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ON CONSERVATIVE UNIFORM SPACES

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D. Bushaw in his paper [1] examined boundedness - conservative uniform spaces (see Definition 3 below). The boundedness in the sense used in [1] has some disadvantages, e.g. a finite set need not be bounded. In this paper we deal with spaces conservative with respect to other properties, namely to the boundedness in the sense of [2] which is implied by total boundedness, and to accessibility (see Definition 2) which is near to embedding into a connected set. The boundedness in the sense of [1] is our boundedness together with accessibility. Such a point of view enables another proofs and a slight generalization of some results of [1]. An attention is also given to some relations between uniformly local possessing properties and conservativity.

For uniform spaces, we use the terminology of [3]. If V is a relation on a set S, we put $V^1 = V$, $V^n = V \circ V^{n-1}$ and $V^\infty = \bigcup_{n=1}^{\infty} V^n$. Let us begin with definitions.

__Definition 1.__ [2] Let $(S, \mathcal{U})$ be a uniform space. A set $X \subset S$ is called bounded in $(S, \mathcal{U})$ (shortly "bounded") if for each U in $\mathcal{U}$ there exists a finite subset K of S and a natural number n such that $X \subset U^n[K]$.

__Definition 2.__ Let $(S, \mathcal{U})$ be a uniform space. A set

$X \subset S$ is called accessible in $(S, \mathcal{U})$ (shortly "accessible") if for each U in $\mathcal{U}$ there exists a point x in S such that $X \subset U^{\infty}[x]$.

A set X which is accessible in the space $(X, \mathcal{U}_X)$ is called chained.

Remarks. If X is a bounded resp. accessible set and $Y \subset X$, then the same holds for the sets $\overline{X}, Y$. In the above definitions we may suppose that $K \subset X$ or $x \in X$ (if $X \neq \emptyset$). If $X \subset T \subset S$ and X is bounded resp. accessible in the subspace T, then the same holds in the space S; the converse does not hold in general, but the following proposition is valid.

Theorem 1. Let T be a dense subspace of a uniform space S. If a set $X \subset T$ is bounded resp. accessible in S, it is also bounded resp. accessible in T.

Proof. For boundedness see 1.2o in [2]; the proof of accessibility is quite similar.

Theorem 2. A set X is both bounded and accessible in a uniform space $(S, \mathcal{U})$ if and only if for each U in $\mathcal{U}$ there exist x in S and a natural n such that $X \subset U^{n}[x]$.

Proof. "If" is clear. If $X \subset U^{n}[K]$ with a finite $K \subset X$ and $X \subset U^{\infty}[x]$ then clearly $K \subset U^{m}[x]$ for some natural m and hence $X \subset U^{n+m}[x]$.

Thus we have shown that the sets bounded in the sense of [1] are those which are both bounded and accessible.

Now we proceed to conservativity.

Definition 3. [1] Let P be a property of subsets of a uniform space $(S, \mathcal{U})$. We say that an entourage $U \in \mathcal{U}$ is P-conserving if for each subset X of S having the property P the set $U[X]$ has also the property P. If there exists a P-conserving entourage, we say that the uniform space $(S, \mathcal{U})$ is P-conservative.

Recall that a uniform space $(S, \mathcal{U})$ is said to possess a property P uniformly locally if there exists U in $\mathcal{U}$ such that $U[x]$ has the property P for each x in S.

Remark. Let P be any property possessed by all one-point subsets of a uniform space $(S, \mathcal{U})$. If $U \in \mathcal{U}$ is P-conserving then clearly $U[x]$ has the property P for each x in S; hence a P-conservative uniform space has the property P uniformly locally. The converse does not hold in general. If $U[x]$ is accessible for each x in S then U is also accessibility-conserving. Now we shall show what is the situation with total boundedness and boundedness.

Theorem 3. Let $(S, \mathcal{U})$ be a uniform space, $U \in \mathcal{U}$ and let $U^2[x]$ be totally bounded for each point x of S. Then U is total boundedness-conserving and each bounded set is totally bounded. If a space is uniformly locally totally bounded then it is total boundedness-conserving and boundedness-conserving.

Proof. The first assertion - see 1.17 and 1.18 in [2], the second one is clear.

Example. Let S be the set of all pairs (m, t) with m positive integer and $-1 \leq t \leq 1$. Put

$$\rho((n,t),(n,t')) = |t-t'| \text{ for } t \leq 0, t' \leq 0,$$
$$= -t + \sqrt[n]{t'} \text{ for } t < 0 < t' \quad (t' < 0 < t \text{ similarly}),$$
$$= \sqrt[n]{|t-t'|} \text{ for } t \geq 0, t' \geq 0 .$$

Now we identify all the pairs $(n, -1)$ and denote so obtained element by a. If $n \neq n'$ we put

$$\rho((n,t),(n',t')) = \rho(a,(n,t)) + \rho(a,(n',t')) .$$

Clearly (S, ρ) is a metric space, the collection of all sets $V_\varepsilon = \mathcal{E}\{(x,y) \mid \rho(x,y) < \varepsilon\}$ with positive ε is a base of the uniformity induced by ρ. Put $A = \mathcal{E}\{(n,0) \mid n = 1, 2, \dots\}$. It is easy to prove that S is chained, A is bounded (and hence also bounded in the sense of [1]) and for each x in S the set $V_1[x]$ is bounded; moreover for each x in A this set is totally bounded. We shall show that no $V_\varepsilon[A]$ is bounded. Suppose the contrary, let $0 < \sigma < \varepsilon$. As $V_\varepsilon[A]$ is bounded and the space S is chained, there exists a point x and a natural k such that $V_\varepsilon[A] \subset V_\sigma^k[x]$. But this implies that $V_\varepsilon[A] \subset V_\sigma^{2k}[y]$ for any point y of $V_\varepsilon[A]$. Choose a natural m and a positive s so that $\sqrt[m]{2k}\,\sigma < \sqrt[m]{s} < \varepsilon$. Evidently $(m, s) \in V_\varepsilon[A]$. The inequality $2k\sigma^m < s$ implies $V_\sigma^{2k}[(m,s)] \subset \mathcal{E}\{(m,t) \mid t > 0\}$ which is a contradiction.

<u>Convention</u>. In the following text, we shall use some abbreviations. A denotes accessibility, B denotes boundedness, T denotes total boundedness. If P, R are two properties, we denote by PR the property meaning that both these properties are possessed.

Recall that a family $\{X_\alpha\}$ of subsets of a uniform space $(S, \mathcal{U})$ is called U-discrete (where $U \in \mathcal{U}$) if $U[X_\alpha] \cap X_\beta = \emptyset$ for any $\alpha \neq \beta$; it is called uniformly discrete if it is U-discrete for some U in $\mathcal{U}$.

Theorem 4. Let P be a property possessed by all one-point subsets of a uniform space $(S, \mathcal{U})$. Let a symmetric $V \in \mathcal{U}$ be AP-conserving. Then there exist S_α such that

(*) $S = \bigcup\{S_\alpha\}$, S_α are chained, $\{S_\alpha\}$ is V-discrete.

If (*) is fulfilled for some entourage $V \in \mathcal{U}$ then V is A-conserving.

Proof. If $x \in S$ then the sets $V^n[x]$ for all natural n possess the property AP ; they have a common point x and therefore $V^\infty[x]$ is accessible. If $x \sim y$ denotes $x \in V^\infty[y]$ then $\sim$ is an equivalence on S which defines a decomposition $S = \bigcup\{S_\alpha\}$. Evidently this family is V-discrete and therefore each S_α is chained. Let (*) be fulfilled. If $X \subset S$ is accessible then $X \subset S_\alpha$ for one α only; hence $V[X] \subset V[S_\alpha] = S_\alpha$ and $V[X]$ is also accessible.

Corollary. A uniform space is A-conservative if and only if it is the union of a uniformly discrete family of chained subsets.

Theorem 5. Let $(S, \mathcal{U})$ be a uniform space, $U \in \mathcal{U}$. If U is AB (resp. AT)-conserving it is also B (resp. T)-conserving.

Proof. The entourage $V = U \cap U^{-1}$ is symmetric and AB (resp. AT)-conserving. Take the decomposition (*) from Theorem 4. If $X \subset S$ is bounded (resp. totally bounded), then the sets $X_\alpha = X \cap S_\alpha$ are non-void for a finite number of α's , denote them by $\alpha_1, \ldots, \alpha_m$. Therefore $X = X_{\alpha_1} \cup \ldots \cup X_{\alpha_m}$, each set X_{α_i} is both bounded (resp. totally bounded) and accessible. The same holds for $\mathcal{U}[X_{\alpha_i}]$ and hence $\mathcal{U}[X] = \mathcal{U}[X_{\alpha_1}] \cup \ldots \cup \mathcal{U}[X_{\alpha_m}]$ is bounded (resp. totally bounded).

<u>Remark</u>. We have proved (Theorems 4,5) that AB -conserving symmetric entourages are exactly those which are both A -conserving and B -conserving. Hence a space is AB -conservative if and only if it is A -conservative and B -conservative. For example, a bounded space is AB-conservative if and only if it is A -conservative.

Using the corollary of Theorem 4, we obtain

<u>Theorem 6</u>. A bounded uniform space is A -conservative (= AB -conservative) if and only if it is the union of a uniformly discrete finite family of chained subsets.

Now we obtain the result of [1], Theorem 2:

<u>Corollary</u>. A totally bounded separated uniform space is A -conservative if and only if its completion has a finite number of components.

<u>Proof</u> follows from these facts: (1) A compact set is chained if and only if it is connected. (2) A family $\{X_\alpha\}$ is uniformly discrete if and only if $\{\overline{X}_\alpha\}$ is uniformly discrete.

Recall that a uniform space $(S, \mathcal{U})$ is called nonarchimedean if $\mathcal{U}$ has a base each element of which is an equivalence. Nonarchimedean AB -conservative spaces were characterized in [1].

Theorem 7. Let $(S, \mathcal{U})$ be a nonarchimedean uniform space. Then (1) Each bounded subset of S is totally bounded. (2) If $V \in \mathcal{U}$ is A-conserving then it is the smallest element of $\mathcal{U}$. (3) If V is the smallest element of $\mathcal{U}$ then it is A-conserving and B-conserving.

Proof. If $U \in \mathcal{U}$ is an equivalence then $U^n = U$ for each natural n which proves (1). Let $V \in \mathcal{U}$ be A-conserving. For each x in S the set $V[x]$ is accessible, therefore for any equivalence U we have $V[x] \subset U^{\infty}[x] = U[x]$ which proves (2). Let V be the smallest element of $\mathcal{U}$. Then for each x in S the $V[x]$ is accessible and therefore V is A-conserving. Moreover $V^2[x]$ is clearly bounded, hence totally bounded, therefore V is T-conserving and also B-conserving.

Corollary. A separated nonarchimedean uniform space $(S, \mathcal{U})$ is A-conservative if and only if $\mathcal{U}$ is discrete.

References

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