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NOTES ON QUOTIENT MAPS

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Summary: The relations between several properties of quotient maps are studied; in particular, an internal characterization of commutativity with formation of products is exhibited (Proposition 2).

Let P, Q be topological spaces, and $e : Q \rightarrow P$ a continuous map onto. (These assumptions will be preserved throughout this paper; the terminology and notation is usually that of [2].) The following properties and appellations are quite current:

(closed) e is a closed map, i.e. $e[Y]$ is closed in P whenever Y is closed in Q;

(open) e is an open map, i.e. $e[Y]$ is open in P whenever Y is open in Q;

(sectionable) there exists a section to e, i.e. a continuous map $s : P \rightarrow Q$ with $e \circ s = 1_P$ (the identity map of P);

(quotient) e is a quotient map, i.e. X is closed in P if $e^{-1}[X]$ is closed in Q.

Also consider the following properties:

(cl) $\bar{X} = e[\overline{e^{-1}[X]}]$ for all $X \subset P$;

(int) $\operatorname{Int} X = e[\operatorname{Int} e^{-1}[X]]$ for all $X \subset P$;

(limit lifting) Whenever $x_i \to x$ in P, there exists a subnet $\{x_j\}$ and also $y_j \to y$ in Q with $e y_j = x_j$, $e y = x$.

It is a simple exercise to verify the implications in the following diagram:

$$\begin{tikzcd}
\text{(closed)} \arrow[d, Rightarrow] & \text{(sectionable)} \arrow[d, Rightarrow] & \text{(open)} \arrow[dl, Rightarrow] \arrow[d, Rightarrow] \\
(cl) \arrow[dr, Rightarrow] & \text{(limit lifting)} \arrow[l, Rightarrow] & (int) \arrow[dl, Rightarrow] \\
 & \text{(quotient)} &
\end{tikzcd}$$

We proceed to present several slightly less elementary interrelations (it is still assumed that $e : Q \to P$ is continuous onto). One of these, namely (cl) et $(int) \Rightarrow$ (open) is contained in Proposition 1 below; this yields, inter alia, that in general $(cl) \not\Rightarrow (int)$, since a closed map need not be open, etc. The example in [1,I,§ 9,11)] shows that e can be closed and not limit lifting.

<u>Proposition 1</u>. Each of the following properties is equivalent with (open):

1° (cl) et (int);

2° $e^{-1}[\overline{X}] = \overline{e^{-1}[X]}$ for all $X \subset P$;

3° $e^{-1}[\operatorname{Int} X] = \operatorname{Int} e^{-1}[X]$ for all $X \subset P$;

4° (limit covering) Whenever $x_i \to x$ in P and $x = ey$, there exists a subnet $\{x_j\}$ and y_j in Q with $y_j \to y$ and $e y_j = x_j$;

5^o (bi-open) For every topological space R, the map $e \times 1_R : Q \times R \to P \times R$ is open.

Proof. Obviously (bi-open) $\Longrightarrow$ (open); the opposite implication is well-known [1,I,§ 9, prop.9]. Next, (open) $\Longrightarrow$ (limit covering) may be established similarly as (open) $\Longrightarrow$ (limit lifting); and the opposite implication is easily obtained e.g. by contradiction. Obviously $2^o \Longleftrightarrow 3^o$; and (open) $\Longrightarrow 1^o$ is in the diagram above. Thus it only remains to prove that

$$1^o \Longrightarrow 2^o, \quad 3^o \Longrightarrow \text{(open)}.$$

Assume 1^o, and take any $X \subset P$; then

$$Int\,(P-X) = P - \overline{X} = P - e[\overline{e^{-1}[X]}]$$

$$= e[Int\,e^{-1}[P-X]] = e[Q - \overline{Q - e^{-1}[P-X]}] = e[Q - \overline{e^{-1}[X]}],$$

so that the set $Y = \overline{e^{-1}[X]}$ has

$$P - e[Y] = e[Q - Y].$$

Now take complements and inverse images:

$$Y \subset e^{-1}[e[Y]] = Q - e^{-1}[Q-Y] \subset Q - (Q-Y) = Y;$$

thus $Y = e^{-1}[e[Y]]$ and returning to X,

$$\overline{e^{-1}[X]} = e^{-1}[e[\overline{e^{-1}[X]}]] = e^{-1}[\overline{X}]$$

having applied ($c\ell$) again. This establishes 2^o as required.

To prove that $3^o \Longrightarrow$ (open), first note that for any G open in Q there is

$$e[G] = e[Int\,e^{-1}[e[G]]] \tag{1}$$

since $G \subset Int\,e^{-1}[e[G]]$ from openness of G, and

$$e[Int\,e^{-1}[e[G]]] \subset e[e^{-1}[e[G]]] = e[G].$$

Now, using 3^o, the set

$$e[\mathrm{Int}\, e^{-1}[e[G]]] = e[e^{-1}[\mathrm{Int}\, e[G]]] = \mathrm{Int}\, e[G]$$

is open in P; with (i) this yields that indeed e is an open map, and this completes the proof.

Proposition 2. Assume that P is a Hausdorff space; then each of the following properties is equivalent to (limit lifting):

1^o For every topological space R, the map

$$e \times 1_R : Q \times R \to P \times R$$

is limit lifting;

2^o (bi-quotient) $e \times 1_R$ is a quotient map for every topological space R;

3^o $e \times 1_R$ is a quotient map for all compact Hausdorff spaces R with a unique non-isolated point.

Proof. Easily or obviously, (limit lifting) $\Longrightarrow 1^o \Longrightarrow 2^o \Longrightarrow 3^o$; thus it remains to prove that, e.g., non (limit lifting) $\Longrightarrow$ non 3^o. Assume the premiss; thus there is a convergent net $\{x_i \mid i \in I\}$ in P, say $x_i \to x$, such that for every subnet $\{x_j\}$ one has that $e^{-1}[x_j] \ni y_j \to y$ implies $ey \neq x$. Moreover, since P is Hausdorff, it even follows that no net $y_j \in e^{-1}[x_j]$ converges.

Now take for R the one-point compactification $R = I \cup (\infty)$ of the directed set I, topologized in the obvious manner: all $i \in I$ are isolated, each neighbourhood of ∞ intersects I in a residual subset. For X take the set $\{(x_i, i) : i \in I\} \subset P \times R$; obviously $(x, \infty) \in \overline{X} - X$.

To prove non 3^{o} it is only needed to verify that $(e\times 1_R)^{-1}[X]$ is closed.

Thus, let

$$(e\times 1_R)^{-1}[X]\ni (y_j, i_j)\to (y, i).$$

Due to the special topology in R, one has the following alternative. Either eventually $i_j = i \neq \infty$, so that $y_j \in e^{-1}[x_i]$ and hence also $y\in e^{-1}[x_i]$, $(y,i)\in(e\times 1_R)^{-1}[X]$. Or $i=\infty$, but then $\{x_{i_j}\}$ is a subnet of $\{x_i\}$, and by assumption the $y_j\in e^{-1}[x_{i_j}]$ cannot converge. Thus only the first case obtains, and hence $(e\times 1_R)^{-1}[X]$ is closed but X is not. This concludes the proof.

Corollary 3. If P is a Hausdorff space with countable character, then (quotient) $\Longleftrightarrow$ ($c\ell$) $\Longleftrightarrow$ (limit lifting) $\Longleftrightarrow$ (bi-quotient).

Proof. On using the diagram and the preceding proposition, it suffices to prove that (quotient) $\Longrightarrow$ (limit lifting). Take any $x_i \to x$ in P_ι and then a (countable) sequence $\{x_n\}$ which is a subnet of $\{x_i\}$. If no subsequence of any $y_n\in e^{-1}[x_n]$ converges, then the set X of terms of $\{x_n\}$ would be non-closed with closed $e^{-1}[X]$, and hence e could not be a quotient map.

R e f e r e n c e s

[1] N. BOURBAKI: Topologie générale (2nd ed.), Paris, 1951.

[2] E. ČECH: Topological Spaces (rev.ed.), Academia, Prague, 1966.

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