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# CONSISTENCY THEOREMS CONNECTED WITH SOME COMBINATORIAL PROBLEMS

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The main purpose of this note is to prove the consistency of the positive solution of a problem of G. Kurepa. The terminology and notation are those of [2],[3]. For notions from partition calculus see [1].

We say that the set $X$ possesses property $(K,\alpha)$ iff

(1,$\alpha$) $X \subseteq \mathcal{P}(\omega_\alpha)$,

(2,$\alpha$) $\overline{\overline{X}} > \aleph_{\alpha+1}$,

(3,$\alpha$) $(\forall y)(y \subseteq \omega_\alpha \,\&\, \overline{\overline{y}} < \aleph_\alpha . \rightarrow \overline{\overline{\{x \cap y : x \in X\}}} < \aleph_\alpha)$.

G. Kurepa has stated the following problem:

Is there a set $X$ with property (K,1)?

The positive solution of this problem leads to many other theorems (for example $\aleph_2 \not\rightarrow [\aleph_1]^2_{\aleph_1, \aleph_0}$ - see[1], p.154). If $\aleph_\alpha$ is strongly inaccessible, then every set with properties (1,$\alpha$) and (2,$\alpha$) also possesses property (3,$\alpha$).

<u>Theorem</u>. Suppose that

(4) $\omega_\alpha$ is an inaccessible cardinal in the sense of Gödel's $\Delta$-model,

(5) in the $\Delta$-model, there is no cardinal between $\omega_\alpha$ and $\omega_{\alpha+1}$,

(6) $\omega_\alpha$ is regular.

Then the set $X = \mathcal{P}(\omega_\alpha) \cap L$ (i.e. the set of all constructible subsets of $\omega_\alpha$ ) possesses the property $(K, \alpha)$.

Proof. From (5), $\overline{\overline{X}} = \aleph_{\alpha+1}$ Let $y \subseteq \omega_\alpha$, $\overline{\overline{y}} < \aleph_\alpha$. Since $\omega_\alpha$ is regular, then there is a $\beta \in \omega_\alpha$ such that $y \subseteq \beta$. Using (4), we may suppose that $\beta$ is a cardinal number in the sense of the $\Delta$ -model. We have to prove that $Y = \{x \cap y : x \in X\}$ is of power less than $\aleph_\alpha$. Set $f(x \cap y) = \beta \cap x$ for $x \in X$. Thus $f$ is a one-to-one mapping of $Y$ into $\mathcal{P}(\beta) \cap L$ ( $L$ is the class of all constructible sets). Let $\gamma$ be the first cardinal number greater than $\beta$ in the sense of the $\Delta$ -model. Then there is a one-to-one mapping of $\mathcal{P}(\beta) \cap L$ onto $\gamma$. Hence there is a one-to-one mapping of $Y$ into $\gamma$. Since $\gamma \in \omega_\alpha$ (using (4)), $\overline{\overline{Y}} < \aleph_\alpha$. This completes the proof.

Conditions (4) and (5) hold in the model $\nabla$ constructed in [4] (with $\alpha = \Lambda$, see p.441). Thus, we have the following

Metatheorem. Let $\Lambda$ be a particular ordinal number (in the sense of [3]) such that the regularity of $\omega_\Lambda$ is provable in the set theory $\Sigma^*$. If the theory $\Sigma^*$ with the axiom "there is an inaccessible cardinal greater than $\omega_\Lambda$ " is consistent, then the theory $\Sigma^*$ with the axiom "there is a set with property $(K, \Lambda+1)$ " is also consistent.

Corollary. If the existence of an inaccessible cardinal greater than $\omega_\Lambda$ is consistent with $\Sigma^*$, then in $\Sigma^*$ it cannot be proved that

$$\aleph_{\Lambda+2} \rightarrow [\aleph_{\Lambda+1}]^2_{\aleph_{\Lambda+1}, \aleph_\Lambda}$$

Proof. It suffices to prove that the existence of a set $X$ with property $(K, \alpha \dot{+} 1)$ implies $\aleph_{\alpha+2} \not\rightarrow [\aleph_{\alpha+1}]^2_{\aleph_{\alpha+1}, \aleph_\alpha}$.

This is well known. I shall sketch the proof suggested to me by Mr. Hajnal.

By definition, $\aleph_{\alpha+2} \twoheadrightarrow [\aleph_{\alpha+1}]^2_{\aleph_{\alpha+1}, \aleph_\alpha}$ is equivalent to the following sentence:

There is a partition $J_\nu$, $\nu \in \omega_{\alpha+1}$ of $[X]^2$, $\bar{\bar{X}} = \aleph_{\alpha+2}$ such that for every $A \subseteq X$, $D \subseteq \omega_{\alpha+1}$, if $\bar{\bar{A}} = \aleph_{\alpha+1}$, $\bar{\bar{D}} \leq \aleph_\alpha$, then $[A]^2 \nsubseteq \bigcup_{\nu \in D} J_\nu$ (see [1], p.144).

Now, we define such a partition. Let $X$ possess the property $(K, \alpha+1)$. Set

$$\{x, y\} \in J_\nu \equiv . x, y \in X \;\&\; ((x-y) \cup (y-x)) \ni \nu \quad \text{for } \nu \in \omega_{\alpha+1}$$

Since $x \in X \rightarrow x \subseteq \omega_{\alpha+1}$, one has $\bigcup_{\nu \in \omega_{\alpha+1}} J_\nu = [X]^2$. Suppose that there are $A \subseteq X$, $D \subseteq \omega_{\alpha+1}$, $\bar{\bar{A}} = \aleph_{\alpha+1}$, $\bar{\bar{D}} \leq \aleph_\alpha$ such that $[A]^2 \subseteq \bigcup_{\nu \in D} J_\nu$. Thus, if $x, y \in A$, then $((x-y) \cup (y-x)) \cap D \neq 0$. Set $Y = \{x \cap D : x \in A\}$. If $x, y \in A$, then $x \cap D \neq y \cap D$, therefore $\bar{\bar{Y}} = \aleph_{\alpha+1}$ - a contradiction with $(3, \alpha+1)$.

Consistency of many other assertions may be proved, for example the following

<u>Metatheorem</u>. If the existence of an inaccessible cardinal is consistent with $\Sigma^*$, then $\Sigma^*$ with the axiom $\aleph_3 \twoheadrightarrow [\aleph_1]^2_{\aleph_2, \aleph_0}$ (and $2^{\aleph_0} = \aleph_2$, $2^{\aleph_1} = \aleph_3$) is consistent.

<u>Proof</u>. From [4], [6] it follow that there is a model of the theory $\Sigma^*$ in which: $2^{\aleph_0} = \aleph_2$, $2^{\aleph_1} = \aleph_3$, $\omega_1$ is an inaccessible cardinal in the sense of the $\Delta$-model, there are no cardinals in the sense of the $\Delta$-model between $\omega_1$, $\omega_2$ and between $\omega_2$, $\omega_3$, there is a perfect class $M$

(i.e. $M$ is almost universal, complete and closed with respect to the fundamental operations, see [3], p.324) such that $\overline{\overline{\mathcal{P}(\omega_1) \cap M}} = \aleph_3$, $\omega_1$ is (strongly) inaccessible in the sense of $M$.

To prove the metatheorem, it suffices to define a partition of $[\mathcal{P}(\omega_1) \cap M]^2$:

$$\mathcal{I}_x = \{\{y,z\} : y,z \in \mathcal{P}(\omega_1) \cap M \;\&\; ((y-z) \cup (z-y)) \cap x \neq 0\} \text{ for } x \subseteq \omega_1, \overline{\overline{x}} = \aleph_0 .$$

The connection between Kurepa's problem and Mycielski's axiom of determinateness (see [5]) may be interesting, because Mycielski's axiom (A) implies (4) for $\alpha = 1$.

Some generalizations of results of this paper will be published later.

I should like to express my thanks to Mr. Hajnal for valuable advice.

References

[1] P. ERDÖS, A. HAJNAL, R. RADO: Partition relations for cardinal numbers, Acta Math.Acad.Sci.Hung.,16 (1965),93-196.

[2] K. GÖDEL: The consistency of the axiom of choice and of the generalized continuum hypothesis with the axioms of set theory, Ann.of Math.Studies, Princeton 1940.

[3] A. HAJNAL: On a consistency theorem connected with the generalized continuum problem, Acta Math.Acad. Sci.Hung.,12 (1961),321-376.

[4] K. HRBÁČEK: Model $\nabla[\omega_\alpha \to \omega_\beta]$ in which $\beta$ is limit number, CMUC, 6(1965),439-442.

[5] J. MYCIELSKI: On the axiom of determinateness, Fund. Math.,53(1964),205-224.

[6] P. VOPĚNKA: $\nabla$ -models in which the generalized continuum hypothesis does not hold, Bull.Acad. Sci.Polon.,14(1966),95-99.

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