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Ivo Marek

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AN INFINITE DIMENSIONAL ANALOGUE OF R.S. VARGA'S LEMMA

Ivo MAREK, Praha

1. Introduction.

The purpose of this note is a generalization of the following lemma of R.S. Varga and some of its consequences.

Let S be the class of square matrices B, for which the following conditions hold:

(a) B is nonnegative $(b_{jk}) = B \geq \theta$, where θ denotes the null-matrix and $b_{jj} = 0$.

(b) B is irreducible and convergent, i.e. $\rho(B) < 1$, where $\rho(B)$ is the spectral radius of B.

(c) B is symmetric.

For $B \in S$, put $B = L_B + L_B^T$, where L_B is a strictly lower triangular matrix and L_B and L_B^T are conjugated.

<u>Lemma</u> ([7], p. 118). If $B \in S$, put

$$A_B(\alpha) = e^{\alpha} L_B + e^{-\alpha} L_B^T + \gamma I ,$$

where I is the identity matrix and α, γ are nonnegative numbers; then

$$\rho(A_B(\alpha_1)) \leq \rho(A_B(\alpha_2))$$

for $0 \leq \alpha_1 \leq \alpha_2$. Moreover, if

$$\rho(A_B(\hat{\alpha}_1)) < \rho(A_B(\hat{\alpha}_2))$$

for some particular values $0 \leq \hat{\alpha}_1 \leq \hat{\alpha}_2$, then it is true for all $0 \leq \alpha_1 < \alpha_2$.

2. Notation and definitions.

Let Y be a real Hilbert space and $\mathcal{X}$ its complex extension. The inner product in $\mathcal{X}$ is an obvious extension of the inner product in Y. If (x_j, y_j) denotes the inner product of $x_j, y_j \in Y$, $j = 1, 2$, then $(z_1, z_2) = [(x_1, y_1) + (x_2, y_2)] + i[(x_2, y_1) - (x_1, y_2)]$ is the value of the inner product of $z_1, z_2 \in \mathcal{X}$, $z_j = x_j + i y_j$.

We assume that there is a cone K in Y and that this cone is generating, normal ([2]) and self-adjoint, i.e. $K' = K$, where K' is the adjoint cone ([2]). If $y - x \in K$, $x, y \in Y$, we write $x \prec y$ or $y \succ x$.

The symbol $[Y]$ ($[\mathcal{X}]$ respectively), denotes the space of linear bounded transformations of Y ($\mathcal{X}$ respectively) into itself. Topologies in these spaces are given by the usual norms; thus, $[Y]$ and $[\mathcal{X}]$ are Banach spaces. Let I denote the identity operator and θ the null-operator. If $T \in [Y]$, then $\tilde{T}$ denotes the complex extension of T, i.e. we have $\tilde{T}z = Tx + iTy$ for $z = x + iy$.

The spectrum of $T \in [\mathcal{X}]$ is denoted by $\sigma(T)$. In the particular case, if $T \in [Y]$, we define $\sigma(T) = \sigma(\tilde{T})$. Similarly, put $\rho(T) = \rho(\tilde{T})$, where $\rho(A)$ is the spectral radius of an operator $A \in [\mathcal{X}]$.

An operator $T \in [Y]$ is called positive [2], if $Tx \in K$ for $x \in K$. A positive operator $T \in [Y]$ is called <u>semi-non-support</u> [5], if for every pair $x \in K$, $x \neq \sigma$, $x' \in K'$, $x' \neq \sigma$, where σ denotes the null-vector in

Y , there exists a positive integer $p = p(x,x')$ such that $(T^p x, x') > 0$. We say that a vector $x \in K$ is <u>non-support</u> element of the cone K [5], if $(x,x') \neq 0$ for every $x' \in K'$, $x' \neq \sigma$. A positive operator $T \in [Y]$ is called <u>strict non-support</u> [5], if for every $x \in K$, $x \neq \sigma$, there is a positive integer $p = p(x)$ such that the vectors $T^n x, n \geq p$, are non-support elements of the cone K .

Finally we define the class V of operators from $[Y]$.

<u>Definition 1</u>, The operator $T \in V$ iff $T \in [Y]$ satisfies the following properties:

1. B is semi-non-support;
2. B has the form

$$B = L_B + L_B^* ,$$

where L_B^* is conjugated with L_B and L_B and L_B^* are positive operators.

<u>Definition 2</u>. An operator $T \in [\mathfrak{X}]$ has property (S) , if every point $\lambda \in \sigma(T)$, $|\lambda| = \rho(T)$, is an isolated pole of $R(\lambda, T) = (\lambda I - T)^{-1}$.

<u>3. Generalized Varga's lemma and its consequences</u>.

<u>Lemma 3.1</u>. Suppose

(i) $B \in V$.

(ii) $A_B(\alpha)$, where

$$A_B(\alpha) = e^{\alpha} L_B + e^{-\alpha} L_B^* + \gamma I ,$$

and γ is a nonnegative constant, have property (S) for all $\alpha \geq 0$.

Then for $0 \leq \alpha_1 \leq \alpha_2$

(3.1) $\rho(A_B(\alpha_1)) \leq \rho(A_B(\alpha_2))$.

Moreover, if

(3.2) $\rho(A_B(\hat{\alpha}_1)) < \rho(A_B(\hat{\alpha}_2))$

for some particular values $0 \leq \hat{\alpha}_1 < \hat{\alpha}_2$, then it is true for all $0 \leq \alpha_1 < \alpha_2$.

Before proving this lemma we shall prove the following proposition.

<u>Proposition 3.1.</u> Assume $T \in [Y]$ is a positive operator having property (S) and $\mu = \rho(T)$ is a dominant proper value of T, i.e. the inequality $|\lambda| < \mu$ holds for every $\lambda \in \sigma(T)$, $\lambda \neq \mu$. Then we have

$$\rho(T) = \lim_{n \to \infty} (T_x^n, x)^{\frac{1}{n}} , \tag{3.3}$$

where x is any non-support element of the cone K.

<u>Proof</u>. According to [3], theorem 1, it holds in the norm of $[\mathfrak{X}]$ that

$$\lim_{n \to \infty} n^{-q+1} \mu^{-n} T^n = \frac{\mu^{-q+1}}{(q-1)!} B_q , \tag{3.4}$$

where q is the multiplicity of μ and

$$R(\lambda, T) = \sum_{k=0}^{\infty} A_k (\lambda - \mu)^k + \sum_{k=1}^{q} B_k (\lambda - \mu)^{-k} ,$$

$$A_k \in [\mathfrak{X}], \quad B_k \in [\mathfrak{X}] ,$$

is the Laurent expansion of $R(\lambda, T)$ in a neighbourhood of $\mu = \rho(T)$ ([6], p.305).

From the formula (3.4) it follows that

$$(T^n x, x) = n^{q-1} \mu^n \frac{\mu^{-q+1}}{(q-1)!} (B_q x, x) + o(n^{q-1}) ,$$

where $g(n) = o(f(n))$ means

$$\lim_{p\to\infty} \frac{q(p)}{f(p)} = 0 .$$

Since $(B_q x, x) > 0$ the following formula is meaningful

$$(T^p x, x)^{\frac{1}{p}} = \mu \exp\{ \frac{q-1}{p} \log p + \frac{1}{p} \log \frac{\mu}{(q-1)!} + \frac{1}{p} \log (B_q x, x)\} + o(1)$$

and hence

$$\lim_{p\to\infty} (T^p x, x)^{\frac{1}{p}} = \mu ,$$

which was to be proved.

Proof of generalized Varga's lemma. Let $\alpha \geq 0$ and let the point $\mu(\alpha) = \rho(A_B(\alpha))$ be a dominant proper value of $A_B(\alpha)$. According to the previous proposition

(3.5) $$\mu(\alpha) = \lim_{p\to\infty} ([A_B(\alpha)]^p x, x)^{\frac{1}{p}} .$$

On the other hand,

$$[A_B(\alpha)]^p = \sum_{k=1}^{p} [e^{k\alpha} L_k^{(p)} + e^{-k\alpha} (L_k^{(p)})^*] + Z_0^{(p)} ,$$

where the adjoint $(Z_0^{(p)})^* = Z_0^{(p)}$ and $(L_B^* L_k^{(p)})^* = L_B (L_k^{(p)})^*$ for $k = 1, 2, \ldots, p$, from which it follows that

(3.6) $$([A_B(\alpha)]^p x, x) = \sum_{k=0}^{p} \xi_k^{(p)} \cosh k\alpha , \quad p = 1, 2, \ldots ,$$

with nonnegative coefficients $\xi_k^{(p)}$. We see that $([A_B(\alpha)]^p x, x)$ depends monotonically on $\alpha \in \in \langle 0, +\infty)$. Thus (3.1) is true for $0 \leq \alpha_1 \leq \alpha_2$.

It remains to prove the strict monotonicity of $\varphi(A_B(\alpha))$ assuming the validity of (3.2). Clearly the assertions of Varga's lemma hold for the operator-function $A_B = A_B(\alpha)$ iff this lemma is valid for the operator-function $D_B = D_B(\alpha) \equiv \nu A_B(\alpha)$, where $\nu > 0$ is such that $\varphi(D_B(0)) > 1$. Thus, we may assume $\varphi(A_B(0)) > 1$.

It follows from (3.6) that for $0 < \hat{\alpha} < \alpha < \beta$

$$([A_B(\beta)]^p x, x) = \sum_{k=0}^{p} \xi_k^{(p)} \cosh k\beta =$$
$$= \sum_{k=0}^{p} \xi_k^{(p)} [\cosh k\alpha + k(\beta-\alpha) \sinh k \nu_k],$$

where $\nu_k = \alpha + \vartheta_k(\beta-\alpha), \quad 0 < \vartheta_k < 1$.

The assumptions $\varphi(A_B(0)) > 1$ and (3.2) guarantee the validity of $\tau_p(\alpha) \to +\infty$, where

$$\tau_p(\alpha) = \sum_{k=0}^{p} \xi_k^{(p)} \cosh k\alpha .$$

Further we have

$$\frac{\sinh \lambda}{\cosh \lambda} \geq \frac{\sinh \hat{\alpha}}{\cosh \hat{\alpha}} \qquad \text{for } \lambda \geq \hat{\alpha} > 0,$$

thus

$$\sinh k\alpha \geq \varkappa \cosh k\alpha, \quad \alpha \geq \hat{\alpha} > 0,$$

with some positive constant $\varkappa$ independent of k. Then

$$([A_B(\beta)]^p x, x) \geq \tau_p(\alpha) + \varkappa(\beta-\alpha)\eta_p(\alpha),$$

where

$$\eta_p(\alpha) = \sum_{k=0}^{p} \xi_k^{(p)} k \cosh k\alpha .$$

From (3.2) and according to $\varphi(A_B(0)) > 1$, it follows that

$$\lim_{p\to\infty} \frac{\eta_p(\alpha)}{\tau_p(\alpha)} = +\infty .$$

Hence

$$\varrho(A_B(\beta)) = \lim_{p\to\infty} ([A_B(\beta)]^p x, x)^{\frac{1}{p}} \geq$$
$$\geq \lim_{p\to\infty} \sup \{1 + \frac{\eta_p(\alpha)}{\tau_p(\alpha)}\}^{\frac{1}{p}} ([A_B(\alpha)]^p x, x)^{\frac{1}{p}} >$$
$$> \varrho(A_B(\alpha)) .$$

We shall prove that $\varrho(A_B(0)) < \varrho(A_B(\alpha))$ for $\alpha > 0$. The equality $\varrho(A_B(0)) = \varrho(A_B(\hat{\alpha}))$ for some $\hat{\alpha}$ implies the validity of equalities $\varrho(A_B(\alpha)) = \varrho(A_B(0))$ for all $\alpha \in \langle 0, \hat{\alpha} \rangle$. But these equalities are impossible according to the preceding investigation. Hence the Varga's lemma is proved in the case of the operators $A_B(\alpha), \alpha \geq 0$, which have dominant proper values.

The validity of Varga's lemma in general case follows from the proved version of this lemma, since for arbitrary pair $\alpha, \beta, 0 \leq \alpha \leq \beta$, the operators $A_B(\alpha) + \sigma I$ and $A_B(\beta) + \sigma I$ have dominant proper values for sufficiently great $\sigma > 0$ and

$$\varrho(A_B(\alpha) + \sigma I) = \varrho(A_B(\alpha)) + \sigma .$$

The proof is completed.

In agreement with Varga [7], p.117, we define the operator-functions

$$(3.7) \qquad M_B(\sigma) \equiv \sigma L_B + L_B^*, \quad \sigma \geq 0,$$

and the scalar-functions m_B, h_B :

$$(3.8) \qquad m_B(\sigma) \equiv \varrho(M_B(\sigma)),$$

$$(3.9) \qquad h_B(\log \sigma) \equiv \frac{m_B(\sigma)}{\sqrt{\sigma}\,\varrho(B)}, \quad \sigma > 0 .$$

Similarly as in the finite dimensional case the following lemma and theorem hold.

Lemma 3.2. Let $B \in V$ and $\alpha = \log \sigma$, $\sigma > 0$. Then $h_B(\alpha) = h_B(-\alpha)$ for all real α and specifically, $h_B(0) = 1$.

Theorem 3.1. If $B \in V$, then either $h_B(\alpha) \equiv 1$ for all real α, or h_B is strictly increasing for $\alpha \geq 0$. Moreover, for any $\alpha \neq 0$

$$1 \leqq h_B(\alpha) < \cosh \frac{\alpha}{2} .$$

Now we may introduce a class of operators which play the same role as the class of consistently ordered cyclic matrices of index 2 in finite dimensional spaces.

Definition 3. The operator $B \in VCO$, if $B \in [Y]$ has the following properties:

1. $B \in V$;

2. The function $h_B = h_B(\alpha)$ is constant in $\langle 0, +\infty)$.

4. Comparison theorems for Gauss-Seidel and successive approximation methods.

Let $B \in [Y]$ be expressed as $B = B_1 + B_2$, where $B_1, B_2 \in [Y]$ and $\rho(B) < 1$. We define the operator

$$H = (I - B_1)^{-1} B_2 , \tag{4.1}$$

assuming $(I - B_1)^{-1} \in [Y]$.

If the components B_1 and B_2 in (4.1) are positive transformations such that the operator B is semi-nonsupport, then there holds the generalized Stein-Rosenberg theorem (for finite dimensional spaces see [7],p.70).

<u>Proposition 4.1</u>. Assume that

(a) $B = B_1 + B_2$, B_1 and B_2 are positive transformations, B has property (S).

(b) $\rho(B_1) < 1$.

(c) The operator $(I - B)^{-1} B_2$ has property (S).

(d) The operator B is semi-non-support.

Then $\rho(H) = 0$ iff $B_2 = \theta$.

<u>Proof</u>. It is easily seen that $\rho(H) \geq 1$, if $\rho(B) \geq 1$. Hence let $\rho(B) < 1$ and $\theta \prec P \prec B_2$, $P \neq \theta$, where $P \prec B_2$ means that $B_2 - P$ is a positive transformation.

Evidently

$$(I - B_1)^{-1} - (I - B_1 - P)^{-1} = -(I - B_1)^{-1} P (I - B_1 - P)^{-1}$$

and hence

$$H = [I - (I - B_1)^{-1} P](I - B_1 - P)^{-1} B_2 .$$

In particular, if $P = B_2$, we have

(4.2) $$H = (I - H)(I - B)^{-1} B_2 .$$

We shall prove that the operator $(I - B)^{-1} B_2$ has a positive spectral radius. Let $y_0 = [\rho(B)]^{-1} B y_0$, $y_0 \in K$, $y_0 \neq \sigma$. The existence of such vector follows from assumption (d) [5]. Then y_0 is a non-support element of the cone K and thus $B_2 y_0 \neq \sigma$. The vector $z_1 = (I - B)^{-1} B_2 y_0 = \sum_{k=0}^{\infty} B^k B_2 y_0$ also is a non-support element of the cone K. Hence $B_2 z_1 \neq \sigma$. By induction, it follows that $[(I - B)^{-1} B_2]^k y_0 \neq \sigma$ for $k = 1, 2, \ldots$.

This means that T cannot be a nilpotent operator. This fact and property (S) imply the existence of a vector $v_0 \neq \sigma$ for which $(I - B)^{-1} B_2 v_0 = \rho v_0$ and $\rho = \rho((I - B)^{-1} B_2) > 0$. According to (4.2), we have

$$H v_0 = \rho (I - H) v_0 ,$$

or

$$H v_0 = \frac{\rho}{1+\rho} v_0$$

and consequently

$$\rho(H) \geq \frac{\rho}{1+\rho} > 0 .$$

The proposition just proved and the generalized minimax-principle for semi-non-support operators [4] form a base for the following theorem.

Theorem 4.1 (Stein-Rosenberg). In addition to assumptions (a) to (c) of proposition 4.1 let also

(e) B is strict non-support operator.

(f) $\rho(H)$ is a proper value of H and to it there corresponds a proper vector of H within K.

(g) The operator B has property (S).

(h) $B_2 \neq \theta$.

Then one of the three following conditions holds:

(4.3) (a) $0 < \rho(H) < \rho(B) < 1$,

(b) $\rho(H) = 1 = \rho(B)$,

(c) $1 < \rho(B) < \rho(H)$.

The proof of theorem 4.1 is analogous as for the finite dimensional case (see [1], p.105).

The sharpened form of statement (4.3) (a) of Stein-Rosenberg theorem 4.1 for the set of operators V is contained in the following theorem.

Theorem 4.2. If $B \in V$, then

$$\rho^2(B) \leq \rho(H) < \frac{\rho(B)}{2-\rho(B)} .$$

The equality is possible only if $B \in VCO$.

Theorem 4.2 is a consequence of a more general theorem concerning successive over relaxation methods. Successive over relaxation methods in Banach spaces will be studied in our another paper, where the theorem mentioned above will be proved amongst other assertions.

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