Václav Havel

Join systems and closure spaces

Commentationes Mathematicae Universitatis Carolinae, Vol. 7 (1966), No. 3, 335--341

Commentationes Mathematicae Universitatis Carolinae
7,3 (1966)

JOIN SYSTEMS AND CLOSURE SPACES

Václav HAVEL, Brno

The purpose of the present paper is to prove that some join systems can be interpreted as closure spaces satisfying the axioms (1)-(5) (cf. Definition 1), and conversely.

In [2], K. Čulík showed that closure spaces satisfying (1)-(6) (cf. Definition 1) are models of Hilbert incidence spaces, and conversely.

Definition 1. (1) A (general) closure space is a couple $\langle S, cl \rangle$ where S is a set and cl is a map of $\mathcal{P}(S)$ (set of all subsets in S) into $\mathcal{P}(S)$. If $\langle S, cl \rangle$ is a closure space, then a) the sets $\mathcal{A} \in \mathcal{P}(S)$, $card\,\mathcal{A} = 1$ will be called points, b) the sets $cl\,(\mathcal{A} \cup \mathcal{B})$, where $\mathcal{A}, \mathcal{B}$ are distinct points, will be called lines and c) the sets $cl(\mathcal{A} \cup \mathcal{B} \cup \mathcal{C})$, where $\mathcal{A}, \mathcal{B}, \mathcal{C}$ are distinct points with $\mathcal{C} \notin cl(\mathcal{A} \cup \mathcal{B})$ will be called planes. If $\langle S, cl \rangle$ is a closure space, then one may formulate the following conditions:

(1) $\mathcal{A} \subseteq cl\,\mathcal{A}$ for $\mathcal{A} \in \mathcal{P}(S)$,

(2) $\mathcal{A} \subseteq \mathcal{B} \Rightarrow cl\,\mathcal{A} \subseteq cl\,\mathcal{B}$ for $\mathcal{A}, \mathcal{B} \in \mathcal{P}(S)$,

---

(1) Cf. [2], p. 83 and p. 85, respectively.

(3) $cl(cl\,\mathcal{A}) = cl\,\mathcal{A}$ for $\mathcal{A} \in \mathcal{P}(S)$,

(4) $\mathcal{A} = cl\,\mathcal{A}$ for $\mathcal{A} \in \mathcal{P}(S)$ with $card\,\mathcal{A} = 0$, $card\,\mathcal{A} = 1$, respectively,

(5) $\mathcal{A} \subseteq \mathcal{B} \Rightarrow \mathcal{A} = \mathcal{B}$ if $\mathcal{A}, \mathcal{B}$ are both points, lines and planes, respectively,

(6) if a point is contained in two planes then a line is contained in these planes; there is a point and a plane which are disjoint.

Definition 2. A join-system is a couple $\langle S, + \rangle$ where S is a set, all subsets of which consisting of exactly one element are called the points, certain subsets of S consisting of at least two elements are called lines, and the validity of two following axioms is supposed:

(7) if A, B are distinct points, then there is precisely one line (denoted by $A + B$) containing both A, B,

(8) if a, b are distinct lines then $card\,(a \cap b)$ is 0 or 1.

If $\langle S, + \rangle$ is a join system then we shall denote subsets in S by letters, points by upper-case and by lower-case letters.

Proposition 1. Any join system $\langle S, + \rangle$ satisfies the following conditions:

(9) $A + B = B + A$ for $A \neq B$,

(10) $A \neq B \neq C \neq D$, $A + B = C + D \Rightarrow A + B = B + C$,

(11) $A \neq B$, $A \neq C$, $A + B \neq A + C \Rightarrow (A + B) \cap (A + C) = A$.

In the definition of a join system one may replace equivalently (8) by (10).

Proof. Obviously (7) $\Longrightarrow$ (9). - Next, (7) and (8) imply (10) since for $A \neq B \neq C \neq D$, $A+B = C+D \neq B+C$, it follows $(C+D) \cap (C+B) = C$ and $(A+B) \cap (C+B) = C$. Thus $B \neq C$ lie simultaneously on $A+B$ and $C+B \neq A+B$, contrary to (8). - Let (7) and (10) be fulfilled, and suppose that points $P \neq Q$ are contained simultaneously in a line a and in a line $b \neq a$. By (4), $a = P+Q$ and $b = P+Q$ contrary to the hypothesis $a \neq b$. Thus (8) holds. - Finally, (7) and (8) $\Longrightarrow$ (11).

Proposition 2. Let $\langle S, + \rangle$ be a couple such that S is a set and $+$ a commutative composition on $\mathcal{P}(S)$ satisfying the following conditions

(12) $\mathcal{A} + \emptyset = \mathcal{A}$ for $\mathcal{A} \in \mathcal{P}(S)$,

(13) $A + A = A$ for $A \in \mathcal{P}(S)$, $\operatorname{card} A = 1$,

(14) $\mathcal{A} + \mathcal{B} = \bigcup\limits_{\substack{A \subseteq \mathcal{A} \\ B \subseteq \mathcal{B} \\ \operatorname{card} A = \operatorname{card} B = 1}} (A+B)$ for $\mathcal{A}, \mathcal{B} \in \mathcal{P}(S) \setminus \{\emptyset\}$,

(15) $\mathcal{A} \subseteq \mathcal{A} + \mathcal{B}$ for $\mathcal{A}, \mathcal{B} \in \mathcal{P}(S)$,

(16) if $A \neq C \subseteq A + B$ for $A, B, C \in \mathcal{P}(S)$ with $\operatorname{card} A = \operatorname{card} B = \operatorname{card} C = 1$, then $A + B = A + C$. (2)

---

(2) The multigroups defined in [1] are a special case of systems $\langle S, + \rangle$ satisfyijng (12) to (16)

Let $\dotplus$ be the restriction of $+$ to all pairs (A, B) with $A, B \in \mathcal{P}(S)$, $\operatorname{card} A = \operatorname{card} B = 1$, $A \neq B$. Then $\langle S, \dotplus \rangle$ is a join system. Conversely, if $\langle S, + \rangle$ is a join system, then there is a unique extension $\hat{+}$ of $+$ to all pairs $(\mathcal{A}, \mathcal{B})$ with $\mathcal{A}, \mathcal{B} \in \mathcal{P}(S)$, such that $\langle S, \hat{+} \rangle$ satisfies (12) to (16).

__Proof__. Let there be given a system $\langle S, + \rangle$ satisfying (12) to (16). Define the lines as all subsets $A + B \subseteq S$ with $A, B \in \mathcal{P}(S)$, $\operatorname{card} A = \operatorname{card} B = 1$, $A \neq B$. Then (15) $\Rightarrow$ (7) and (16) $\Longrightarrow$ (8), so that $\langle S, \dotplus \rangle$ is a join system. - Conversely, let $\langle S, + \rangle$ be a join system. Here, $+$ is defined only for pairs (A, B), $A \neq B$. Define $A + A = A$ for all A, $\mathcal{A} + \emptyset = \mathcal{A}$ for all $\mathcal{A} \in \mathcal{P}(S)$ and $\mathcal{A} + \mathcal{B} = \bigcup\limits_{\substack{A \subseteq \mathcal{A} \\ B \subseteq \mathcal{B}}} (A + B)$ for $\mathcal{A}, \mathcal{B} \in \mathcal{P}(S) \setminus \{\emptyset\}$. Then (1) $\Longrightarrow$ $A \subseteq A + B$, and by (14) it follows that (15) to (11) imply (16).

__Remark__. Given any join system $\langle S, + \rangle$, we shall denote by $+$ also the extended composition, in the sense of Proposition 3.

__Proposition 3__. In any join system $\langle S, + \rangle$ the following conditions are fulfilled:

(17) $\mathcal{A} \subseteq \mathcal{B},\ \mathcal{C} \subseteq \mathcal{D} \Longrightarrow \mathcal{A} + \mathcal{C} \subseteq \mathcal{B} + \mathcal{D}$,

(18) $\mathcal{A} + \mathcal{A} = \mathcal{A},\ \mathcal{B} \subseteq \mathcal{A} \Longrightarrow \mathcal{A} + \mathcal{B} = \mathcal{A}$,

(19) $\mathcal{A} + \mathcal{A} = \mathcal{A},\ \mathcal{B} \subseteq \mathcal{A},\ \mathcal{C} \subseteq \mathcal{A} \Longrightarrow \mathcal{B} + \mathcal{C} \subseteq \mathcal{A}$,

(20) $(A+B)+(A+B)=A+B$,

(21) $A+B \subseteq C+D,\ A \neq B \Rightarrow A+B=C+D$,

(22) if $(\mathcal{A}_\gamma)_{\gamma\in\Gamma}$ is a family of subsets in S, then $\mathcal{A}_\gamma + \mathcal{A}_\gamma = \mathcal{A}_\gamma$ for all $\gamma \in \Gamma$ implies $\bigcap_{\gamma\in\Gamma} A_\gamma + \bigcap_{\gamma\in\Gamma} A_\gamma = \bigcap_{\gamma\in\Gamma} A_\gamma$,

(23) $A \subseteq B \Rightarrow A = B$.

Proof. (17) follows from the definition of the sum of two subsets in S. - (18): The assumptions imply $\mathcal{A}+\mathcal{B} \subseteq \mathcal{A}$ (by(17)), so that $\mathcal{A}+\mathcal{B}=\mathcal{A}$ by (9). - (19): (17) $\Rightarrow$ (19). - (20): By (15), we have $A+B \subseteq (A+B)+(A+B)$. Each element of $(A+B)+(A+B)$ belongs to some $C+D$ with $C\in A+B$, $D\in A+B$, so that by (17), $C+D \subseteq A+B$. - (21): The assumptions imply $C \neq D$, and from (16) there follows $C+D=C+A=A+B$. - (22): By (15), $\bigcap_{\gamma\in\Gamma} A_\gamma \subseteq \bigcap_{\gamma\in\Gamma} A_\gamma + \bigcap_{\gamma\in\Gamma} A_\gamma$, and the assumptions also yield $\bigcap_{\gamma\in\Gamma} A_\gamma + \bigcap_{\gamma\in\Gamma} A_\gamma \subseteq \bigcap_{\gamma\in\Gamma} A_\gamma$. - (23) is trivial.

Definition 3. Let $\langle S,+\rangle$ be a join system. Then $\mathcal{A} \in \mathcal{P}(S)$ is said to be closed if $\mathcal{A}+\mathcal{A}=\mathcal{A}$. Define a closure map $cl : \mathcal{P}(S) \to \mathcal{P}(S)$ in such a manner that, for any $\mathcal{A} \in \mathcal{P}(S)$, $cl\ \mathcal{A}$ is the intersection of all closed subsets in S which contain $\mathcal{A}$. The planes of $\langle S, cl\rangle$ (Definition 1) will also be called the planes of $\langle S,+\rangle$.

Proposition 4. Let $\langle S,+\rangle$ be a join system and $\langle S, \mathcal{cl}\rangle$ the closure space constructed in Definition 3. Then $\mathcal{cl}\,\mathcal{A}$ is closed for all $\mathcal{A} \in \mathcal{P}(S)$. Furthermore, conditions (1) to (4) are satisfied, and

(24) $A + B \subseteq \mathcal{C}$ for any A, B contained in a plane $\mathcal{C}$.

Proof. From $\mathcal{cl}\,\mathcal{A} = \bigcap_{\substack{\mathcal{B} \supseteq \mathcal{A} \\ \mathcal{B}+\mathcal{B}=\mathcal{B}}} \mathcal{B}$, it follows by (22) that $\mathcal{cl}\,\mathcal{A} \bigcap_{\substack{\mathcal{B} \supseteq \mathcal{A} \\ \mathcal{B}+\mathcal{B}=\mathcal{B}}} \mathcal{B} + \bigcap_{\substack{\mathcal{B} \supseteq \mathcal{A} \\ \mathcal{B}+\mathcal{B}=\mathcal{B}}} \mathcal{B}$. - (1) follows from the definition of closure maps (Definition 3). - (2): If $\mathcal{A} \subseteq \mathcal{B}$, then $\mathcal{A} \subseteq \mathcal{cl}\,\mathcal{B}$ and $\mathcal{cl}\,\mathcal{B}$ is closed, so that, by the definition of $\mathcal{cl}\,\mathcal{A}$, $\mathcal{cl}\,\mathcal{A} = \mathcal{cl}\,\mathcal{B}$. - (22) $\Longrightarrow$ (3). - (12) and (13) $\Longrightarrow$ (4). - (24) is evident.

Proposition 5. Let $\langle S,+\rangle$ be a join system satisfying

(25) $\mathcal{A} \subseteq \mathcal{B} \Longrightarrow \mathcal{A} = \mathcal{B}$ if $\mathcal{A}, \mathcal{B}$ are planes.

Then the corresponding closure space $\langle S, \mathcal{cl}\rangle$ (constructed in Definition 3) satisfies (1) to (5). - Conversely, if a given closure space $\langle S, \mathcal{cl}\rangle$ satisfies (1) to (5), then there is a join structure $\langle S,+\rangle$, the lines of which are precisely the lines of $\langle S, \mathcal{cl}\rangle$ determined according to Definitionl. This join structure $\langle S,+\rangle$ satisfies (25). - Let $\langle S, \mathcal{cl}\rangle$ be a closure space satisfying (1) to (5), $\langle S,+\rangle$ a join system constructed to $\langle S, \mathcal{cl}\rangle$ as above and $\langle S, \mathcal{cl}^*\rangle$ the closure space corresponding to $\langle S,+\rangle$. Then $\mathcal{cl} = \mathcal{cl}^*$.

Proof. The first part only repeats the matter of Proposition 4 (note that (5) consists of (21),(23) and (25)). In the second part, (7) and (8) follow easily if the composition $+$ is defined by $A + B = cl(A \cup B)$ for $A, B \in \mathcal{P}(S)$, $card\ A = card\ B = 1$, $A \neq B$ in a given closure system $\langle S, cl \rangle$ satisfying (1) to (5). - The rest of Proposition 5 is verified easily, using the definitions of the systems $\langle S, + \rangle$ and $\langle S, cl^* \rangle$.

R e f e r e n c e s

[1] W. PRENOWITZ: Projective geometries as multigroups, Amer.Journ.of Math., 65(1943),235-256.

[2] K. ČULÍK: Topologische Auffassung der Axiome der Verknüpfung von D.Hilbert, Zeitschr.f. math.Logik und Grundlagen d.Math., 11 (1965),81-88.

(Received March 21,1966)