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ON LATTICE POINTS IN HIGH-DIMENSIONAL ELLIPSOIDS

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(Preliminary communication)

Let

$$Q(u) = Q(u_i) = \sum_{i,j=1}^{r} a_{ij} u_i u_j$$

be a positive definite quadratic form, whose discriminant will be denoted by D and $M_i > 0$, b_i, α_i be real numbers $(i = 1, 2, \dots, r)$. In order to simplify the formulation of our results suppose that $r > 5$. For $x > 0$, consider the function

$$A(x) = \sum e^{2\pi i \sum_{i=1}^{r} \alpha_i u_i} \tag{1}$$

where the summation runs over all systems $u = (u_1, u_2, \dots, u_r)$ of real numbers, satisfying

$$u_i \equiv b_i \ (\mathrm{mod}\ M_i) \ \text{for} \ i = 1, 2, \dots, r$$

and

$$Q(u) \leqq x . \tag{2}$$

In the particular case when

$$\alpha_i = 0,\ b_i = 0,\ M_i = 1 \quad \text{for} \quad i = 1, 2, \dots, r \tag{3}$$

(1) gives the number of the lattice points in the closed ellipsoid (2). Put

$$V(x) = \frac{\pi^{\frac{r}{2}} x^{\frac{r}{2}} e^{2\pi i \sum_{i=1}^{r} \alpha_i b_i}}{\sqrt{D} \prod_{i=1}^{r} M_i \, \Gamma(\frac{r}{2} + 1)} \, \sigma . \tag{4}$$

(where $\delta = 1$ if all numbers $\alpha_1 M_1, \alpha_2 M_2, \dots, \alpha_r M_r$ are integers and $\delta = 0$ otherwise), and

$$P(x) = A(x) - V(x);$$

then

$$P(x) = O(x^{\frac{r}{2} - \frac{r}{r+1}}) \tag{5}$$

and, if $A(x) \not\equiv 0$, also

$$P(x) = \Omega(x^{\frac{r-1}{4}}) \tag{6}$$

as shown by Landau in [1].

The function $P(x)$ (especially, under assumption (3)) has been investigated by many authors (e.g. Jarník, Landau, Müntz, Petersson, Walfisz). In what follows, consider the case when all numbers a_{ij}, M_i and b_i $(i, j = 1, 2, \dots, r)$ are integers (cf. Walfisz [4]). Expressing the function (1) by means of the corresponding theta-function (Jarník [2] and [3]) and making use of transformational relations we can prove the following theorems.

<u>Theorem 1</u>. (A generalization of so-called First Petersson Theorem.) Let $\alpha_1, \alpha_2, \dots, \alpha_r$ be rational numbers and let H denote the least common denominator of $\alpha_1 M_1, \alpha_2 M_2, \dots, \alpha_r M_r$. For a natural k and integer h such that

$$k \equiv 0 \pmod{H} \text{ and } (h, k) = 1$$

define $S_{h,k}$ by

$$S_{h,k} = \sum_{a_1, a_2, \dots, a_r = 1}^{k} e^{-2\pi i \frac{h}{k} Q(a_i M_i + b_i) + 2\pi i \sum_{i=1}^{r} \alpha_i (a_i M_i + b_i)}$$

Put

(7) $$H_j(x)=\frac{(-1)^j}{(2\pi i)^{j+1}\Gamma(\frac{r}{2}-j)}\sum_{\substack{k=1\\ k\equiv 0 (\mathrm{mod}\, H)}}^{\infty}\frac{1}{k^{r-j-1}}\lim_{T\to\infty}\sum_{\substack{(h,k)=1\\ 0<|h|<T}}\frac{S_{h,k}e^{2\pi i\frac{h}{k}x}}{h^{j+1}}$$

for any real number x and $0 \leq j < \frac{r}{4} - 1$. Then, the above series converges for all x (absolutely if $j > 0$),

$$H_j(x) = O(1)$$

and the formula

(8) $$\frac{1}{2}(P(x+0)+P(x-0))=\frac{\pi^{\frac{r}{2}}}{\sqrt{D}\prod_{i=1}^{r}M_i}\sum_{0\leq j<\frac{r}{4}-1}x^{\frac{r}{2}-j-1}H_j(x)+O(x^{\frac{r}{4}}\lg x)$$

holds.

Thus, in this case, we have

$$P(x) = O(x^{\frac{r}{2}-1})$$

and if, in addition, for some h, k $S_{h,k} \neq 0$, then

$$P(x) = \Omega(x^{\frac{r}{2}-1});$$

this has been shown first by Walfisz.

In "singular" case, i.e. if $S_{h,k} = 0$ for all h, k then we get - as a consequence of (7) and (8) -

$$P(x) = O(x^{\frac{r}{4}}\lg x).$$

The question of the exact order of the function $P(x)$ remains in the latter case open (Walfisz [4], Linnik [5]). However, for the function

$$M(x) = \int_0^x |P(y)|^2 dy$$

the matter is settled by the following

<u>Theorem 2</u>.

$$M(x)=\frac{\pi^r x^{r-1}}{4D\pi^2(r-1)\prod_{i=1}^{r}M_i^2\Gamma^2(\frac{r}{2})}\sum_{\substack{k=1\\ k\equiv(\mathrm{mod}\, H)}}^{\infty}\sum_{\substack{(h,k)=1\\ h\neq 0}}\frac{|S_{h,k}|^2}{k^{2r-2}h^2}+O(x^{r-2}).$$

In the "singular" case,

$$M(x) = O(x^{\frac{r}{2}+\frac{1}{2}})$$

and, if moreover $A(x) \not\equiv 0$, also

$$M(x) = \Omega(x^{\frac{r}{2}+\frac{1}{2}}).$$

The mean value

$$\sqrt{\frac{1}{x} M(x)}$$

of the function $P(x)$ is therefore in the second case of order $x^{\frac{r-1}{4}}$ (comp. (6)).

If at least one of the numbers $\alpha_1, \alpha_2, \dots, \alpha_r$ is irrational, we get results of a different type.

<u>Theorem 3</u>. (see [6]) a) If at least one of the numbers $\alpha_1, \alpha_2, \dots, \alpha_r$ is irrational then

$$P(x) = o(x^{\frac{r}{2}-1}). \tag{9}$$

b) For any positive decreasing function $\varphi(x)$ defined for $x > 0$ such that

$$\varphi(x) = o(1),$$

there exists a system $(\alpha_1, \alpha_2, \dots, \alpha_r)$ such that (9) holds and

$$P(x) = \Omega(x^{\frac{r}{2}-1}\varphi(x)).$$

c) There exists a set $M \subset E_r$ of zero Lebesgue measure such that for any system $(\alpha_1, \alpha_2, \dots, \alpha_r) \notin M$

$$P(x) = O(x^{\frac{r}{4}+\varepsilon})$$

holds for an arbitrary $\varepsilon > 0$.

On the other hand, the following interesting statement can be proved.

Theorem 4.
$$\int_0^1\int_0^1\cdots\int_0^1 |P(x)|^2 d\alpha_1 d\alpha_2\cdots d\alpha_r = \frac{\pi^{\frac{r}{2}} x^{\frac{r}{2}}}{\sqrt{D}\prod_{i=1} M_i \Gamma(\frac{r}{2}+1)} + O(x^{\frac{r}{2}-1}).$$

Restricting ourselves to the case

(10) $\alpha_1=\alpha_2=\dots=\alpha_r=\alpha$ (α irrational), $\ell_1=\ell_2=\dots=\ell_r=0$

we can express the relation between the arithmetic character of α and the evaluation of the upper and lower bounds of the function $P(x)$ very exactly.

Theorem 5. Let (10) hold, and let γ $(\gamma=\gamma(\alpha))$ be the supremum of those numbers β, $\beta>0$ for which the inequality
$$\min_{p\ \text{integer}} |\alpha k-p| < \frac{c}{k^\beta}$$
with a suitable c holds for infinitely many natural numbers k[1].

Put
$$f=\left(\frac{r}{4}-\frac{1}{2}\right)\frac{2\gamma+1}{\gamma+1}$$
(if $\gamma=+\infty$, put $f=\frac{r}{2}-1$). Then
$$P(x)=O(x^{f+\varepsilon})$$
and
$$P(x)=\Omega(x^{f-\varepsilon})$$
for an arbitrary positive ε, i.e.
$$\limsup_{x\to\infty}\frac{\lg|P(x)|}{\lg x}=f$$

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(1) If $q_0, q_1, q_2, \dots$ are the partial denominators of α then
$$\gamma(\alpha)=\limsup_{n\to\infty}\frac{\lg q_{n+1}}{\lg q_n}$$

The above results were read by the author at the International Congress of Mathematicians in Moscow, August 16-26,1966. The proofs and some further results on the subject will appear in Czechoslovak Mathematical Journal and Acta Arithmetica.

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