Commentationes Mathematicae Universitatis Carolinae

Miroslav Hušek

Remarks on reflections

Commentationes Mathematicae Universitatis Carolinae, Vol. 7 (1966), No. 2, 249--259

Persistent URL: http://dml.cz/dmlcz/105057

Terms of use:

© Charles University in Prague, Faculty of Mathematics and Physics, 1966

Institute of Mathematics of the Academy of Sciences of the Czech Republic provides access to digitized documents strictly for personal use. Each copy of any part of this document must contain these *Terms of use*.

This paper has been digitized, optimized for electronic delivery and stamped with digital signature within the project *DML-CZ: The Czech Digital Mathematics Library* http://project.dml.cz

Commentationes Mathematicae Universitatis Carolinae

7, 2 (1966)

REMARKS ON REFLECTIONS

Miroslav HUŠEK, Praha

Recently there have appeared several papers dealing with modifications (reflections or coreflections) of objects of a category in its subcategory. Almost simultaneously there were obtained several theorems concerning the existence of special modifications $\langle Y, f \rangle$ (f is mapped on an identity by a given functor). In special categories (of closure, uniform and proximity spaces), this problem was treated by Frolík in [1] by means of projective and inductive generation. This method was carried over to general categories in the Notes of [1] and in the author's Thesis. Kennison used a slightly different method in [2]. Unfortunately his proof of the main general theorem 2.8 is based on lemma 2.6 which does not hold under the given conditions, but theorem 2.8 is true in a more general form. The aim of this paper is state this generalization.

First we shall state theorem 1 on the existence of special modifications (see above). By a method similar to that used in [2], this result is then extended (by means of theorem 2) to theorem 3, dealing with the existence of general modifications in categories. Because it is sometimes necessary to investigate modifications of some object only, theorems 2 and 3 treat the case of modifications of precisely one object. The last part of this paper is devoted to examples in categories of closure spaces (e.g. in compact spaces). Some results differ considerably from the known for topological spaces.

The assertions are stated for reflections (upper modifications) only; the corresponding assertions for coreflections (lower modifications) are dualizations of those stated.

The notation and terminology from [1] will be used. Let us mention some which will be used more frequently.

Let $\mathcal{F}$ be a functor (we omit the term "covariant") from a category $\mathcal{K}$ into a category $\mathcal{M}$. The functor $\mathcal{F}$ is said to be product-stable if a product of a family $\{X_i\}$ exists in $\mathcal{K}$ whenever a product of $\{\mathcal{F}X_i\}$ exists in $\mathcal{M}$ and if $\mathcal{F}$ preserves these products (we shall work only with products of non-void families). There is given the following quasi-order $\leq_{\mathcal{F}}$ on the class $obj\ \mathcal{K}$:

$$X \leq_{\mathcal{F}} Y \quad \text{if } \mathcal{F}f = 1_{\mathcal{F}X} \quad \text{for some } f \in Hom_{\mathcal{K}} \langle X, Y \rangle$$

(the inverse images of identities will be called $\mathcal{F}$-identities).

In the sequel, the classes $\mathcal{F}^{-1}[A]$ are always taken with this quasi-order.

Now, we can define an upper (lower) $\mathcal{F}$-modification of an object X from $\mathcal{K}$ in a subcategory $\mathcal{K}'$ as the least (greatest) object from $\mathcal{K}'$ greater (smaller) than X.

<u>Theorem 1</u>. Let $\mathcal{F}$ be a faithful functor of a category $\mathcal{K}$ into $\mathcal{M}$, let $\mathcal{K}'$ be a full subcategory of $\mathcal{K}$. Assume that, in $\mathcal{K}$, products of objects from $\mathcal{K}'$ exist. Then each object X of $\mathcal{K}$ has an upper $\mathcal{F}$-modification in $\mathcal{K}'$, which is (with the corresponding $\mathcal{F}$-identity) a reflection of X in $\mathcal{K}'$, if and only if

(a) the embedding of $\mathcal{K}'$ into $\mathcal{K}$ is product-stable;

(b) for each $X \in obj\ \mathcal{K}$ there is a monomorphism from X into an object of $\mathcal{K}'$;

(c) for each $X \in obj\ \mathcal{K}$ there is a down-cofinal set in the class $\mathcal{E}\{Y \mid Y \in obj\ \mathcal{K}', X \leqq_{\mathcal{F}} Y\}$;

(d) each monomorphism f with $\mathcal{E} f \in obj\ \mathcal{K}'$ can be factorized as $f_1 \circ f_2$ where f_2 is an $\mathcal{F}$-identity and f_1 is a morphism of $\mathcal{K}'$.

<u>Proof</u>- The necessity of (a),(b),(c) and (d) is obvious even without any assumption on the existence of products. Now, let $X \in obj\ \mathcal{K} - obj\ \mathcal{K}'$, let M_X be a down-cofinal set in the class $\mathcal{E}\{Y \mid Y \in obj\ \mathcal{K}', X \leqq_{\mathcal{F}} Y\}$ ($M_X \neq \emptyset$ by (b),(d)), Y' a product of M_X, $Y' \in obj\ \mathcal{K}'$ (this is possible by (a)), and $f: X \to Y'$ the reduced product of the $\mathcal{F}$-identities $X \to Y$. By (d) there exists an object $Y'' \in obj\ \mathcal{K}'$, $X \leqq_{\mathcal{F}} Y''$ such that f can be factorized over the $\mathcal{F}$-identity $X \to Y''$.

Evidently Y'' is an upper $\mathcal{F}$-modification of X in $\mathcal{K}'$. Hence it is sufficient to prove that each morphism $g: X \to Z$, $Z \in obj\ \mathcal{K}'$, can be factorized over some $\mathcal{F}$-identity $X \to Y$, $Y \in obj\ \mathcal{K}'$. If g is a monomorphism, one has case (d). In the opposite case apply (d) to the monomorphism $h : X \to Z \times Y''$, being the reduced product of g and an $\mathcal{F}$-identity $X \to Y''$.

<u>Remark</u>. (1) If it is not supposed that $\mathcal{K}'$ is full in $\mathcal{K}$, one must add, as a further assumption, that every invertible $\mathcal{F}$-identity $Y_1 \to Y_2$, $Y_i \in obj\ \mathcal{K}'$, is a morphism of $\mathcal{K}'$ (this is fulfilled e.g. if each invertible $\mathcal{F}$-identity is an identity, i.e. $\leq_{\mathcal{F}}$ is an order).

(2) If the term "monomorphism" is omitted in (d), then it need only be supposed that there exist products of families $\{X_i\}$ from $obj\ \mathcal{K}'$ such that $card\ \mathcal{E}\{\mathcal{F}X_i\} = 1$.

(3) Assume that $\mathcal{F}$ preserves monomorphisms and that projective generation by monomorphism obtains in $\mathcal{K}$. Then condition (d) means that $\mathcal{K}'$ is $\mathcal{F}$-hereditary in the following sense: if f is a monomorphism with $\mathcal{E} f \in obj\ \mathcal{K}'$, then there is an object of $\mathcal{K}'$ projectively generated by f.

In the sequel we shall investigate general reflections by means of theorem 1. Sometimes it is evident that $\mathcal{K}'$ is not reflective in $\mathcal{K}$, but we desire to recognize those objects of $\mathcal{K}$ which do have a reflection in $\mathcal{K}'$. Hence we shall consider rather the subcategory $\mathcal{K}_X$ of $\mathcal{K}$ for fixed $X \in obj\ \mathcal{K} - obj\ \mathcal{K}'$, described by:

$obj\ \mathcal{K}_X = obj\ \mathcal{K}' \cup (X)$, $\mathcal{K}'$ is a full subcategory of $\mathcal{K}_X$, $Hom_{\mathcal{K}_X}\langle X, X\rangle = 1_X$ and $Hom_{\mathcal{K}_X}\langle X, Y\rangle = Hom_{\mathcal{K}}\langle X, Y\rangle$, $Hom_{\mathcal{K}_X}\langle Y, X\rangle = \emptyset$ for $Y \in obj\ \mathcal{K}'$.

Clearly the problem of existence of reflections of X in $\mathcal{K}'$ has the same answer in $\mathcal{K}$ as in $\mathcal{K}_X$.

If $\langle Y, f \rangle$ is a reflection of X in $\mathcal{K}'$ then f is an epimorphism in $\mathcal{K}_X$ which is "least" in the class $\mathcal{E}\{g \mid g$ is an epimorphism in $\mathcal{K}_X$, $\mathcal{D}g = X$, $\mathcal{E}g \in obj\ \mathcal{K}'\}$. Therefore we shall construct a further category $\widetilde{\mathcal{K}}_X$: $\widetilde{\mathcal{K}}_X$ is the full subcategory of the category of morphisms from $\mathcal{K}_X$ (which is sometimes denoted by $\mathcal{K}_X^2$ or $Morph\ \mathcal{K}_X$ or $([\rightarrow], \mathcal{K}_X)$; see [3]) generated by all epimorphisms g of $\mathcal{K}_X$ with $\mathcal{D}g = X$ and by all identities of $\mathcal{K}'$.
Let $\mathcal{F}_X$ be the functor from $\widetilde{\mathcal{K}}_X$ into $\mathcal{K}_X$ assigning $\mathcal{D}f$ to $f \in obj\ \widetilde{\mathcal{K}}_X$ ($\mathcal{F}_X$ is faithful and preserves monomorphisms). Denote by $\widetilde{\mathcal{K}}_X'$ the full subcategory of $\widetilde{\mathcal{K}}_X$ generated by $obj\ \widetilde{\mathcal{K}}_X - (1_X)$. Now we summarize what was indicated above.

<u>Theorem 2</u>. The following statements are equivalent:
(1) X has a reflection in $\mathcal{K}'$;
(2) 1_X has an upper $\mathcal{F}_X$ -modification in $\widetilde{\mathcal{K}}_X'$ which is (with the corresponding $\mathcal{F}_X$ -identity) a reflection of 1_X in $\widetilde{\mathcal{K}}_X'$;
(3) 1_X has an upper $\mathcal{F}_X$ -modification in $\widetilde{\mathcal{K}}_X'$ and each $f \in Hom_{\mathcal{K}} \langle X, Y \rangle$, $Y \in obj\ \mathcal{K}'$, can be factorized as $f_1 \circ f_2$ with $f_2 \in obj\ \widetilde{\mathcal{K}}_X'$.

<u>Proof</u>. If $\langle Y, f \rangle$ is a reflection of X in $\mathcal{K}'$, then $\langle f, \langle 1_X, f \rangle \rangle$ is a reflection of 1_X in $\widetilde{\mathcal{K}}_X'$; and also conversely. The equivalence of (2) and (3) is evident.

We are prepared to apply theorem 1 to the general case (we shall use remark (2), because in $\widetilde{\mathcal{K}}_X$ there need not exist products of objects from $\widetilde{\mathcal{K}}_X'$ even in the case of a product-admitting category $\mathcal{K}$). The direct proof may be simpler than that exhibit.

Theorem 3. Assume that in $\mathcal{K}$ products of objects from $\mathcal{K}'$ exist. Then X has a reflection in $\mathcal{K}'$ if and only if

(a) the embedding of $\mathcal{K}'$ into $\mathcal{K}_X$ is product-stable;

(b) $Hom_{\mathcal{K}}\langle X, Y\rangle \neq \emptyset$ for some $Y \in obj\ \mathcal{K}'$;

(c) there is a cofinal set in the class $\mathcal{F}_X^{-1}[X] - (1_X)$;

(d) each $f \in Hom_{\mathcal{K}}\langle X, Y\rangle$, $Y \in obj\ \mathcal{K}'$, can be factorized as $f_1 \circ f_2$ where $f_2 \in \mathcal{F}_X^{-1}[X] - (1_X)$.

Remark. (1) Under some additional conditions on $\mathcal{K}'$ and $\mathcal{F}_X$, this theorem follows from the "adjoint functor theorem" of [3] (cofinal sets of $\mathcal{F}_X^{-1}[X] - (1_X)$ are solution sets). It is easy to generalize theorem 3 to the case of existence of left adjoints for faithful functors T such that every $T^{-1}[Z]$ has a least element and that every Tf is some Tg with given $\mathcal{E}g$.

(2) In special cases it is possible to improve theorem 3, mainly conditions (c) and (d) (e.g. under certain assumptions on hereditariness of $\mathcal{K}'$ in $\mathcal{K}$ one may restrict $\mathcal{F}_X^{-1}[X]$ to the epimorphism in $\mathcal{K}$).

Examples. (1) Let $\mathcal{K} = \mathcal{Cl}$ be the category of closure spaces, $\mathcal{A}$ a non-void productive and hereditary full subcategory of $\mathcal{Cl}$ such that, for each closure space X, $\mathcal{F}_X^{-1}[X]$ contains only surjections; and let $\mathcal{K}' = \mathcal{A} \cap \mathcal{Comp}$ be the full subcategory of $\mathcal{Cl}$ generated by compact spaces from $\mathcal{A}$. (E.g. take for $\mathcal{A}$ the categories $\mathcal{Cl}$, $\mathcal{Cl}_{T_0}$, $\mathcal{Cl}_{SU}$ (the category of semi-uniformizable or symmetric spaces), $\mathcal{Cl}_{T_1}$, $\mathcal{Cl}_{Reg}$, $\mathcal{Cl}_U$ (uniformizable spaces), $\mathcal{Top}$, $\mathcal{Top}_{T_0}$, $\mathcal{Top}_{SU}$, $\mathcal{Top}_{T_1}$, $\mathcal{Top}_{Reg}$.)

Hence, by theorem 3, a closure space X has a reflection in $\mathcal{A} \cap \mathcal{Comp}$ if and only if for each continuous mapping $f : X \to Y$ with $Y \in obj(\mathcal{A} \cap \mathcal{Comp})$, the subspace $f[X]$ of Y is compact.

Consequently if one can embed each space from $\mathcal{A}$ into a compact space from $\mathcal{A}$, then the closure space X has a reflection in $\mathcal{A} \cap \mathcal{Comp}$ if and only if its reflection in $\mathcal{A}$ is compact ($\mathcal{A}$ is reflective in $\mathcal{Cl}$, e.g. by theorem 3). We shall show that this situation obtains for all categories $\mathcal{A}$ indicated above except for $\mathcal{A} = \mathcal{Top}_{Reg}$ (here X has a reflection in $\mathcal{A} \cap \mathcal{Comp}$ if and only if its uniformizable modification is compact), and possibly excepting $\mathcal{A} = \mathcal{Cl}_{Reg}$.

Proof. The case of $\mathcal{A} = \mathcal{Cl}_U = \mathcal{Top}_U$ is known just as the cases of $\mathcal{A} = \mathcal{Top}, \mathcal{Top}_{T_0}, \mathcal{Top}_{SU}, \mathcal{Top}_{T_1}$ (here it is sufficient to take one-point compactifications with some neighborhood systems of the ideal point). A similar one-point compactification may be performed also in the cases of closure spaces. However, for these categories $\mathcal{A}$

we shall construct augmentation-separated compactifications (this is not possible for topological spaces). Let $\mathcal{P} = \langle P, u \rangle$ be a closure space, $\xi = \mathcal{E}\{\mathcal{X} \mid \mathcal{X}$ is an ultrafilter on P without limit points in $\mathcal{P}\}$, let $\mathcal{Q} = \langle Q, v \rangle$ be a compact separated space such that $P \cap Q = \emptyset$ and that there exists a one-to-one mapping φ of ξ onto Q. Now it is sufficient to put $R = P \cup Q$, $\mathcal{R} = \langle R, w \rangle$, where $\mathcal{P}$ is an open subspace of $\mathcal{R}$, and the neighborhood system of $\varphi\mathcal{X}$ in $\mathcal{R}$ is the smallest filter in R containing both $\mathcal{X}$ and the neighborhood system of $\varphi\mathcal{X}$ in $\mathcal{Q}$. The closure space $\mathcal{R}$ is compact and each point of Q is separated from any other point of R.

(2) Let $\mathcal{K} = C\ell$, $\mathcal{K}' = C\ell_{T_2}$. The category $\mathcal{K}'$ is reflective in $\mathcal{K}$; this follows e.g. from theorem 3. Here we may restrict $\mathcal{F}_X^{-1}[X]$ to surjections (see remark 3 following theorem 3) which form a set. But we shall show that the equivalence classes $(f \leq_{\mathcal{F}_X} f', f' \leq_{\mathcal{F}_X} f)$ in the whole $\mathcal{F}_X^{-1}[X]$ form a class. Epimorphisms in $\mathcal{K}_X$ are continuous mappings onto "topologically dense" sets (i.e. for $f: X \to Y$ it is the case whenever $f[X]$ is dense in the topological modification of Y — Y is the only closed set containing $f[X]$). Hence if Y is topological then $\overline{f[X]} = Y$ and thus $\operatorname{card} Y \leq \exp\exp \operatorname{card} f[X]$. If Y is not topological, $\operatorname{card} Y$ may be arbitrarily large.

Proof. Let X be a separated space having an infinite set of ultrafilters without limit points. We shall construct a transfinite sequence of separated spaces $\{Y_\alpha \mid \alpha \in \operatorname{Ord}\}$ such that $Y_0 = X$, for $\alpha > \beta$ Y_β is a topologically dense subspace of Y_α and $\operatorname{card} M_\alpha >$

$> card\ M_\beta$ where $M_\alpha = \mathcal{E}\{ \mathcal{X} \mid \mathcal{X}$ is an ultrafilter on Y_α without limit points $\}$.

Assume that all Y_β for $\beta < \alpha$ have already been constructed.

(a) If $\alpha = \beta + 1$ take for Y_α the space $\mathcal{R}$ from the preceding proof, where $\mathcal{P} = Y_\beta$ and $\mathcal{Q}$ is a discrete space. Evidently $card\ M_\alpha = exp\ exp\ card\ M_\beta$.

(b) If $\alpha = sup\{\beta \mid \beta < \alpha\}$, put $Y'_\alpha = \bigcup\{Y_\beta \mid \beta < \alpha\}$ with the closure structure of Y'_α so defined that every Y_β is an open subspace of Y'_α. In this case $card\ M'_\alpha \geq$ $\geq card\ Y'_\alpha$ (because each filter $\{F_\beta \mid \beta < \alpha\}$ in Y'_α, such that $F_\beta \subset Y'_\alpha - Y_\beta$ for $\beta < \alpha$ and $F_{\beta'} \subset F_\beta$ for $\beta < \beta' < \alpha$, has no accumulation points in Y'_α). Now it is sufficient to take for Y_α the space constructed from Y'_α as in (a) Y_α from Y_β.

Hence $card\ Y_\alpha > card\ Y_\beta$ starting from some $\alpha_0 \in Ord$.

(3) Let $\mathcal{K} = Cl$, $\mathcal{K}' = Cl_{T_2} \cap Comp$. All the conditions of theorem 3 are fulfilled except (c). Indeed, according to the preceding two examples, the equivalence classes of $\mathcal{F}_X^{-1}[X]$ may be a class; if there were a smallest element in $\mathcal{F}_X^{-1}[X] - (1_X)$, its range would have to be of greatest cardinality. This is the case if X is a separated space having an infinite set of ultrafilters without limit points. Each separated space with only a finite set of non-converging ultrafilters (such spaces exist) has a reflection in $Cl_{T_2} \cap Comp$, the space Y_1 from preceding example. (Evidently a closure space has a reflection in $Cl_{T_2} \cap Comp$ if and only if its

upper modification in $\mathcal{Cl}_{T_2}$ has such a reflection.)

This result implies the following interesting theorem:

In the category $\mathcal{Cl}_{T_2} \cap \mathcal{Comp}$, all injective spaces are one-point spaces.

Proof. Let I be an injective space in $\mathcal{Cl}_{T_2} \cap \mathcal{Comp}$, $\operatorname{card} I > 1$. Let $\mathcal{C}$ be the full subcategory of $\mathcal{Cl}$ generated by closure spaces which are projectively generated by mappings into I (i.e. $\operatorname{proj}(I)$ in the notation of [1]). It is almost obvious that $\mathcal{Cl}_{T_2} \cap \mathcal{Comp}$ is a subcategory of $\mathcal{C}$ and that each object of $\mathcal{C}$ has a reflection in $\mathcal{Cl}_{T_2} \cap \mathcal{Comp}$ (by the same method as used for construction of the Čech-Stone compactification). But this is a contradiction, since the infinite discrete space is an object of $\mathcal{C}$ and, by preceding, has no reflection in $\mathcal{Cl}_{T_2} \cap \mathcal{Comp}$.

A characterization of projective objects in $\mathcal{Cl}_{T_2} \cap \mathcal{Comp}$ is the open problem (it seems that only finite spaces are projective in $\mathcal{Cl}_{T_2} \cap \mathcal{Comp}$).

(4) Except theorem 2 I do not know any general theorem on the existence of reflections (or coreflections) in the case that products (sums, respectively) in $\mathcal{K}$ do not exist.

Let $\mathcal{K}$ be a category of closure spaces with one-to-one continuous mappings as morphisms. In this category there do not exist sums of families (of non-void spaces) with cardinality greater than one. For the full subcategory $\mathcal{K}'$ of $\mathcal{K}$ generated by the compact spaces, conditions (a),(b),(c) and (d) of the dual to theorem 3 are fulfilled, but no non-compact space has a coreflection in $\mathcal{K}'$. On the other hand, e.g. the full subcategory $\mathcal{K}'$ of $\mathcal{K}$ generated by dense-it-themselves spaces is coreflective in $\mathcal{K}$ (this example is due to Katětov).

R e f e r e n c e s

[1] E. ČECH: Topological Spaces, Academia Prague, 1966.

[2] J.F. KENNISON: Reflective functors in general topology and elsewhere, Trans.Amer.Math.Soc.118 (1965),303-315.

[3] P. FREYD: Abelian Categories, Harper & Row, 1964.

(Received April 13,1966)