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COMPLETION OF CERTAIN Λ - STRUCTURES

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1° In this paper, a linear space $(E, \mathcal{T})$ will mean a vector space E endowed with a separated (i.e. Hausdorff) locally convex topology $\mathcal{T}$. In general all continuous structures which occur in the sequel are supposed to be separated. We denote by $\pi(E, \mathcal{T})$ the completion of the space $(E, \mathcal{T})$. The following concept of Λ-structures is due to Prof. Katětov: Let $\mathcal{X}$ be a set and denote by $\Lambda\mathcal{X}$ the free modul on $\mathcal{X}$ over the real numbers, i.e. the vector space of all finite formal linear combinations $\sum_{i=1}^{n} \lambda_i x_i$, where λ_i are real numbers and $x_i \in \mathcal{X}$. If $(\Lambda\mathcal{X}, \mathcal{T})$ is a linear space, then the pair $(\mathcal{X}, \mathcal{T})$ will be called a Λ-structure or a Λ-space. We say that $(\mathcal{X}, \mathcal{T})$ is a weak or Mackey Λ-structure if $\mathcal{T}$ is the weak or Mackey topology on $\Lambda\mathcal{X}$, respectively. There is a $1-1$ correspondence between the mappings of the set $\mathcal{X}$ into a vector space E and the linear mappings of $\Lambda\mathcal{X}$ into E. The linear mapping $f: \Lambda\mathcal{X} \to E$ which corresponds to the mapping $h: \mathcal{X} \to E$ is called the linear extension of h, and we write $f = \Lambda h$. We say that a mapping $f: (\mathcal{X}, \mathcal{T}) \to (\mathcal{Y}, \mathcal{S})$ is Λ-morphic if its extension $\Lambda f: (\Lambda\mathcal{X}, \mathcal{T}) \to (\Lambda\mathcal{Y}, \mathcal{S})$ is continuous. Often (but not always) we will not distinguish between Λf and f, thus omitting the letter Λ.

2° We will consider some Λ-structures which are defined in the following manner. Let $\mathfrak{X}$ be endowed initially with some topology or uniformity $\mathcal{S}$ and let Φ be a suitable subspace of the vector space of all $\mathcal{S}$-continuous functions on $\mathfrak{X}$. Then one may define in a natural manner a duality of the pair $\{\Lambda\mathfrak{X}, \Phi\}$: For $x = \sum_{i=1}^{n} \lambda_i x_i \in \Lambda\mathfrak{X}$, $f \in \Phi$ put

(1) $$\langle x, f \rangle = \sum_{i=1}^{n} \lambda_i f(x_i);$$

(evidently Φ must be such that for every $x \neq 0$, $x \in \Lambda\mathfrak{X}$, there is a $f \in \Phi$ with $\langle x, f \rangle \neq 0$). Denote by μ the Mackey topology of the pair $\{\Lambda\mathfrak{X}, \Phi\}$, and let ν be the finest topology on $\Lambda\mathfrak{X}$ such that $(\Lambda\mathfrak{X}, \nu)$ is a linear space and the natural imbedding of $\mathfrak{X}$ (with the continuous structure induced by $\mathcal{S}$) in $(\Lambda\mathfrak{X}, \nu)$ is a homeomorphism. Now a natural question arises, whether it is possible to describe in a simple way the completions $\pi(\Lambda\mathfrak{X}, \mu)$ and $\pi(\Lambda\mathfrak{X}, \nu)$?

3° The following special but important cases were mentioned in [1]:

a) Let $\mathfrak{X}$ be a compact space and $\Phi = \mathcal{C}(\mathfrak{X})$ the space of all continuous functions on $\mathfrak{X}$. The proof of

(2) $$\pi(\Lambda\mathfrak{X}, \nu) = \mathcal{C}'(\mathfrak{X})$$

was sketched in [1], and it was there stated that also

(3) $$\pi(\Lambda\mathfrak{X}, \mu) = \mathcal{C}'(\mathfrak{X}),$$

although in general μ is strictly finer than ν. For a detailed proof of (2) see [2]. (The proof of (2) and (3) was also given by S. Tomášek and will appear in this Journal.)

b) Let $\mathfrak{X}$ be the segment $\langle 0, 1 \rangle$ of the real line with its usual topology, and let $\phi = \mathcal{E}(\mathfrak{X})$ where $\mathcal{E}(\mathfrak{X})$ is the space of all infinitely differentiable functions on $\langle 0, 1 \rangle$. According to [1],

(4) $$\pi(\Lambda \mathfrak{X}, \mu) = \mathcal{E}'(\mathfrak{X})$$

where $\mathcal{E}'(\mathfrak{X})$ is the space dual to $\mathcal{E}(\mathfrak{X})$, i.e. $\mathcal{E}'(\mathfrak{X})$ is the space of all distributions on $\mathfrak{X}$. We shall give another and simple proof of (2). Then, following an idea of Prof. Katětov, we prove (3) and finally (4). In the course of this we obtain some further generalizations. The proofs of (2) and (4) are based on the well known

4° Grothendieck's theorem, [3]. The completion of the linear space E is (algebraically) isomorphic to the vector space of all those linear forms on E' which are $\sigma(E', E)$ - continuous on every equicontinuous set in E'. -

An immediate consequence of this theorem is

5° Corollary. If (E, μ_1), (E, μ_2) are linear spaces, $(E, \mu_1)' = (E, \mu_2)'$ and $\mu_1 \subset \mu_2$, then $\pi(E, \mu_2)$ is algebraically isomorphic to a subspace of (E, μ_1).

We begin with the 3°, case a).

6° Let $\mathfrak{X}$ be an uniform space and let ϕ be the space of all uniformly continuous functions on $\mathfrak{X}$. Raikov [4] proved the following

Theorem. Let $\mathfrak{X}$ be the system of all weakly (i.e. pointwise) bounded uniformly equicontinuous sets of functions from ϕ. Then ν is the topology of the uniform convergence on the system $\mathcal{H}$ and $(\Lambda \mathfrak{X}, \nu)' = \phi$.

It should be emphasized that one must distinguish between uniformly equicontinuous sets in ϕ and equicontinuous sets in $(\Lambda \mathfrak{X}, \nu)' = \Lambda \phi$: $H \subset \phi$ is called a uniformly equicontinuous set iff for every $\varepsilon > 0$ there is a neighborhood $\mathcal{U}_\varepsilon$ (of the diagonal in the uniformity on $\mathfrak{X}$) such that

$\mathcal{U}_\varepsilon \subset \{[x_1, x_2] \in \mathfrak{X} \times \mathfrak{X} : |f(x_1) - f(x_2)| < \varepsilon$ for all $f \in \mathcal{H}\}$

Denote by $\mathcal{H}_1$ the system of all sets of the form $\Gamma(\Lambda H)$ for $H \in \mathcal{H}$. (Here $\Lambda H = \{g = \Lambda f : f \in H\}$ and $\Gamma(M)$ denotes the convex hull of M.) Now the set $L \subset (\Lambda \mathfrak{X}, \nu)'$ is equicontinuous iff there is a neighborhood (of 0 in $(\Lambda \mathfrak{X}, \nu)$) $\mathcal{U}$ such that $L \subset \mathcal{U}^\circ$. The system of all these sets will be denoted by $\mathcal{H}_2$. Now Raïkov's theorem states that $H_2 \in \mathcal{H}_2$ iff there exists a $H_1 \in \mathcal{H}_1$ such that $H_2 \subset H_1^{\circ\circ}$, i.e. $\mathcal{H}_1 = \mathcal{H}_2$.

7^0 __Proposition.__ If $(\mathfrak{X}, \mathcal{U})$ is a totally bounded uniform space and if ϕ is the space of all uniformly continuous functions on $\mathfrak{X}$, then

$$\pi(\Lambda \mathfrak{X}, \nu) = \phi'.$$

Here ϕ' is the space of all linear forms on ϕ continuous in the norm topology ($\|f\| = \sup_{x \in \mathfrak{X}} |f(x)|$).

__Proof__: Let $\varphi \in \pi(\Lambda \mathfrak{X}, \nu)$. Using the theorems of Raikov and Grothendieck we see that φ is $\sigma(\phi, \Lambda \mathfrak{X})$ -continuous on every $H \in \mathcal{H}_2$. Let $f_n \to 0$, i.e. $\|f_n\| = \sup_{x \in \mathfrak{X}} |f_n(x)| \to 0$. Put $H_0 = \{f_n\}_{n \geq 1} \cup \{0\}$; evidently $H_0 \in \mathcal{H}$, so that

$\Lambda H_0 \in \mathcal{H}_2$ for $\Lambda H_0 \subset (\Gamma(\Lambda H_0))^{oo}$ and $\Gamma(\Lambda H_0) \in \mathcal{H}_1$. Now $f_n \to 0$ in the topology $\sigma(\Phi, \Phi')$ and thus also in the $\sigma(\Phi, \Lambda \mathcal{X})$ -topology, and the f_n belong to a set from $\mathcal{H}_2$. We conclude that $\varphi(f_n) \to 0$, i.e. $\varphi \in \Phi'$.

Conversely, let $\varphi \in \Phi'$ and $H \in \mathcal{H}_2$. If $f_\iota \in H$ ($\iota \in J$) is a net converging to zero in the topology $\sigma(\Phi, \Lambda \mathcal{X})$, then by [3, chap.III, § 3, proposition 5] our net converges to zero uniformly on all totally bounded subsets $Y \subset \Lambda \mathcal{X}$; in particular for $Y = \mathcal{X}$ we obtain $\| f_\iota \| \to 0$, so that $\varphi(f_\iota) \to 0$, i.e. $\varphi \in \pi(\Lambda \mathcal{X}, \nu)$ by Grothendieck's theorem.

8° We recall an interesting ~~theorem~~ theorem of Pták, ~~[5]:~~

<u>Theorem</u>. Let $\mathcal{X}$, $\mathcal{Y}$ be completely regular spaces, $\mathcal{X}$ pseudocompact and $\mathcal{Y}$ countably compact. Then every bounded separately continuous real function B on $\mathcal{X} \times \mathcal{Y}$ can be extended to a separately continuous bilinear form B on $\mathcal{C}'(\mathcal{X}) \times \mathcal{C}'(\mathcal{Y})$, in the following manner: Let k be the mapping of $\mathcal{Y}$ into $\mathcal{C}(\mathcal{X})$ defined by the relation $\langle x, k(y) \rangle = f(x, y)$. Extend every $k(y)$ to $\mathcal{C}'(\mathcal{X})$ and then, for $p \in \mathcal{C}'(\mathcal{X})$, put $\langle h(p), y \rangle = \langle p, k(y) \rangle$ $(y \in \mathcal{Y})$. Then h is a mapping of $\mathcal{C}'(\mathcal{X})$ into $\mathcal{C}(\mathcal{Y})$; extending every $h(p)$ to $\mathcal{C}'(\mathcal{Y})$ put $B(p, q) = \langle h(p), q \rangle$.

9° <u>Proposition</u>. Let $\mathcal{X}$ be a pseudocompact completely regular space, and let E be a linear space which is complete in its Mackey topology $\tau(E, E')$. Then every continuous mapping $f : \mathfrak{X} \to (E, \sigma)$, where $\sigma =$ $= \sigma(E, E')$, may be extended to a linear continuous mapping $h : \mathcal{C}'(\mathfrak{X}) \to (E, \sigma)$.

Proof: First let E be a Banach space; then we may regard E as a subspace of $\mathcal{C}(\mathcal{Y})$ for some compact $\mathcal{Y}$. ($\mathcal{Y}$ is the unit sphere in E' with its week topology). Since $f(\mathfrak{X}) \subset \mathcal{C}(\mathcal{Y})$ is bounded, the function $B(x, y) =$ $= \langle f(x), y \rangle$ is bounded and evidently separately continuous, and may be extended to a bilinear form $B(p, q) =$ $= \langle h(p), q \rangle$ on $\mathcal{C}'(\mathfrak{X}) \times \mathcal{C}'(\mathcal{Y})$. Now h is obviously an extension of f on $\mathcal{C}'(\mathfrak{X})$ and is weakly continuous. The set $\Lambda\mathfrak{X}$ is dense in $\mathcal{C}'(\mathfrak{X})$ so that $h(\mathcal{C}'(\mathfrak{X})) \subset \bar{E} = E$ since the weak closure of the convex set E coincides with its closure and E is complete. In the case of general E one imbeds (E, τ) into the product of Banach spaces and then proceeds in the obvious manner.

10° Proposition. Let $\mathfrak{X}$ be pseudocompact completely regular space, $\Phi = \mathcal{C}(\mathfrak{X})$. Then, in the sense of algebraic isomorphism, $\pi(\Lambda\mathfrak{X}, \mu) = \mathcal{C}'(\mathfrak{X})$.

Proof: According to 5° and using 7° where the uniformity projectively generated by $\mathcal{C}(\mathfrak{X})$ is taken for $\mathfrak{U}$, it is only needed to show that there is an injective mapping of $\mathcal{C}'(\mathfrak{X})$ into $\pi(\Lambda\mathfrak{X}, \mu)$. But this is an immediate consequence of 9°, since for $0 \neq \xi \in \mathcal{C}'(\mathfrak{X})$ there is an $f \in \mathcal{C}(\mathfrak{X})$ such that $\langle f, \xi \rangle \neq 0$, and $\Lambda\mathfrak{X}$ being $\sigma(\mathcal{C}'(\mathfrak{X}), \mathcal{C}(\mathfrak{X}))$ - dense in $\mathcal{C}'(\mathfrak{X})$, there is $\langle f, h(\xi) \rangle = \lim \langle f, h(\xi_\alpha) \rangle = \lim \langle f, \xi_\alpha \rangle = \langle f, \xi \rangle \neq 0$

where $\xi_\alpha \in \Lambda \mathcal{X}$, $\xi_\alpha \to \xi$.

11° Now we turn to the case b) from 3°. Let $\mathcal{X}$ be an n-dimensional cube in euclidean n-space E_n. Put $\Phi = \mathcal{E}(\mathcal{X})$, i.e. Φ is the space of all infinitely differentiable functions on $\mathcal{X}$. We consider $\mathcal{E}(\mathcal{X})$ in its Fréchet topology ω defined by the seminorms $p_\iota(f) = \sup_{x \in \mathcal{X}} |D^\iota f(x)|$ where ι is an arbitrary multiindex and D^ι the corresponding derivative. We denote by τ the Mackey topology of the pair $\{\Lambda \mathcal{X}, \Phi\}$.

12° Proposition. Let E be a complete linear space. Then the following conditions are equivalent

(a) The mapping $f : \mathcal{X} \to E$ is ∞-differentiable

(b) The mapping $\Lambda f : (\Lambda \mathcal{X}, \tau) \to E$ is continuous.

Proof: (a) $\Longrightarrow$ (b): Suppose that f is ∞-differentiable. It is sufficient to prove that for every $\varphi \in E'$ the form $\varphi \circ \Lambda f$ is weakly continuous on $\Lambda \mathcal{X}$, since $\Lambda \mathcal{X}$ has the Mackey topology. But for every $\varphi \in E'$ there is $\varphi \circ f = \chi \in \mathcal{E}(\mathcal{X})$, and thus from the uniqueness of the linearization it follows that $\varphi \circ \Lambda f = \Lambda(\varphi \circ f) = \Lambda \chi \in (\Lambda \mathcal{X}, \tau)'$.

(b) $\Longrightarrow$ (a): We shall prove by induction the existence of derivatives $D^\iota f$ for all multiindexes ι. For $\iota_0 = (0,\dots,0)$ there is $D^{\iota_0} f = f$; we proceed to prove the existence of $\frac{\partial}{\partial x_1} f$ (the general case is similar). Choose a fixed $x \in \mathcal{X}$ and put

$$\varphi(h) = (f(x+h) - f(x))\, h_1^{-1}$$

for all $h = [h_1, 0, \dots, 0]$ for which $x + h \in \mathfrak{X}$ and $h \neq 0$. Now $\varphi \circ f \in \mathcal{E}(\mathfrak{X})$ for all $\varphi \in E'$ since Λf is continuous and $\varphi \circ \Lambda f = \Lambda(\varphi \circ f) \in (\Lambda\mathfrak{X}, \tau)' = \mathcal{E}(\mathfrak{X})$. A twofold application of the mean-value theorem yields $|\varphi(g(h) - g(h'))| \leq C_\varphi \max(|h|, |h'|)$ where C_φ is a constant depending only on φ. The set $\{g \in E : g = (g(h) - g(h'))(\max |h|, |h'|)^{-1}, x + h \in \mathfrak{X}, x + h' \in \mathfrak{X}, h \neq 0 \neq h'\}$ is therefore weakly bounded and also bounded in E, which means that $\{g(h)\}_{x+h \in \mathfrak{X}, h \neq 0}$ is a Cauchy net in E. From completeness of E there then follows the existence of $\lim\limits_{h \to 0} g(h) = \frac{\partial}{\partial x_1} f(x)$. From $\varphi \circ \Lambda \frac{\partial f}{\partial x_1} = \Lambda \frac{\partial}{\partial x_1}(\varphi \circ f)$ there follows the continuity of the mapping $\Lambda \frac{\partial f}{\partial x_1}$ and we can continue as above.

13° <u>Proposition</u>. Every set $H \subset \Phi$ equicontinuous with respect to the topology τ is bounded in the Fréchet space $\mathcal{E}(\mathfrak{X})$.

<u>Proof</u>: It follows from Arzelà theorem that $p_{\iota_0}(H) = \sup\limits_{\substack{x \in \mathfrak{X} \\ f \in H}} |f(x)| < \infty$. We must prove that for every multiindex ι there is $p_\iota(H) < \infty$. It suffices to prove this for $\iota_1 = [1, 0, \dots, 0]$ since the general case is again similar. One may suppose that $H = \Gamma(\Lambda H)$. Let E be the subspace generated by the set H in Φ. For $f \in E$ put $|f|_E = \sup\{|f(x)| : x \in H^\circ\}$, so that

$$|f|_E \leq 1 \iff f \in H.$$

Recall that H° is a neighborhood in $\Lambda\mathfrak{X}$ and thus it "swallows" each point of $\Lambda\mathfrak{X}$, especially each point of $\mathfrak{X}$ so that $|f|_E = 0$ implies $f = 0$ and so $|\cdot|_E$ is really the norm. Denote by F the completion of the conjugate space of E with the norm $|\xi|_F = \sup\{|\langle f, \xi\rangle| : f \in \in H\}$. Now the canonical mapping of $\Lambda\mathfrak{X}$ into F, which we denote by $x \to \delta_x$ (δ_x is the evaluation at x, $\langle f, \delta_x\rangle = f(x)$, so that for $x \in \mathfrak{X}$, δ_x is in fact the Dirac measure at x), is continuous: for $x \in H^o$ there is $|\delta_x|_F = \sup_{f\in H} |f(x)| \leq 1$. By the preceding proposition this means that $x \to \delta_x$ is a ∞-differentiable mapping of $\mathfrak{X}$ into F. Now compute $D^{L_1}\delta_x = \frac{\partial}{\partial x_1}\delta_x$: $0 = \lim_{h\to 0} |\frac{\partial \delta_x}{\partial x_1} - (\delta_{x+h} - \delta_x)h_1^{-1}|_F = \lim_{h\to 0} \sup_{f\in H} |(f(x+h) - f(x))h_1^{-1} - \frac{\partial}{\partial x_1} f(x)|$, and thus, uniformly in f, $\langle f, \frac{\partial \delta_x}{\partial x_1}\rangle = \langle \frac{\partial f}{\partial x_1}, \delta_x\rangle$.

The following implications hold: δ_x is ∞-differentiable $\Rightarrow \frac{\partial \delta_x}{\partial x_1}$ is ∞-differentiable $\Rightarrow \frac{\partial \delta_x}{\partial x_1}$ is a continuous mapping of $(\Lambda\mathfrak{X}, \tau)$ into F (see 12°). The mapping $\frac{\partial \delta_x}{\partial x_1}$ is therefore bounded on the weakly compact subset $\mathfrak{X}$ of $\Lambda\mathfrak{X}$, i.e. $p_{L_1}(H) = \sup\{|\frac{\partial f}{\partial x_1}(x)| : x \in \mathfrak{X}, f \in H\} < +\infty$.

14° We have just used the following fact: the topology $\tau/_{\mathfrak{X}}$ coincides with the usual topology ρ on $\mathfrak{X}$. (Evidently $\tau/_{\mathfrak{X}}$ is coarser than ρ and $(\mathfrak{X}, \rho)$ is compact. Thus $(\mathfrak{X}, \tau/_{\mathfrak{X}})$ is also compact and $\tau/_{\mathfrak{X}} = \rho$.)

15° <u>Proposition</u>. Under our assumptions we have the algebraic equality

$$\pi(\Lambda\mathfrak{X}, \tau) = \mathcal{E}'(\mathfrak{X})$$

where $\mathcal{E}'(\mathcal{X})$ is the space dual to $\mathcal{E}(\mathcal{X})$, i.e. the space of all distributions on $\mathcal{X}$.

Proof: Let $\varphi \in \mathcal{E}'(\mathcal{X})$ and let H be a τ-equicontinuous set in Φ. Then by 13°, H is bounded and thus relatively compact in $\mathcal{E}(\mathcal{X})$ (the latter space is a Montel space), and therefore the topology induced by $\mathcal{E}(\mathcal{X})$ in H coincides with the topology $\sigma(\Phi, \Lambda\mathcal{X})$. Using Grothendieck's theorem we see that $\mathcal{E}'(\mathcal{X}) \subset \pi(\Lambda\mathcal{X}, \tau)$. Conversely, let φ be a linear form on Φ which is $\sigma(\Phi, \Lambda\mathcal{X})$ -continuous on every τ-equicontinuous subset of Φ. If $h_n \to 0$ in $\mathcal{E}(\mathcal{X})$, then $H = \Gamma(\{h_n\}_{n\geq 1})$ is bounded and thus relatively compact in $\mathcal{E}(\mathcal{X})$ and also relatively compact in $\sigma(\Phi, \Lambda\mathcal{X})$; but this means that H is τ-equicontinuous in Φ, and evidently $\sigma(h_n) \to 0$. By Grothendieck's theorem $\pi(\Lambda\mathcal{X}, \tau) \subset \mathcal{E}'(\mathcal{X})$.

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