F_2, G \rangle (X)$ onto $\langle F_1, G \rangle (X)$ for every X. We shall prove that T' is a transformation. If F_1, F_2 are covariant, G, $\langle F_1, G \rangle$ and $\langle F_2, G \rangle$ are contravariant and we have

$$\langle F_1, G \rangle (f) \cdot T'_Y = \langle F_1(f), G(f) \rangle \cdot \langle T_Y, id_{G(X)} \rangle = \langle T_Y \cdot F_1(f), G(f) \rangle = \langle F_2(f) \cdot T_X, G(id_X) \cdot G(f) \rangle = \langle T_X, id_{G(X)} \rangle \cdot \langle F_2(f), G(f) \rangle = T'_X \cdot \langle F_2, G \rangle (f).$$

If F_1, F_2 are contravariant, G, $\langle F_1, G\rangle$ and $\langle F_2, G\rangle$ are covariant and we have

$$\langle F_1, G\rangle(f)\circ T_X' = \langle F_1(f), G(f)\rangle\circ\langle T_X, id_{G(X)}\rangle = \langle F_2(f)\circ T_Y, id_{G(Y)}\circ G(f)\rangle = \langle T_Y, id_{G(Y)}\rangle\circ\langle F_2(f), G(f)\rangle = T_Y'\circ\langle F_2, G\rangle(f) .$$

If $T: F_2 \to F_1$ is an epitransformation, we prove analogously that T', defined by $T_X' = \langle T_X, id_{G(X)}\rangle$ is a monotransformation of $\langle F_1, G\rangle$ into $\langle F_2, G\rangle$.

2) Let $T: F_1 \to F_2$ ($T: F_2 \to F_1$ resp.) be a monotransformation (an epitransformation). We prove easily that T' defined by $T_X' = \langle id_{G(X)}, T_X\rangle$ is a monotransformation of $\langle G, F_1\rangle$ into $\langle G, F_2\rangle$ (an epitransformation of $\langle G, F_2\rangle$ into $\langle G, F_1\rangle$, resp.).

§ 4. Set functors V_A, K_A, Q_A, P_A, P^-, P^+ .

In the present paragraph we shall discuss some naturally defined set functors and deduce some majorisation rules concerning them. The identical set functor will be denoted by I .

4.1. Definition: Let A be a set. The set functor V_A is defined by $V_A(X) = X \vee A$ and $V_A(f) = f \vee id_A$. Let A be non-void. K_A is defined by $K_A(X) = X \times A$ and $K_A(f) = f \times id_A$, Q_A by $Q_A(X) = \langle A, X\rangle$ and $Q_A(f) = \langle id_A, f\rangle$, P_A by $P_A(X) = \langle X, A\rangle$ and $P_A(f) = \langle f, id_A\rangle$. P^+ and P^- are defined by $P^+(X) = P^-(X) = \{Z \mid Z \subset X\}$ and, for $f: X \to Y$, $P^+(f)(Z) = f(Z)$ for every $Z \subset X$ and, finally, $P^-(f)(Z) = f^{-1}(Z)$ for every $Z \subset Y$.

Remarks: 1) Evidently P^- is naturally equivalent with P_2 (where $2 = \{0, 1\}$). Q_2 is naturally equivalent with

Q defined by $Q(X) = X \times X$, $Q(f) = f \times f$.

2) $V_\emptyset$, K_1 and Q_1 are naturally equivalent with I.

3) If $\mathrm{card}\, A = \mathrm{card}\, B$, V_A is naturally equivalent with V_B, K_A with K_B, Q_A with Q_B and P_A with P_B.

4.2. Theorem: The functors V_A, K_A and Q_A are selective; under the assumption that there is no measurable cardinal, P^- is selective x).

Proof: For a definition of selectivity and for the proof concerning Q_A and P^- see [2]. By the previous remark and by Theorem 1 from [2] it suffices to prove the selectivities of V_δ and K_δ for ordinals δ.

Let $\Delta = \{\alpha_\beta \mid \beta < \gamma\}$. We have to find types Δ' and Δ'', and full embeddings $\Phi : S(I, \Delta) \Longrightarrow S(I, \Delta')$ and $\Psi : S(I, \Delta) \Longrightarrow S(I, \Delta'')$ such that $\square \circ \Phi = V_\delta \circ \square$ and $\square \circ \Psi = K_\delta \circ \square$ ($\square$ are the forgetful functors).

Put $\Delta' = \{\alpha_\beta \mid \beta < \gamma + \delta + 1\}$ where $\alpha_\beta = 1$ for $\beta \geq \gamma$. Construct Φ as follows: Let $\langle X, R \rangle$ be an object from $S(I, \Delta)$. Thus, $R = \{r_\beta\}$ is a relational system of the type Δ on X. Define $\bar{R} = \{\bar{r}_\beta\}$ on $V_\delta(X)$ by

----

x) That assumption may be replaced by the following, weaker one: (M)There exists a cardinal δ such that every δ-additive measure is γ-additive for any γ.
The assumption of non-existence of a measurable cardinal is equivalent with the assumption that ω_0 may be taken for δ. The question whether there has to be an assumption on measurable cardinals at all seems to be open.

$\{(x_\iota, 0)\} \in \bar{r}_\beta$ iff $\{x_\iota\} \in r_\beta$ for $\beta < \gamma$,

$(x, i) \in \bar{r}_{\gamma+\varkappa}$ iff $(x, i) = (\varkappa, 1)$ for $\varkappa < \sigma$,

$(x, i) \in \bar{r}_{\gamma+\sigma}$ iff $i = 0$.

Put $\Phi(X, R) = (V_\sigma(X), \bar{R})$, $\Phi(f) = V_\sigma(f)$. It is easy to prove that Φ is a full embedding of $S(I, \Delta)$ into $S(I, \Delta')$.

Now, put $\Delta'' = \{\alpha_\beta \mid \beta < \gamma + \sigma + 1\}$ where, this time, $\alpha_\beta = 2$ for $\beta \geq \gamma$. Let R be a relational system of the type Δ on X. Define a relational system $\bar{R} = \{\bar{r}_\beta \mid \beta < \gamma + \sigma + 1\}$ on $X \times \sigma$ as follows:

$\{(x_\varkappa, \iota)\} \in \bar{r}_\beta$ iff $\{x_\varkappa\} \in r_\beta$ (for $\beta < \gamma, \iota < \sigma$)

$((x, \varkappa), (y, \lambda)) \in \bar{r}_{\gamma+\iota}$ iff $\varkappa = \lambda = \iota$ (for $\iota < \sigma$)

$((x, \varkappa), (y, \lambda)) \in \bar{r}_{\gamma+\sigma}$ iff $x = y$.

Put $\Psi(X, R) = (K_\sigma(X), \bar{R})$, $\Psi(f) = K_\sigma(f)$. It is easy to prove that Ψ is a one-to-one functor into $S(I, \Delta'')$. It remains to be shown that for every morphism $g : (X \times \sigma, \bar{R}) \to (Y \times \sigma, \bar{S})$ there is a morphism $f : (X, R) \to (Y, S)$ with $\Psi(f) = g$. Let $g : (X \times \sigma, \bar{R}) \to (Y \times \sigma, \bar{S})$ be a morphism. The formula

$$g(x, 0) = f(x, \mu)$$

determines (uniquely) a mapping $f : X \to Y$. Denote $(y, \nu) = g(x, \lambda)$. Since $((x, \lambda), (x, 0)) \in \bar{r}_{\gamma+\sigma}$, we have $((y, \nu), (f(x), \mu)) \in \bar{s}_{\gamma+\sigma}$ and hence $y = f(x)$. Since $((x, \lambda), (x, \lambda)) \in \bar{r}_{\gamma+\lambda}$, we have $((y, \nu), (y, \nu)) \in \bar{s}_{\gamma+\lambda}$ and hence $\nu = \lambda$. Thus, $g(x, \lambda) = (f(x), \lambda)$ for every (x, λ), i.e. $g = K_\sigma(f)$. Considering $\bar{r}_\beta$, $\bar{s}_\beta$ with $\beta < \gamma$ we see easily that f is RS-compatible.

4.3. <u>Lemma</u>: Let $\operatorname{card} A \leq \operatorname{card} B$. Then $V_A < V_B$, $K_A < K_B$, $Q_A < Q_B$ and $P_A < P_B$.

Proof: Let $\varphi : A \to B$ be a one-to-one mapping. Define transformations $T : V_A \to V_B$, $T' : K_A \to K_B$, $T'' : Q_B \to Q_A$ and $T''' : P_A \to P_B$ by $T_X = id_X \vee \varphi$, $T'_X = id_X \times \varphi$, $T''_X = \langle \varphi, id_X \rangle$ and $T'''_X = \langle id_X , \varphi \rangle$. By 2.1, T_X, T'_X and T'''_X are one-to-one mappings and T''_X are mappings onto.

4.4. Lemma: $I < V_A$, $I < K_A$, $I < Q_A$.

Proof: This follows by 4.3 and Remark 2) in 4.1.

4.5. Lemma: a) $P^+ < P^- \circ P^-$, b) $I < P^- \circ P^-$.

Proof: a) First, notice that for every $f : X \to Y$, $M \subset X$, $N \subset Y$

$$f(M) \subset N \quad \text{iff} \quad M \subset f^{-1}(N) .$$

For $A \subset X$ define $\mathfrak{M}(X, A) = \{M \subset X \mid A \subset M\}$. Let $f : X \to Y$ be a mapping. We have

$$\begin{aligned} P^-(P^-(f))(\mathfrak{M}(X,A)) &= \{N \in P^-(Y) \mid P^-(f)(N) \in \\ \in \mathfrak{M}(X,A)\} &= \{N \subset Y \mid f^{-1}(N) \in \mathfrak{M}(X,A)\} = \\ &= \{N \subset Y \mid A \subset f^{-1}(N)\} = \{N \subset Y \mid f(A) \subset N\} = \\ &= \mathfrak{M}(Y, P^+(f)(A)) . \end{aligned}$$

Now, define $T_X : P^+(X) \to P^-(P^-(X))$ by $T_X(A) = \mathfrak{M}(X, A)$. We have $(P^- \circ P^-)(f) \circ T_X(A) = P^-(P^-(f))(\mathfrak{M}(X,A)) = \mathfrak{M}(Y, P^+(f)(A)) = T_Y \circ P^+(f)(A)$. Thus, T is a transformation. As $A = \bigcap \mathfrak{M}(X, A)$, all T_X are obviously one-to-one.

b) follows by a) and the transformation $T : I \to P^+$, $T_X : X \to P^+(X)$ defined by $T_X(x) = \{x\}$.

4.6. Lemma: Denote by $\cong$ natural equivalences of functors. We have: a) $V_A \circ V_B \cong V_{B \vee A}$, b) $K_A \circ K_B \cong K_{B \times A}$, c) $K_B \circ V_A \cong V_{A \times B} \circ K_B$.

Proof is trivial.

4.7. **Lemma**: a) $P_{\langle A,2\rangle} \cong P^{-} \circ K_A$, b) $P_A < P^{-} \circ K_A$,

c) $V_B \circ Q_A < Q_A \circ V_B$, d) $V_A \circ P^{-} < P^{-} \circ V_A$.

Proof: a) We shall prove $P_{\langle A,2\rangle} \cong P_2 \circ K_A$. Let $\varphi : X \to \langle A, 2 \rangle$ be a mapping; define $\varphi' : X \times A \to 2$ by $\varphi'(x, a) = (\varphi(x))(a)$. Let $\psi : X \times A \to 2$ be a mapping, define $\overline{\psi} : X \to \langle A, 2 \rangle$ by $(\overline{\psi}(x))(a) = \psi(x, a)$. Now, define $T_X : \langle X, \langle A, 2\rangle\rangle \to \langle X \times A, 2 \rangle$ by $T_X(\varphi) = \varphi'$, $T'_X : \langle X \times A, 2 \rangle \to \langle X, \langle A, 2 \rangle\rangle$ by $T'_X(\psi) = \overline{\psi}$. We see easily that this defines transformations $T : P_{\langle A,2\rangle} \to P_2 \circ K_A$ and $T' : P_2 \circ K_A \to P_{\langle A,2\rangle}$ such that both $T \circ T'$ and $T' \circ T$ are the identical transformations.

b) follows easily from a) and 4.3. Namely, we have $P_A < P_{\langle A,2\rangle} \cong P^{-} \circ K_A$.

c) First, define $j_X : X \to X \vee B$ by $j_X(x) = (x, 0)$. We see easily that $(f \vee id_B) \circ j_X = j_Y \circ f$ for any $f : X \to Y$. Now, define $T_X : \langle A, X \rangle \vee B \to \langle A, X \vee B \rangle$ as follows:

$$T_X(\varphi, 0) = j_X \circ \varphi, \; T_X(b, 1) = const_b \; .$$

It is easy to prove that this defines a transformation $T : V_B \circ Q_A \to Q_A \circ V_B$ and that T_X are one-to-one.

d) Define $p : 2 \vee 2 \to 2$ by $p(i, j) = i$, and, for every $a \in A$, $\chi_a : A \to 2$ by $\chi_a(a) = 1$, $\chi_a(b) = 0$ for $b \neq a$. Define $T_X : \langle X, 2 \rangle \vee A \to \langle X \vee A, 2 \rangle$ by $T_X(\varphi, 0) = p \circ (\varphi \vee const_0)$, $T_X(a, 1) = p \circ (const_0 \vee \chi_a)$. Let $f : X \to Y$ be a mapping; we have

$$P^{-} \circ V_A(f) \circ T_Y(\varphi, 0) = \langle f \vee id_A, id_2 \rangle (p \circ (\varphi \vee const_0)) =$$

$$= p \circ (\varphi \vee const_0) \circ (f \vee id_A) = p \circ (\varphi \circ f \vee const_0),$$

$$T_X(V_A \circ P^{-}(f)(\varphi, 0)) = T_X(\varphi \circ f, 0) = p \circ (\varphi \circ f \vee const_0),$$

$$P^- \circ V_A(f)(T_Y(a,1)) = P^- \circ V_A(f)(p \circ (const_0 \vee \chi_a)) =$$
$$= p \circ (const_0 \vee \chi_a) \circ (f \vee id_A) = p \circ (const_0 \vee \chi_a),$$
$$T_X(V_A \circ P^-(f)(a,1) = T_X(a,1) = p \circ (const_0 \vee \chi_a).$$

Evidently, every T_X is one-to-one.

4.8. Lemma: a) $Q_A < P^+ \circ K_A$, b) $K_A < P^+ \circ V_A$.

Proof: a) define $T_X : Q_A(X) \to P^+ \circ K_A(X)$ by $T_X(\varphi) = \{(\varphi(a), a) \mid a \in A\}$. Every T_X is one-to-one. If $f : X \to Y$, we have

$$(P^+ \circ K_A)(f) \circ T_X(\varphi) = (P^+ \circ K_A)(f)\{(\varphi(a), a) \mid a \in A\} =$$
$$= \{(f \circ \varphi(a), a \in A\} = T_Y(f \circ \varphi) = T_Y(Q_A(f)(\varphi)).$$

b) Define $T_X : K_A(X) \to P^+ \circ V_A(X)$ by $T_X(x,a) = \{(x,0),(a,1)\}$. Every T_X is one-to-one. If $f : X \to Y$, we have

$$(P^+ \circ V_A)(f) \circ T_X(x,a) = (P^+ \circ V_A)(f)\{(x,0),(a,1)\} =$$
$$= \{(f(x),0),(a,1)\} = T_Y(f(x),a) = T_Y \circ K_A(f)(x,a).$$

§ 5. Constructive functors and their majorisation.

It is easy to prove

5.1. Lemma: The set functors $I, V_A, K_A, Q_A, P_A, P^+$ are nice.

5.2. Theorem: For every composition G of functors I, V_A, K_B, Q_C, P_D, P^+ there is a natural number k and a set M such that $G < (P^-)^k \circ V_M$.

Proof: We shall use 3.7, 4.4 and 4.5 b) without further mentioning.

I. $K_A < (P^-)^2 \circ V_A$ by 4.8 b), 4.5 a) ;

$Q_A < P^+ \circ K_A < (P^+)^2 \circ V_A < (P^-)^4 \circ V_A$ by 4.8 a)b), 4.5 a) ;

$P_A < P^- \circ K_A < (P^-)^3 \circ V_A$ by 4.7 b), 4.8 b), 4.5 a) ;

$P^+ < (P^-)^2 \circ V_\emptyset$ by 4.5 a) .

II. Let the statement hold for compositions of at most n functors; let G be a composition of $n+1$ functors. Hence $G = G' \circ H$, where $G' < (P^-)^k \circ V_N$ and H is some of the functors V_A, K_A, Q_A, P_A, P^+ .

1) $H = V_A : G < (P^-)^k \circ V_N \circ V_A \cong (P^-)^k \circ V_M$ where $M = N \vee A$ (by 4.6 a)) ;

2) $H = K_A : G < (P^-)^k \circ V_N \circ K_A < (P^-)^k \circ V_{N \times A} \circ K_A \cong$

$$\cong (P^-)^k \circ K_A \circ V_N < (P^-)^{k+2} \circ V_A \circ V_N \cong (P^-)^{k+2} \circ V_M$$

where $M = N \vee A$ (by 4.3 a), 4.6 c), 4.8 b), 4.5 a) and 4.6 a)).

3) $H = Q_A : G < (P^-)^k \circ V_N \circ Q_A < (P^-)^k \circ Q_A \circ V_N < (P^-)^{k+4} \circ V_M$ where $M = A \vee N$ (by 4.7 c), 4.8, 4.5 a) and 4.6 a)) ;

4) $H = P_A : G < (P^-)^k \circ V_N \circ P_A < (P^-)^k \circ V_N \circ P^- \circ K_A < (P^-)^{k+1} \circ V_N \circ K_A < (P^-)^{k+3} \circ V_M$ where $M = N \vee A$ (by 4.7 b)d) and 2) in this proof) ;

5) $H = P^+ : G < (P^-)^k \circ V_N \circ P^+ < (P^-)^k \circ V_N \circ (P^-)^2 < (P^-)^{k+2} \circ V_N$ (by 4.5 a) and 4.7 d)).

5.3. Constructive functors are defined recursively as follows:

(1) I, V_A, K_A, Q_A, P_A, P^+ are constructive functors,

(2) If F, G are constructive, $F \circ G$, $F \times G$, $F \vee G$, $\langle F, G \rangle$ are constructive whenever defined,

(3) If F is constructive and $G \cong F$, then G is constructive.

Remark: Thus, constructive functors are "polynomials produced from I, V_A, K_A, Q_A, P_A, P^+ under the operations $\circ$, $\times$, $\vee$, $\langle\ \ \rangle$".

5.4. Lemma: Let G be a set functor. Then $G \times G \cong Q_2 \circ G$, $G \vee G = K_2 \circ G$.

Proof: $(G \vee G)(X) = G(X) \vee G(X) = G(X) \times \{0\} \cup G(X) \times \{1\} = G(X) \times 2 = K_2 \circ G(X)$, $(G \vee G)(f)(z, i) = (G(f)(z), i) = (G(f) \times id_2)(z, i) = K_2 \circ G(f)(z), i)$.

By remark 4.1, $Q_2 \cong Q$. We have
$(G \times G)(X) = G(X) \times G(X) = (Q \circ G)(X)$, $(G \times G)(f) = G(f) \times G(f) = (Q \circ G)(f)$.

5.5. Lemma: The functors in three variables, F_1, F_2 defined by

$$F_1(X, Y, Z) = \langle X, \langle Y, Z \rangle\rangle, \quad F_1(f, g, h) = \langle f, \langle g, h \rangle\rangle,$$

$$F_2(X, Y, Z) = \langle X \times Y, Z \rangle, \quad F_2(f, g, h) = \langle f \times g, h \rangle.$$

are naturally equivalent.

Proof: This is a well known fact; we see easily that formulae $(T_{X,Y,Z}(\varphi))(x, y) = (\varphi(x))(y)$, $((T'_{X,Y,Z}(\psi))(x))(y) = \psi(x, y)$ define transformations $T: F_1 \to F_2$, $T': F_2 \to F_1$ which are mutually inverse.

5.6. Lemma: Let G be a set functor. Then

$$\langle G, P^- \circ G \rangle \cong P_2 \circ Q_2 \circ G.$$

Proof: We shall use the following evident facts: 1) If $G_1 \cong G_2$, then $\langle G, G_1 \rangle \cong \langle G, G_2 \rangle$, 2) Superpositions of naturally equivalent functors are naturally equivalent.

Thus, $\langle G, P^- \circ G \rangle \cong \langle G, P_2 \circ G \rangle$. Let F_1, F_2 be the functors from 5.5. For every X we have

$$\langle G, P_2 \circ G \rangle (X) = \langle G(X), P_2(G(X)) \rangle = \langle G(X), \langle G(X), 2 \rangle\rangle = F_1(G(X), G(X), 2),$$

$$(P_2 \circ Q \circ G)(X) = \langle Q(G(X)), 2 \rangle = \langle G(X) \times G(X), 2 \rangle = F_2(G(X), G(X), 2).$$

For every mapping $f : X \to Y$

$$\langle G, P_2 \circ G \rangle (f) = \langle G(f), P_2(G(f)) \rangle = \langle G(f), \langle G(f), id_2 \rangle\rangle = F_1(G(f), G(f), id_2),$$

$$(P_2 \circ Q \circ G)(f) = P_2(G(f) \times G(f)) = \langle G(f) \times G(f), id_2 \rangle = F_2(G(f), G(f), id_2).$$

Thus, by 5.5, $\langle G, P_2 \circ G \rangle \cong P_2 \circ Q \circ G$ and hence $\langle G, P^- \circ G \rangle \cong P_2 \circ Q_2 \circ G$.

5.7. Theorem: For every constructive G there is a natural number k and a set M such that

$$G < (P^-)^k \circ V_M .$$

Proof: For I, V_A, K_A, Q_A, P_A, P^+ the statement holds by 5.2. Let $G < (P^-)^m \circ V_A$, $H < (P^-)^n \circ V_B$. Then $G \circ H < P_2^m \circ V_A \circ P_2^n \circ V_B < (P^-)^k \circ V_M$ by 5.2. Let G, H be either both covariant or both contravariant. Put $\ell = \max(m, n)$, $C = A \cup B$. We see easily that

$$G < (P^-)^m \circ V_A < (P^-)^\ell \circ V_C\,,\; H < (P^-)^n \circ V_B < (P^-)^\ell \circ V_C\,,$$

since $|m-n|$ has to be even.

Thus,

$$G \times H < P_2^\ell \circ V_C \times P_2^\ell \circ V_C \cong Q_2 \circ P_2^\ell \circ V_C < (P^-)^k \circ V_M\,,$$

$$G \vee H < P_2^\ell \circ V_C \vee P_2^\ell \circ V_C \cong K_2 \circ P_2^\ell \circ V_C < (P^-)^k \circ V_M$$

by 5.4 and 5.2.

Let either G be covariant and H contravariant, or G contravariant and H covariant; again, let $G < (P^-)^m \circ V_A$, $H < (P^-)^n \circ V_B$. Put $\ell = \max(m, n-1)$, $C = A \cup B$. As $|m-(n-1)|$ has to be even, we have $G < (P^-)^m \circ V_A <$ $< (P^-)^\ell \circ V_C$, $H < (P^-)^n \circ V_B < (P^-)^{\ell+1} \circ V_C$.

Hence, by 5.6 and 5.2,

$$\langle G, H\rangle < \langle (P^-)^\ell \circ V_C, (P^-)\circ(P^-)^\ell \circ V_C\rangle \cong P_2 \circ Q_2 \circ P_2^\ell \circ V_C < (P^-)^k \circ V_M\,.$$

5.8. Using the statement of 5.7 and repeating the part of its proof concerning the operations we obtain easily

Theorem: Let $G_1, \ldots, G_n$ be set functors, each of them majorised by a constructive functor. Let G be a functor obtained from $G_1, \ldots, G_n$ as a polynomial in the operations $\circ$, $\times$, $\vee$, $\langle\ \ \rangle$. Then there is a natural number k and a set M such that

$$G < (P^-)^k \circ V_M\,.$$

5.9. Definition: If $G_1, \ldots, G_n$ are set functors majorised by constructive functors and G is a functor, obtained from $G_1, \ldots, G_n$ as a polynomial in the operations $\circ$, $\times$, $\vee$, $\langle\ \ \rangle$, we say that G is constructively majorisable.

§ 6. Majorisation and realisations of categories.

6.1. Theorem: Let $F < G$. Then $S(F) \rightrightarrows S(G)$.

Proof: According to 1.2 it suffices to prove that $S(F) \rightrightarrows S(G)$ whenever $F \dashv G$.

First, let $T: F \to G$ be a monotransformation. Let (X, s) be an object from $S(F)$. Put $\Phi(X,s) = (X, \bar{s})$ with $\bar{s} = T_X(s)$. We have to prove that $f: X \to Y$ is a morphism from (X, s) into (Y, t) iff it is a morphism from $(X, \bar{s})$ into $(Y, \bar{t})$ i.e. that

(1)
$$F(f)(s) \subset t \quad \text{iff} \quad G(f)(\bar{s}) \subset \bar{t} \quad \text{for covariant } F, G,$$

$$F(f)(t) \subset s \quad \text{iff} \quad G(f)(\bar{t}) \subset \bar{s} \quad \text{for contravariant}$$
F, G.

Let F, G be covariant. Let $F(f)(s) \subset t$. If $a \in \bar{s}$, we have $a = T_X(b)$ for some $b \in s$. Hence, $G(f)(a) = G(f)(T_X(b)) = T_Y(F(f)(b)) \in \bar{t}$, as $F(f)(b) \in t$. Thus, $G(f)(\bar{s}) \subset \bar{t}$. Now, let $G(f)(\bar{s}) \subset \bar{t}$. If $a \in s$, we have $T_X(a) \in \bar{s}$ and hence $G(f)(T_X(a)) \in \bar{t}$; thus, $T_Y(F(f)(a)) = G(f)(T_X(a)) = T_Y(b)$ with $b \in t$. As T_Y is one-to-one, $F(f)(a) = b \in t$. The proof for contravariant F, G is quite analogous.

Now, let $T: G \to F$ be an epitransformation. Let (X, s) be an object from $S(F)$. Put $\Phi(X,s) = (X, \bar{s})$ with $\bar{s} = T_X^{-1}(s)$. Again, we have to prove the validity of the formulae (1). Let F, G be covariant. Let $F(f)(s) \subset t$. If $a \in \bar{s}$, we have $T_X(a) \in s$ and hence $T_Y(G(f)(a)) = F(f)(T_X(a)) \in t$, i.e. $G(f)(a) \in T_Y^{-1}(t) = \bar{t}$. Let $G(f)(\bar{s}) \subset \bar{t}, a \in s$. As T_X is onto, $a = T_X(b)$ for some

$b \in G(X)$. We have $b \in \bar{t}$ and hence $F(f)(a) = F(f)(T_X(b)) = T_Y(G(f)(b)) \in t$, since $G(f)(b) \in \bar{t} = T_Y^{-1}(t)$. Analogously, for contravariant F, G.

6.2. Lemma: Let $F_1, \dots, F_n$ be covariant set functors, $\Delta_1, \dots, \Delta_n$ types, $\Delta_i = \{\alpha^i_\beta \mid \beta < \gamma^i\}$. Then there exist sets A_i, B_i ($i = 1, \dots, n$) such that

$$S((F_1, \Delta_1), \dots, (F_n, \Delta_n)) \Rightarrow S(K_{A_1} \circ Q_{B_1} \circ F_1, \dots, K_{A_n} \circ Q_{B_n} \circ F_n).$$

We may put $A_i = \gamma^i$, $B_i = \alpha^i = \sup \Delta_i$.

Proof: For every couple (i, β) (where $\beta < \gamma^i$) choose a mapping p^i_β of α^i onto α^i_β. We shall consider α-nary relations on Z as subsets of $\langle \alpha, Z \rangle$.

Let $(X, \{R^i, i = 1, \dots, n\})$ be an object from $S((F_1, \Delta_1), \dots, (F_n, \Delta_n))$, $R^i = \{r^i_\beta \mid \beta < \gamma^i\}$. Define $\bar{R}^i \subset K_{\gamma^i} \circ Q_{\alpha^i} \circ F_i(X) = \langle \alpha^i, F_i(X) \rangle \times \gamma^i$ by

$(\varphi, \beta) \in \bar{R}^i$ iff there is a $\psi \in r^i_\beta$ with $\varphi = \psi \circ p^i_\beta$.

Since evidently $\bar{R}^i_1$, $\bar{R}^i_2$ are distinct whenever R^i_1, R^i_2 are distinct, it suffices to show that for any mapping $f: X \to Y$ and objects $(X, \{R^i\})$, $(Y, \{S^i\})$ from $S((F_1, \Delta_1), \dots, (F_n, \Delta_n))$ the following two statements are equivalent:

(1) for every i, $F_i(f)$ is $R^i S^i$-compatible,

(2) for every i, $(K_{\gamma^i} \circ Q_{\alpha^i} \circ F_i)(f)(\bar{R}^i) \subset \bar{S}^i$.

Let (1) hold. Let $(\varphi, \beta) \in \bar{R}^i$. Thus $\varphi = \psi \circ p^i_\beta$ with $\psi \in r^i_\beta$. By (1), $F_i(f) \circ \psi \in s^i_\beta$ and hence

$$K_{\gamma^i} \circ Q_{\alpha^i} \circ F_i(f)(\varphi, \beta) = (F_i(f) \circ \psi \circ p^i_\beta, \beta) \in \bar{S}^i.$$

Let (2) hold, $\psi \in r^i_\beta$. We have to prove that $F_i(f) \circ \psi \in s^i_\beta$. Since $(\psi \circ p^i_\beta, \beta) \in \bar{R}^i$, we have $(F_i(f) \circ \psi \circ p^i_\beta, \beta) = (K_{\gamma^i} \circ Q_{\alpha^i} \circ F_i)(f)(\psi \circ p^i_\beta, \beta) \in \bar{S}^i$. Thus, $F_i(f) \circ \psi \circ p^i_\beta =$

$= \chi \circ p^i_\beta$ with $\chi \in s^i_\beta$. As p^i_β is a mapping onto, we have $F_i(f) \circ \psi = \chi \in s^i_\beta$. The proof is finished.

6.3. Lemma: If $F_1, \dots, F_n$ are covariant set functors, then

$$S(F_1, \dots, F_n) \Rightarrow S(\bigvee_{i=1}^{n} F_i).$$

Proof: Let $(X, s_1, \dots, s_n)$ be an object from $S(F_1, \dots, F_n)$. Put $s = \bigcup_{i=1}^{n} s_i \times (i) \subset \bigvee F_i(X)$. If $\{s_i\} \neq \{\bar{s}_i\}$, then evidently $s \neq \bar{s}$. Hence, it suffices to prove that $f : X \to Y$ is a morphism from $(X, \{s_i\})$ into $(Y, \{t_i\})$ iff it is a morphism from (X, s) into (Y, t). If $F_i(f)(s_i) \subset t_i$ for every i and $(a, j) \in s$, we have

$$(\bigvee F_i)(f)(a, j) = (F_j(f)(a), j) \in t_j \times (j) \subset t.$$

If $(\bigvee F_i)(f)(s) \subset t$ and $a \in s_i$, then $(a, i) \in s$ and hence $(F_i(f)(a), i) \in t$ so that $(F_i(f)(a), i) \in t_i \times (i)$. Thus, $F_i(f)(a) \in t_i$.

6.4. Remarks: 1) Of course, in 6.2 and 6.3 it suffices to assume F_i either all covariant or all contravariant.

2) The realisation in 6.3 is an isofunctor.

6.5. Theorem: Let $G_1, \dots, G_n$ be constructively majorisable functors, $\Delta_1, \dots, \Delta_n$ types. Then there is a natural number k and a set M such that

$$S((G_1, \Delta_1), \dots, (G_n, \Delta_n)) \Rightarrow S((P^-)^k \circ V_M).$$

Proof: By Theorem 1.5, 6.2 and 6.3 we obtain

$$S((G_1, \Delta_1), \dots, (G_n, \Delta_n)) \Rightarrow S((F_1, \Delta_1), \dots, (F_n, \Delta_n)) \Rightarrow$$

$$\Rightarrow S(K_{A_1} \circ Q_{B_1} \circ F_1, \dots, K_{A_n} \circ Q_{B_n} \circ F_n) \Rightarrow S(\bigvee_{i=1}^{n} K_{A_i} \circ Q_{B_i} \circ F_i).$$

As G_i are constructively majorisable, $\vee K_{A_i} \circ Q_{B_i} \circ F_i$ is constructively majorisable. By 5.8 there are k and M with $\vee K_{A_i} \circ Q_{B_i} \circ F_i < (P^-)^k \circ V_M$. Thus, by 6.1, $S(\bigvee_{i=1}^{m} K_{A_i} \circ Q_{B_i} \circ F_i) \Rightarrow S((P^-)^k \circ V_M)$.

6.6. <u>Corollary</u>: Let $G_1, \dots, G_n$ be constructively majorisable functors, $\Delta_1, \dots, \Delta_n$ types. Let $\mathfrak{K}$ be fully embedded into $S((G_1, \Delta_1), \dots, (G_n, \Delta_n))$. Then, in any set theory satisfying (M) (see footnote at 4.2), $\mathfrak{K}$ is boundable.

<u>Proof</u>: This follows immediately by 6.5 and [2], as $(P^-)^k \circ V_M$ is a selective functor. (See 4.2.)

§ 7. <u>Some remarks</u>.

7.1. By 4.3 and 6.1 $S(K_A) \Rightarrow S(K_B)$, $S(Q_A) \Rightarrow S(Q_B)$, $S(P_A) \Rightarrow S(P_B)$ whenever $\mathrm{card}\, A \leq \mathrm{card}\, B$. On the other hand there holds

<u>Theorem</u>: The condition $\mathrm{card}\, A \leq \mathrm{card}\, B$ is <u>necessary</u> for any of the following: a) $S(K_A) \Rightarrow S(K_B)$, b) $S(Q_A) \Rightarrow S(Q_B)$, c) $S(P_A) \Rightarrow S(P_B)$.

<u>Proof</u>: a) Let $\mathrm{card}\, A > \mathrm{card}\, B$. Take a set X such that $\mathrm{card}\, X = \mathrm{card}\, A$ and a one-to-one mapping φ of X onto A. Consider the object (X, s) with $s = \{(x, \varphi(x)) \mid x \in X\}$. If $f: (X, s) \to (X, s)$ is a morphism, $K_A(f)(s) \subset s$, and hence, for any $x \in X$, $(f(x), \varphi(x)) \in s$. Since φ is one-to-one, we obtain $f(x) = x$. Thus, there is no non-identical morphism of (X, s) into itself. Let $t \subset K_B(X)$ be such that the only morphism of (X, t) into itself is the identity. Thus, if we define

$f_{xy}: X \to X$ $(x \neq y)$ by $f_{xy}(x) = y$, $f_{xy}(z) = z$ for $z \neq x$, there is $(u, b) \in t$ such that $(f_{xy}(u), b) \notin$ $\notin t$. By the last formula necessarily $u = x$. Find some b with this property and denote it by $\varphi(x)$. For $x \neq y$ we obtain $(y, \varphi(x)) = (f_{xy}(x), \varphi(x)) \notin t$, while $(y, \varphi(y)) \in t$. Thus, $\varphi: X \to B$ is a one-to-one mapping. This is a contradiction, since $card\ B < card\ X$.

b),c) Denote by $C(X)$ the semigroup consisting of the identity mapping of X and of all the mappings of X into itself which are <u>not onto</u>. We shall prove that, for any semigroup $\mathcal{S}$ of morphisms of X into itself containing the identity mapping, if $card\ A = card\ X$, there are $s \subset Q_A(X)$, $t \subset P_A(X)$ such that

$$\mathcal{S} = \{f \mid Q_A(f)(s) \subset s\} = \{f \mid P_A(f)(t) \subset t\}.$$

On the other hand, if $card\ A < card\ X$, there are no s, t with

$$C(X) = \{f \mid Q_A(f)(s) \subset s\},\ C(X) = \{f \mid P_A(f)(t) \subset t\}.$$

First, let $card\ A = card\ X$; take a one-to-one mapping φ of A onto X. Put $s = \{\alpha \circ \varphi \mid \alpha \in \mathcal{S}\}$, $t = \{\varphi^{-1} \circ \alpha \mid$ $\mid \alpha \in \mathcal{S}\}$. If $Q_A(f)(s) \subset s$ ($P_A(f)(t) \subset t$), then, in particular, $f \circ id \circ \varphi \in s$ ($\varphi^{-1} \circ id \circ f \in t$), hence, $f \circ \varphi = \alpha \circ \varphi$, $\alpha \in \mathcal{S}$ ($\varphi^{-1} \circ f = \varphi^{-1} \circ \alpha$, $\alpha \in \mathcal{S}$). As φ is one-to-one mapping onto, $f = \alpha \in \mathcal{S}$. If $f \in \mathcal{S}$, then, for any $\alpha \in \mathcal{S}$, $f \circ \alpha \circ \varphi \in s$, $\varphi^{-1} \circ \alpha \circ f \in t$. Now, let $card\ A < card\ X$. Let s (t resp.) be such that $C(X) =$ $= \{f \mid Q_A(f)(s) \subset s\}$ ($C(X) = \{f \mid P_A(f)(t) \subset t\}$ resp.). As $0 \neq card\ A < card\ X$, $card\ X \geq 2$. Thus, there is a non-identical one-to-one mapping g of X onto iself; hence there is an $\alpha : A \to X$, $\alpha \in \mathfrak{s}$, such that $g \circ \alpha \notin \mathfrak{s}$. By the assumption, $X \setminus \alpha(A) \neq \emptyset$. Choose an $x_0 \in X \setminus \alpha(A)$ and $x_1 \neq x_0$ and define $f : X \to X$ by $f(x_0) = g(x_1)$, $f(x) = g(x)$ otherwise. Now, $f \in C(X)$, while $f \circ \alpha = g \circ \alpha \notin \mathfrak{s}$. This is a contradiction.

Let the t exist; there is a $\beta : X \to A$, $\beta \in t$ such that $\beta \circ g \notin t$. By the assumption, β is not one-to-one. Choose x_0, x_1, $x_0 \neq x_1$ such that $\beta(x_0) = \beta(x_1)$ and define $f(g^{-1}(x_0)) = x_1$, $f(x) = g(x)$ otherwise. We have $\beta \circ f = \beta \circ g \notin t$ and f is not onto, which is a contradiction. The proof is finished.

7.2. The following statement concerning the, in some sense dually defined, Q_A and P_A may be of some interest:

<u>Theorem</u>: I. If $\operatorname{card} A \geq 2$, $S(P_A)$ is realisable in <u>no</u> $S(Q_B)$.

II. On the other hand, however, for every A there is a B such that $S(Q_A) \rightrightarrows S(P_B)$.

<u>Proof</u>: I. By 4.3 and 6.1 it suffices to prove that $S(P_2) \rightrightarrows S(Q_B)$ for no B.

For any set X with $\operatorname{card} X \geq 2$ choose an $x_0 \in X$ and put

$$\mathfrak{s}(X) = \{\varphi : X \to 2 \mid (\varphi(x_0) = 1) \Rightarrow (\exists x \neq x_0, \varphi(x) = 1)\};$$

thus, $\mathfrak{s}(X) \subset P_2(X)$. An important property of $\mathfrak{s}(X)$ is the following one: If $B \subsetneqq X$ and $f' : B \to X$ is any mapping, then there is a morphism $f : (X, \mathfrak{s}) \to (X, \mathfrak{s})$ with $f \mid B = f'$. Really, if $B = X \setminus (x_0)$, choose a $b_0 \in B$ and put $f(x_0) = f'(b_0)$. If $\varphi \in \mathfrak{s}$ and $\varphi \circ f(x_0) = 1$, we have $\varphi \circ f(b_0) = 1$; thus, $\varphi \circ f \in \mathfrak{s}$. If there is

$x_1 \in X \smallsetminus B$, $x_1 \neq x_0$, put $f(x) = f'(x_0)$ for $x \notin B$ (if $x_0 \notin B$ define first arbitrarily $f(x_0)$). If $\varphi \in s$ and $\varphi \circ f(x_0) = 1$ we have $\varphi(f(x_1)) = 1$ and hence $\varphi \circ f \in s$.

Now, let $S(P_2) \not\rightrightarrows S(Q_B)$. Choose an X with $\operatorname{card} X > \operatorname{card} B$. Let $r \subset Q_B(X)$ replace $s(X)$. Choose an $f: X \to X$ which is not a morphism (e.g., choose $x_1 \in X$, $x_1 \neq x_0$ and put $f(x) = x_0$ for $x \neq x_0$, $f(x_0) = x_1$). Hence, there is an $\alpha: B \to X$, $\alpha \in r$, such that $f \circ \alpha \notin r$. We have $\alpha(B) \neq X$ and hence there is a morphism g with $g | \alpha(B) = f | \alpha(B)$. Then $g \circ \alpha = f \circ \alpha \notin r$. This is a contradiction.

II. If e is an equivalence relation on A denote by A_e the set A/e of all the equivalence classes. If $a \in A$, denote by $a(e)$ the element of A_e containing a. Hence, $(a, b) \in e$ iff $a(e) = b(e)$. Denote by E the set of the all equivalence relations on A and put

$$B = (0) \cup (\cup \{A_e \times (e) \mid e \in E\}).$$

If $\varphi: A \to C$ is any mapping, denote by $e(\varphi)$ the equivalence relation defined by: $(a, b) \in e(\varphi)$ iff $\varphi(a) = \varphi(b)$. Evidently $e(\varphi) \subset e(f \circ \varphi)$ whenever $f \circ \varphi$ is defined.

Now, let (X, s) be an object of $S(Q_A)$. We define $\bar{s} \subset P_B(X)$ by

$$\psi \in \bar{s} \text{ iff } \forall \varphi: A \to X ((\exists e \supset e(\varphi) \, \forall a \in A \;\; \psi \circ \varphi(a) = (a(e), e)) \Rightarrow \varphi \notin s).$$

We shall prove that, for $f: X \to Y$, $r \subset Q_A(X)$, $s \subset Q_A(Y)$,

$$Q_A(f)(r) \subset s \quad \text{iff} \quad P_B(f)(\bar{s}) \subset \bar{r}.$$

First, let $Q_A(f)(\kappa) \subset \rho$, $\psi \in \overline{\rho}$. Let, for some $\varphi : A \to X$, there be an $e \supset e(\varphi)$ such that for every $a \in A$ $\psi \circ f \circ \varphi(a) = (a(e), e)$. We have $e \supset e(f \circ \varphi)$, since if $f \circ \varphi(a) = f \circ \varphi(b)$, then $(a(e), e) = \psi \circ f \circ$ $\circ \varphi(a) = \psi \circ f \circ \varphi(b) = (b(e), e)$, hence $a(e) = b(e)$ so that $(a, b) \in e$. As $\psi \in \overline{\rho}$, $f \circ \varphi \notin \rho$ and hence $\varphi \notin \kappa$. Thus, for every $\psi \in \overline{\rho}$, $\psi \circ f \in \overline{\kappa}$.

Now, let $P_B(f)(\overline{\rho}) \subset \overline{\kappa}$. If $Q_A(f)(\kappa) \not\subset \rho$, there is a $\varphi \in \kappa$ with $f \circ \varphi \notin \rho$. Define $\psi : Y \to B$ as follows:

$\psi(y) = (a(e(f \circ \varphi)), e(f \circ \varphi))$ whenever $y = f \circ \varphi(a)$,

$\psi(y) = 0$ otherwise.

(This is correct: if $y = f \circ \varphi(a) = f \circ \varphi(b)$, $(a, b) \in e(f \circ \varphi)$ and hence $a(e(f \circ \varphi)) = b(e(f \circ \varphi))$.) Let $\mu : A \to Y$ be such that there is an $e \supset e(\mu)$ with $\psi(\mu(a)) = (a(e), e)$ for every $a \in A$. If $a \in A$, $\mu(a) \in f \circ \varphi(A)$ by the definition, since otherwise $\psi(\mu(a)) = 0$. Thus, $\mu(a) = f \circ \varphi(b)$ for some $b \in A$. We have $(a(e), e) =$ $= \psi(\mu(a)) = \psi(f \circ \varphi(b)) = (b(e(f \circ \varphi)), e(f \circ \varphi))$ so that $e = e(f \circ \varphi)$ and $a(e) = b(e)$, i.e. $f \circ \varphi(b) = f \circ \varphi(a)$.
Thus, $\mu(a) = f \circ \varphi(a)$. We obtained $\mu = f \circ \varphi \notin \rho$ and consequently $\psi \in \overline{\rho}$. Hence, $\psi \circ f \in \overline{\kappa}$. On the other hand, $(\psi \circ f) \circ \varphi(a) = \psi(f \circ \varphi(a)) = (a(e), e)$ where $e =$ $= e(f \circ \varphi) \supset e(\varphi)$, so that $\varphi \notin \kappa$. This is a contradiction.

7.3. The realisation of $S(Q_A)$ in $S(P_B)$ in the last theorem is not caused by majorisation; Q_A is covariant and P_B contravariant. Of course, combining 1.5 and 6.2 we obtain realisations of $S(F)$ in $S(G)$ with differently variant F, G; that last one, however was of another character. Here is a further statement of this type:

Theorem: $S(P^-) \Rightarrow S(P^+, \{2\})$.

Proof: Let $\kappa \subset P^-(X)$. Define a binary relation $\bar{\kappa}$ on $P^+(X)$ as follows

$(A, B) \in \bar{\kappa}$ iff for every $U \in \kappa$ $A \subset U$ implies $B \cap U \neq \emptyset$.

Let $\kappa \subset P^-(X)$, $\lambda \subset P^-(Y)$, $f: X \to Y$. Let $P^-(f)(\lambda) \subset \kappa$, $(A, B) \in \bar{\kappa}$. If $f(A) \subset V$ with $V \in \lambda$, we have $A \subset f^{-1}(V) \in \kappa$ so that there exists a $b \in B \cap f^{-1}(V)$. Thus, $f(b) \in f(B) \cap V$ and hence $(f(A), f(B)) \in \bar{\lambda}$. Let $P^+(f)$ be $\bar{\kappa}\bar{\lambda}$-compatible, $V \in \lambda$. If $f^{-1}(V) \notin \kappa$, we have $(f^{-1}(V), X \setminus f^{-1}(V)) = (f^{-1}(V), f^{-1}(Y \setminus V)) \in \bar{\kappa}$. Thus, $(ff^{-1}(V), ff^{-1}(Y \setminus V)) \in \bar{\lambda}$. Since $ff^{-1}(V) \subset V$, we have $f(f^{-1}(Y \setminus V)) \cap V \neq \emptyset$. As $ff^{-1}(Y \setminus V) \subset Y \setminus V$, we obtained a contradiction.

Corrollary: $S(P^-) \Rightarrow S((P^+)^3)$.

Proof: By the previous theorem and 6.2, $S(P^-) \Rightarrow S(Q_2 \circ P^+)$. As $Q_2 < (P^+)^2$ (define $T: Q_2 \to (P^+)^2$ by $T_X((x, y)) = \{\{x\}, \{x, y\}\}$), we obtain the statement.

This last statement gives rise to a question, whether in Theorem 6.5 the functor P^- may be replaced by P^+. This question seems to be open.

References

[1] M. KATĚTOV: On continuity structures and spaces of mappings. Comment.Math.Univ.Carolinae 6(1965), 257-278.

[2] Z. HEDRLÍN, A. PULTR: On categorial embeddings of topological structures into algebraic,Comment. Math.Univ.Carolinae 7(1966),377-400.

[3] J.R. ISBELL: Small adequate subcategories, to appear in J.London Math.Soc.

(Received November 16,1966)