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THE CONCEPT OF RANK AND SOME RELATED QUESTIONS IN THE THEORY OF MODULES

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(Preliminary communication)

The present results extend the ideas of [1]; their applications show some new aspects of the theory of modules; in particular, they generalize some results of A.W.GOLDIE [2] and EBEN MATLIS [3]. The results were partly read at the IMC in Moscow, August 16-26, 1966.

Let R be an (associative) ring with an identity. Denote by $\mathcal{L}$ the family of all its proper (i.e. $\neq R$) left ideals, by $\mathcal{J} \subseteq \mathcal{L}$ the subfamily of all irreducible ideals. For $L \in \mathcal{L}$ and $\rho \in R$, the symbol $L:\rho$ stands for the (left) ideal consisting of all $\chi \in R$ such that $\chi\rho \in L$.

Let M be a (unitary left) R -module; put $M_0 = M \setminus \{0\}$. The order of $x \in M$ is denoted by $O(x)$; hence $O(x) \in \mathcal{L}$ if and only if $x \in M_0$.
Evidently, $O(\rho x) = O(x):\rho$ for any $\rho \in R$ and $x \in M_0$.

We refer to [1] for the definitions and some basic facts concerning dependence over modules.

1. Let T be an index set. For $t \in T$, let $\mathcal{P}_t^1 \subseteq \mathcal{L}$ be a subfamily satisfying

$$L \in \mathcal{P}_t^1 \wedge \rho \in R \setminus L \to L : \rho \in \mathcal{P}_t^1 .$$

Then, define $\mathcal{P}_t^{-1} \subseteq \mathcal{L}$ by

$$L \in \mathcal{P}_t^{-1} \leftrightarrow \forall \rho \, (\rho \in R \setminus L \to L : \rho \notin \mathcal{P}_t^1) .$$

Evidently,

$$L \in \mathcal{P}_t^{-1} \wedge \rho \in R \setminus L \to L : \rho \in \mathcal{P}_t^{-1}$$

and

$$\mathcal{P}_t^1 \cap \mathcal{P}_t^{-1} = \emptyset .$$

Put

$$\mathcal{P}_t^0 = \mathcal{P}_t^1 \cup \mathcal{P}_t^{-1} .$$

Now, consider the set 2^T of all functions of the index set T into $\{-1, 1\}$ and, for each $f \in 2^T$, define the subset M_f of an R-module M by

$$x \in M_f \leftrightarrow O(x) \in \bigcap_{t \in T} \mathcal{P}_t^{f(t)} .$$

Clearly, $M_f \; (\subseteq M_0)$ have the following two simple properties:

(i) $x \in M_f \wedge \rho \notin O(x) \to \rho x \in M_f$;

(ii) $f \neq f' \to M_f \cap M_{f'} = \emptyset$.

Hence,

(iii) $x \in M_{f'} \wedge \rho x \in M_f \to f = f'$.

Also

(iv) $x \in \bigcup_{f \in 2^T} M_f \leftrightarrow O(x) \in \bigcap_{t \in T} \mathcal{P}_t^0$.

The following two lemmas are of fundamental importance:

Lemma 1. Let $\mathfrak{M} \subseteq \bigcup_{f \in 2^T} M_f$ be a maximal independent subset of M. Then, for any $f \in 2^T$,

$$\mathfrak{M}_f = \mathfrak{M} \cap M_f$$

is a maximal independent subset of M_f .

Lemma 2. Let $\mathfrak{M}_f$ $(f \in 2^T)$ be an independent subset of M_f. Then,

$$\mathfrak{M} = \bigcup_{f \in 2^T} \mathfrak{M}_f$$

is an independent subset of M. Moreover, if $\mathfrak{M}_f$ are maximal in M_f and if a subfamily $\hat{\mathcal{L}} \subseteq \mathcal{L}$ exists such that

$$L \in \mathcal{L} \rightarrow \exists \wp (\wp \in R \setminus L \wedge L : \wp \in \hat{\mathcal{L}})$$

and

$$\bigcup_{t \in T'} \mathcal{P}_t^{o} \supseteq \hat{\mathcal{L}} \qquad \text{for every infinite } T' \subseteq T,$$

then $\mathfrak{M}$ is maximal in M.

In particular, $\mathfrak{M}$ is a maximal independent subset of M provided

(i) for any $T' \subseteq T$ there is a finite $T'' \subseteq T'$ such that

$$\bigcap_{t \in T'} \mathcal{P}_t^1 = \bigcap_{t \in T''} \mathcal{P}_t^1 , \qquad \text{or}$$

(ii) T is finite.

2. Some applications. (a) Let $T = \{1\}$, $\mathcal{P}_1^1 = \mathcal{I}$. Then, $\mathcal{P}_1^{-1}$ consists of what will be called strongly reducible ideals. Denote the corresponding subsets of M by M_1 and M_{-1}.

There exist maximal independent subsets $\mathfrak{M}$ of M such that $\mathfrak{M} \subseteq M_1 \cup M_{-1}$ and any such $\mathfrak{M}$ is a disjoint union of a maximal independent subset $\mathfrak{M}_1$ of M_1 and a maximal independent subset $\mathfrak{M}_{-1}$ of M_{-1}. The cardinality $card(\mathfrak{M}_1)$ is an invariant of M. On the other hand, any element of $\mathfrak{M}_{-1}$ can be replaced by two elements of M_{-1} so that the resulting subset is again independent.

Define

$$ {}^{i}r(M) = card(\mathfrak{M}_1), \quad {}^{r}r(M) = \sup_{\mathfrak{M}_{-1}} card(\mathfrak{M}_{-1}) $$

and

$$ r(M) = {}^{i}r(M) + {}^{r}r(M) $$

and call ${}^{i}r(M)$ the <u>irreducible rank</u>, ${}^{r}r(M)$ the <u>reducible rank</u> and $r(M)$ the <u>complete rank</u> of the module M.

An R-module M is said to be <u>tidy</u> if ${}^{r}r(M) = 0$. ${}^{r}r(M) = 0$ (i.e. $M_{-1} = \emptyset$) for any R-module M, if and only if R has the property $(\mathcal{J})$ of 1. Thus, the property $(\mathcal{J})$ of a ring R expresses the fact that every R-module is tidy. Since any (left) noetherian ring has $(\mathcal{J})$ (cf.[1]), the above definition of $r(M)$ extends the definition of rank of Goldie [2] to arbitrary R-modules.

(b) Let $T = \{1, 2\}$, $\mathcal{P}_1^1 = \mathcal{J}$ and $\mathcal{P}_2^1$ be the subfamily of all (proper) maxi ideals in R. Here an ideal $L \subseteq R$ is said to be <u>maxi</u> in R if, for every $\rho \in R \setminus L$, there exists $\sigma \in R \setminus (L : \rho)$ such that $L : \sigma\rho$ is essential in R. The ideals of $\mathcal{P}_2^{-1}$ will be called <u>mini</u> (in R).

The particular value of the concept of a maxi ideal rests on the fact that it allows to extend the definition of torsion and torsion-free R-modules to the general case: An R-module M is said to be <u>torsion</u> if the order of each of its elements is maxi. The set of all elements of maxi orders of an arbitrary R-module M is an R-submodule, - the <u>torsion R-submodule</u> T_M of M. M is said to be <u>torsion-free</u> if $T_M = \{0\}$.

The quotient R-module M/T_M is torsion-free for every R-module M.

Denote by $M_{f(1),\, f(2)}$ the subsets of M corresponding to the intersections

$$\mathcal{P}_1^{f(1)} \cap \mathcal{P}_2^{f(2)} .$$

Here, $M_{1,1} \cup M_{1,-1} = M_1$ of (a). There exist maximal independent subsets $\mathfrak{M}$ of M such that $\mathfrak{M} \subseteq \cup M_{f(1), f(2)}$ and any such $\mathfrak{M}$ is a disjoint union of maximal independent subsets $\mathfrak{M}_{f(1),\, f(2)}$ of $M_{f(1),\, f(2)}$.

Again,

$$card\ (\mathfrak{M}_{1,1}) = {}^{it}\kappa\ (M)$$

and

$$card\ (\mathfrak{M}_{1,-1}) = {}^{if}\kappa\ (M)$$

are invariants of M and are called the <u>irreducible torsion rank</u> and <u>irreducible torsion-free rank</u> of M, respectively. Thus,

$$^{i}\kappa(M) = {}^{it}\kappa(M) + {}^{if}\kappa(M),$$

$$^{it}\kappa(M) = {}^{it}\kappa(T_M),$$

$$^{if}\kappa(T_M) = 0$$

and

$$^{if}\kappa(M) = {}^{if}\kappa(M/T_M).$$

In fact, the latter relation is a particular case of the following formula

$$^{if}\kappa(M) = {}^{if}\kappa(N) + {}^{if}\kappa(M/N)$$

which holds for any R-submodule N of M. These results extend again those of [2].

(c) Let $\sim$ be the equivalence defined on the subfamily $\mathcal{J}$ as follows:

$L_1 \sim L_2 \leftrightarrow L_1 : \rho_1 = L_2 : \rho_2 \neq R$ for certain $\rho_1, \rho_2 \in R$.
Denote the corresponding partition of $\mathcal{J}$ by Π:

$$\Pi = \{\pi_t\}_{t \in T} .$$

Π is a refinement of $\{\mathcal{P}_1^1 \cap \mathcal{P}_2^1, \mathcal{P}_1^1 \cap \mathcal{P}_2^{-1}\}$ of (b) and, thus, we can write

$$\Pi = \{\pi_{t_1}\}_{t_1 \in T_1} \cup \{\pi_{t_2}\}_{t_2 \in T_2} ,$$

where $T = T_1 \cup T_2$, $\bigcup_{t_1 \in T_1} \pi_{t_1} = \mathcal{P}_1^1 \cap \mathcal{P}_2^1$ and $\bigcup_{t_2 \in T_2} \pi_{t_2} = \mathcal{P}_1^1 \cap \mathcal{P}_2^{-1}$.

Put $\mathcal{P}_t^1 = \pi_t$ for $t \in T$. Then, besides
$\bigcup_{t \in T} \mathcal{P}_t^1 = \mathcal{P}_1^1$ of (a), also

$$\bigcap_{t \in T} \mathcal{P}_t^{-1} = \mathcal{P}_1^{-1} \quad \text{of (a).}$$

Hence, any maximal independent subset $\mathfrak{M}$ of an R-module M such that $\mathfrak{M} \subseteq M_1 \cup M_{-1}$ (which exists by (a)) is a disjoint union

$$\mathfrak{M} = \bigcup_{t \in T} \mathfrak{M}_t \cup \mathfrak{M}_{-1} ,$$

where $\mathfrak{M}_t$ is a maximal independent subset of the set M_t of all elements of M of orderes belonging to π_t $(t \in T)$ and $\mathfrak{M}_{-1}$ to a maximal independent subset of M_{-1} of (a). Again, for $t \in T$,

$$card(\mathfrak{M}_t) = {}^{\pi_t}\kappa(M)$$

is an invariant of M and will be called the $\underline{\pi_t}$-rank of M.

Thus,

$${}^{it}\kappa(M) = \sum_{t_1 \in T_1} {}^{\pi_{t_1}}\kappa(M) \quad \text{and} \quad {}^{if}\kappa(M) = \sum_{t_2 \in T_2} {}^{\pi_{t_2}}\kappa(M) .$$

In particular, if $\kappa(M) = 1$, then the orderes of all non-zero elements of M belong to the same family π_t for a certain $t \in T$.

Let us remark that in the case when R is a commutative noetherian ring, there is just one prime ideal P_t in every π_t and we can call, in accordance with the terminology of abelian groups, the cardinality ${}^{\pi_t}\kappa(M)$ the P_t -rank of the R-module M.

(d) The latter results can be used to generalize some of the results on injective hulls of R-modules of Matlis [3].

$\mathfrak{M} = \{x_i\}_{i \in I}$ is a maximal independent subset of an R-module M if and only if the direct sum $\bigoplus_{i \in I} R x_i$ is essential in M. Thus, if an R-submodule N is essential in M, then $\mathfrak{M} \subseteq N$ is a maximal independent subset of N if and only if it is a maximal independent subset of M. Since M is essential in its injective hull $H(M)$, we get immediately

$$^{*}\kappa(M) = {}^{*}\kappa(H(M)),$$

where $*$ can be replaced by any of the symbols from $\{i, \kappa, it, if, \pi_t\}$.

Let H be an injective R-module. Then, the elementary properties of dependence yield immediately the equivalence of the following statements (cf.[3]):

(i) H is indecomposable.

(ii) $\kappa(H) = 1$.

(iii) For any $0 \neq x \in H$, $O(x) \in \mathcal{J}$ and $H = H(Rx) \cong H(R/O(x))$.

(iv) $H \cong H(R/L)$ for $L \in \mathcal{J}$.

Also, for $L_1, L_2 \in \mathcal{J}$,

$$H(R/L_1) \cong H(R/L_2)$$

if and only if L_1 and L_2 belong to the same equivalence class π of (c).

Denote the indecomposable injective R-module corresponding to π by $H(\pi)$.

Let $\mathfrak{M} = \{x_i\}_{i \in I}$ be an independent subset of M such that $O(x_i) \in \mathcal{J}$ $(i \in I)$; let $H(M) \geq M$ be an injective hull of M. Let $H(Rx_i)$ be an injective hull of Rx_i in $H(M)$ for $i \in I$. Then

$$< H(Rx_i) >_{i \in I} = \bigoplus_{i \in I} H(Rx_i).$$

Summarizing, we can formulate

<u>Theorem</u>. There is a one-to-one correspondence between the equivalence classes $\pi \in \Pi_R$ and the indecomposable injective R-modules $H(\pi)$. This correspondence amounts in the case of commutative noetherian rings R to a one-to-one correspondence between the prime ideals $P \subseteq R$ and the indecomposable injective R-modules $H(P)$ (cf. [3]).

If M is an R-module and $H(M)$ its injective hull, then $H(M)$ contains a direct sum

$$(*) \quad \bigoplus_{\substack{\pi \in \Pi_R \\ i \in I_\pi}} H_i(\pi) \qquad \text{with } \operatorname{card}(I_\pi) = {}^{\pi}\kappa(H(M)) = {}^{\pi}\kappa(M);$$

on the other hand, any maximal direct sum of indecomposable injective R-modules contained in $H(M)$ has the form $(*)$. In particular, any two direct decompositions of an

R-module into direct sums of indecomposable injective R-modules are isomorphic and can be described by a cardinal-valued function on Π_R (cf.[3]).

Furthermore, if M is tidy (see (a)), then ($*$) is essential in $H(M)$ and thus, $H(M)$ is, up to an isomorphism, uniquely determined by the function f :

$$f(\pi) = {}^{\pi}\kappa(M)$$

defined on Π_R . Again, this latter statement amounts in the case of (commutative) noetherian rings R to the, up to an isomorphism, unique decomposition of an injective R-module M into the direct sum of indecomposable injective R-submodules described by a cardinal-valued function on the family Π_R (the family of prime ideals of R) which is well-defined by any essential submodule of M .

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