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LIMITS IN CATEGORIES AND LIMIT-PRESERVING FUNCTORS

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It need not be emphasized that the existence of direct and inverse limits is one of the most important properties of a category. Of course not all categories have this property. But sometimes it suffices that a category be embedded-able into a category with limits (or with sums or kernels or biproducts etc.) if the embedding functor has convenient properties (full, exact etc.).

A number of various embedding-theorems is well-known, [5], usually for categories satisfying some further properties, most often for additive and abelian categories. For general categories the possibility of full embedding into a category with sums has been proved [3], and also into a category with inverse limits,[10]; for small categories x) the possibility of full embedding into a category with finite sums and finite products has also been proved,[3]. In general, the embedding functor does not preserve the limits already existing in the original category.

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x) In [3]it is not expressly stated that the categories considered are small; the proof proceeds by induction, which cannot be carried out for "large" categories within the framework of the Bernays-Gödel axioms which are used for the present paper.

In the present paper full embeddings of a given category into a category with limits are studied. However, we also require that the embedding functor preserve some or all the limits already present.[x)] Embedding theorems are usually proved by describing or constructing the whole category at one step. The basic idea of the present constructions consists in adding limits one by one.

The present paper is divided into three parts. The first contains some auxiliary lemmas used for the proof of the main lemmas I.8 and I.9. Lemma I.8 states that, roughly speaking, to a given small category one object can be adjoined in such a manner that it is a direct limit of a given diagram, and that all direct and inverse limits originally present are preserved. Lemma I.9 also considers the possibility of extending the given functor. The auxiliary lemmas of this first part also contain some of constructive character which do not expressly mention limits, and might possibly even have further applications. In particular, Lemma I.2 is of this type; it states that to an arbitrary given category one may "add a morphism" such that its composition with some morphisms is prescribed and with the others it is "free". In the second part of the present paper there are proved some embedding-theorems for small categories, obtained by suitable iteration of Lemmas I.8 and

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x) While this paper was being referred, the author obtained a preprint of J.R. Isbell's paper Structure of categories I which is connected with the problems studied here.

I.9 (and of their duals). For example, by means of Lemma I.8 we obtain that every small category may be fully embedded into a complete category such that the embedding-functor preserves all already existing direct and inverse limits (Theorem II.7.B). By means of Lemma I.9 we obtain the following Theorem II.5:

Let $\mathfrak{k}$ be a small category, let $\mathfrak{G}_d$, $\mathfrak{G}_i$ be sets of diagrams in $\mathfrak{k}$, let V_d, V_i be classes of diagram schema. Then there exists a $(\overrightarrow{V_d}, \overleftarrow{V_i})$-complete category K and a full embedding $\iota : \mathfrak{k} \to K$ which is $(\overrightarrow{\mathfrak{G}_d}, \overleftarrow{\mathfrak{G}_i})$-preserving; furthermore, for every $(\overrightarrow{V_d}, \overleftarrow{V_i})$-complete category K' and every $(\overrightarrow{\mathfrak{G}_d}, \overleftarrow{\mathfrak{G}_i})$-preserving functor $\Phi : \mathfrak{k} \to K'$ there exists a $(\overrightarrow{K^{V_d}}, \overleftarrow{K^{V_i}})$-preserving functor $\Psi : K \to K'$ unique up to natural equivalence such that $\iota\, \Psi = \Phi$.

Analogous problems are considered for categories with a system of null morphisms. For example it is proved that every small category $\mathfrak{k}$ with a system of null morphisms may be fully and "exactly" embedded (i.e. the embedding preserves kernels and cokernels already existing in $\mathfrak{k}$) into a small category K with kernels and cokernels such that every "exact" functor from $\mathfrak{k}$ to a category with kernels and cokernels may be extended on K (cf.II.8). In the third part of the present paper the embedding of arbitrary categories is considered. In general it is not possible to embed fully a "large" category into a category with finite sums preserving all finite sums already existing (cf. example III.1). But this is "almost possible"; for almost-categories (obtained by omitting the axiom that all morphisms from one object to another form a set) stated above Theorem II.5 is consistent with the axioms of set-theory (if there exists a strongly inaccessible cardinal number) even if $\mathfrak{G}_d$ and $\mathfrak{G}_i$ are c l a s s e s of diagrams (cf. III.2). Finally, (cf. III.3-7) there is exhibited a construction which, for every class V of diagram schema and for an arbitrary category $\mathfrak{k}$, describes a full embedding $\iota : \mathfrak{k} \to K$ such that $\mathfrak{k}$ is $\overrightarrow{V}$-complete in K (i.e. every $\mathcal{G}\iota$, where $\mathcal{G}$ is a V-diagram in $\mathfrak{k}$, has a direct limit in K) and every functor $\Phi : \mathfrak{k} \to K'$ into a $\overrightarrow{V}$-complete category may be extended to K. If V is the class of all small discrete categories, then K is the category constructed in [3]. If $\mathfrak{k}$ is the class of all diagram schema, then K is the category dual to that constructed in [10].

The present paper is written within the Bernays-Gödel set-theory; thus, we distinguish sets and classes. The axioms are described in [6]. Although the present paper is not written formally (in some details even not quite precisely), these axioms are consistently respected. The axiom of choice is assumed. The existence of a strongly inaccessible cardinal number (i.e. a regular uncountable cardinal $\aleph$ such that $\alpha < \aleph \Rightarrow 2^{\alpha} < \aleph$) is not assumed, unless expressly stated (only in III.2). The results presented may also be carried over into some other set-theories.

The definitions of the basic notions (category, objects and morphisms, full subcategory, category with a system of null morphisms, skeleton, functor and so on) are taken over from [8]. Also the notation of [8] is used; if K is a category, then $H_K(a, b)$ denotes the set of all morphisms of K from an object a to an object b. If

$\alpha \in H_K(a, b)$, $\beta \in H_K(b, c)$, then the composition of α and β is denoted by $\alpha . \beta$. In agreement with this convention, the value of a map (e.g. a functor) φ at an x will be denoted by $(x)\varphi$ instead of the more usual $\varphi(x)$. This also applies to the order in writing the composition of mappings. By an embedding is meant an iso-functor into. If K is a category, then K^σ denotes the class of all its objects, K^m the class of all its morphisms. If $a \in K^\sigma$, denote by e_a the identity-morphism of a. If K^σ is a set, then K is called small and the power of K is meant the power of K^m. Let $\mathcal{J}$, K be categories, $\mathcal{J}$ small, let $\mathcal{F} : \mathcal{J} \to K$ be a functor. We shall term $\mathcal{F}$ a diagram in K, and $\mathcal{J}$ is called a diagram schema; put $card\ \mathcal{F} = card\ \mathcal{J}^m$. If $\mathcal{J}$ is a quasi-ordered set, then it may be considered as a category. In such a case $\mathcal{F}$ will be called a presheaf; and if furthermore $a, b \in \mathcal{J}^\sigma$, $H_\mathcal{J}(a, b) = \{\alpha\}$, then $(\alpha)\mathcal{F}$ will also be denoted by $\mathcal{F}_a^b$. If $\mathcal{J}$ is a category such that $\mathcal{J}^m$ contains only identities, then it is called a discrete category and $\mathcal{F}$ is also called a collection (in K). We recall the well-known definitions, [4], [5], [7], [9].

<u>Definitions</u>: Let $\mathcal{F} : \mathcal{J} \to K$ be a diagram in K. A couple $\langle b; \{\psi_i ; i \in \mathcal{J}^\sigma\} \rangle$ will be called a direct (or inverse) bound of $\mathcal{F}$ (in K) if $\{\psi_i ; i \in \mathcal{J}^\sigma\}$ is a natural transformation of $\mathcal{F}$ into the constant functor $\mathcal{K} : \mathcal{J} \to K$ (or of $\mathcal{K}$ in $\mathcal{F}$ respectively) such that $(\mathcal{J}^\sigma)\mathcal{K} = \{b\}$, i.e. if $b \in K^\sigma$, $\psi_i \in H_K((i)\mathcal{F}, b)$ (or $\psi_i \in H_K(b, (i)\mathcal{F})$) and if $i, i' \in \mathcal{J}^\sigma$, $\sigma \in H_\mathcal{J}(i, i')$ then $\psi_i = (\sigma)\mathcal{F} . \psi_{i'}$, (or $\psi_i . (\sigma)\mathcal{F} = \psi_{i'}$;

respectively). A direct bound (or inverse bound) $\langle a; \{v_i; i \in \mathcal{J}^{\sigma}\}\rangle$ of $\mathcal{F}$ will be called a direct (or inverse) limit of $\mathcal{F}$ and denoted by $\overrightarrow{lim}_K \mathcal{F}$ (or $\overleftarrow{lim}_K \mathcal{F}$) if it has the following property: if $\langle b; \{\psi_i; i \in \mathcal{J}^{\sigma}\}\rangle$ is an arbitrary direct (or inverse) bound of $\mathcal{F}$ then there exists exactly one $f \in H_K(a, b)$ (or $f \in H_K(b, a)$) such that $v_i . f = \psi_i$ (or $f . v_i = \psi_i$, respectively) for all $i \in \mathcal{J}^{\sigma}$. Then f is called the canonical morphism of the direct (or inverse) bound $\langle b; \{\psi_i; i \in \mathcal{J}^{\sigma}\}\rangle$, a is denoted by $|\overrightarrow{lim}_K \mathcal{F}|$ (or $|\overleftarrow{lim}_K \mathcal{F}|$, respectively).

Let now $\mathcal{F}: \mathcal{J} \to K$ be a diagram in K, let $\langle a; \{v_i; i \in \mathcal{J}^{\sigma}\}\rangle$ be its direct (or inverse) limit, let $\Phi: K \to H$ be a functor. We shall say that Φ preserves the direct (or inverse) limit of $\mathcal{F}$ if $\langle (a)\Phi; \{(v_i)\Phi; i \in \mathcal{J}^{\sigma}\}\rangle$ is a direct (or inverse, respectively) limit of $\mathcal{F}\Phi$. The direct (or inverse) limit of a collection is also called its sum (or product, respectively).

<u>Conventions</u>: Let $\mathfrak{G}$ be a class of diagrams in a category K, $\Phi: K \to H$ a functor; the class of all $\mathcal{Y}\Phi$, where $\mathcal{Y} \in \mathfrak{G}$, is denoted by $\mathfrak{G}\Phi$. Let V be a class of diagram schema; every diagram whose schema belongs to V will be called a V-diagram. Let V_d, V_i be classes of diagram schema; every category K in which every V_d-diagram (or V_i-diagram) has a *direct* (or an *inverse*) limit will be called $\overrightarrow{V_d}$-complete (or $\overleftarrow{V_i}$-complete, respectively); every category K which is both $\overrightarrow{V_d}$-complete and $\overleftarrow{V_i}$-complete will be called $(\overrightarrow{V_d}, \overleftarrow{V_i})$-complete or

$(\overleftarrow{V_i}, \overrightarrow{V_d})$-complete. Let K be a category, let $\mathfrak{G}_d$, $\mathfrak{G}_i$ be classes of diagrams in K, let $\Phi : K \to H$ be a functor; we shall say that Φ is $\overrightarrow{\mathfrak{G}_d}$-preserving (or $\overleftarrow{\mathfrak{G}_i}$-preserving) if it preserves direct limits (or inverse limits) of all diagrams of $\mathfrak{G}_d$ (or $\mathfrak{G}_i$, respectively); if Φ is $\overrightarrow{\mathfrak{G}_d}$-preserving and $\overleftarrow{\mathfrak{G}_i}$-preserving, we shall say that it is $(\overrightarrow{\mathfrak{G}_d}, \overleftarrow{\mathfrak{G}_i})$-preserving or $(\overleftarrow{\mathfrak{G}_i}, \overrightarrow{\mathfrak{G}_d})$-preserving. If K is a category, V is a class of diagram schema, denote by K^V the class of all V-diagrams in K. The class of all diagram schema will always be denoted by $\mathbb{Z}$. A $(\overrightarrow{\mathbb{Z}}, \overleftarrow{\mathbb{Z}})$-complete category is called complete. If a direct bound $\langle b; \{\varphi_i ; i \in \mathcal{J}^\sigma\} \rangle$ is denoted by a single letter m, then $\langle (b)\Phi; \{(\varphi_i)\Phi ; i \in \mathcal{J}^\sigma\} \rangle$ is often denoted by $(m)\Phi$.

If K, H are categories with system of null morphisms, then every functor $\Phi : K \to H$ such that $(\alpha)\Phi$ is a null morphism of H whenever α is a null morphism of K will be called a null functor. If L is a set, denote

$$\Delta_L = \{\langle x, x \rangle ; x \in L\} .$$

I. Auxiliary lemmas.

The aim of this section is the proof of lemmas I.8 and I.9 which will be useful in obtaining embedding theorems for small categories (given in II.).

I.1. **Lemma**: Let ℓ be a category, let R be a relation on ℓ^m such that $\alpha R \beta$ implies $\alpha, \beta \in H_\ell(a, b)$ for some $a, b \in \ell^\sigma$. Then there exists a category h and a functor $\pi : \ell \to h$ such that

1) $\ell^\sigma = h^\sigma$, π is identical on ℓ^σ; if $\alpha R \beta$, then

$(\alpha)\pi = (\beta)\pi$;

2) if $\Phi : \mathcal{L} \to K$ is a functor such that $(\alpha)\Phi = (\beta)\Phi$ whenever $\alpha R \beta$, then there exists exactly one functor $\Psi : \mathcal{h} \to K$ such that $\Phi = \pi . \Psi$.

__Proof__: Let S be the smallest equivalence on $\mathcal{L}^m$ such that $R \cup \Delta_{\mathcal{L}^m} \subset S$ and $\alpha . \beta S \alpha' . \beta'$ whenever $\alpha S \alpha'$, $\beta S \beta'$ and either $\alpha . \beta$ or $\alpha' . \beta'$ is defined. Let $\mathcal{h}$ be the category such that $\mathcal{L}^\sigma = \mathcal{h}^\sigma$, $H_{\mathcal{h}}(a, b) = H_{\mathcal{L}}(a, b)_{/S}$; the definition of the composition in $\mathcal{h}$ is evident. π is the functor identical on $\mathcal{L}^\sigma$ and factor-mapping from $\mathcal{L}^m$ onto the decomposition $\mathcal{L}^m / S$. Evidently $\mathcal{h}$ and π have the required properties.

__Note 1__ : Let $\mathcal{L}$ be a category, let $a \in \mathcal{L}^\sigma$, let $\mathcal{k}$ be the full subcategory of $\mathcal{L}$ such that $\mathcal{k}^\sigma = \mathcal{L}^\sigma - \{a\}$. Let R be a relation on $\mathcal{L}^m$ such that if $\alpha R \beta$, then $\alpha, \beta \in H_{\mathcal{L}}(c, a)$ for some $c \in \mathcal{L}^\sigma$ and $\alpha . \rho = \beta . \rho$ for every $\rho \in H_{\mathcal{L}}(a, d)$ whenever $d \in \mathcal{L}^\sigma$, $d \neq a$. Then there exists a category $\mathcal{h}$ satisfying conditions 1) 2) from lemma I.1 and also the following 3): $\mathcal{k}$ is a full subcategory of $\mathcal{h}$, $H_{\mathcal{h}}(a, c) = H_{\mathcal{L}}(a, c)$ for every $c \in \mathcal{L}^\sigma - \{a\}$ and π is identical on $\mathcal{k}$ and on all $H_{\mathcal{L}}(a, c)$, $c \neq a$. In such a case we shall usually write $\mathcal{h} = \mathcal{L}/_R$, $\pi = 1/_R$. (The category constructed in the proof of lemma I.1 does not satisfy 3), but some isomorphic category does.)

__Note 2__ : Evidently, if $\mathcal{L}$ has a system of null morphisms then $\mathcal{h}$ also does, and π is a null functor.

I.2. <u>Lemma A</u>: Let $\mathcal{L}$ be a category, $a, b \in \mathcal{L}^{\sigma}$. For every $c \neq a$ and $\rho \in H_{\mathcal{L}}(a, c)$ let there be given a morphism ${}_{\mu}\rho \in H_{\mathcal{L}}(b, c)$ such that:

a) if $c, d \in \mathcal{L}^{\sigma}, c \neq a \neq d, \rho \in H_{\mathcal{L}}(a, c), \rho' \in H_{\mathcal{L}}(c, d)$, then ${}_{\mu}\rho \cdot \rho' = {}_{\mu}(\rho \cdot \rho')$;

b) if $c \neq a \neq d$, $\alpha \in H_{\mathcal{L}}(a, c)$, $\beta \in H_{\mathcal{L}}(c, a)$, $\gamma \in H_{\mathcal{L}}(a, d)$, $\delta \in H_{\mathcal{L}}(d, a)$, $\alpha \cdot \beta = \gamma \cdot \delta$, then ${}_{\mu}\alpha \cdot \beta = {}_{\mu}\gamma \cdot \delta$.

Then there exists a category $\mathcal{h}$ such that

1) $\mathcal{L}$ is a subcategory of $\mathcal{h}$ (denote by $\iota : \mathcal{L} \to \mathcal{h}$ the inclusion functor); $\mathcal{L}^{\sigma} = \mathcal{h}^{\sigma}$; if $c, d \in \mathcal{L}^{\sigma}$, $d \neq a$, then $H_{\mathcal{L}}(c, d) = H_{\mathcal{h}}(c, d)$; there exists a $\mu \in H_{\mathcal{h}}(b, a)$, such that $\mu \cdot \rho = {}_{\mu}\rho$ for all $\rho \in H_{\mathcal{h}}(a, d)$, $d \neq a$.

2) If $\Phi : \mathcal{L} \to K$ is a functor and if there exists a $\mu' \in H_K((b)\Phi, (a)\Phi)$ such that $\mu' \cdot (\rho)\Phi = ({}_{\mu}\rho)\Phi$ whenever $\rho \in H_{\mathcal{L}}(a, d)$, $d \neq a$, then there exists exactly one functor $\Psi : \mathcal{h} \to K$ with $\mu' = (\mu)\Psi$, $\Phi = \iota\Psi$.

3) If $\aleph$ is an infinite regular cardinal with $\mathrm{card}\, \mathcal{L}^{m} \leqq \aleph$, then $\mathrm{card}\, \mathcal{h}^{m} \leqq \aleph$. Moreover, if $\aleph$ is uncountable and if $H_{\mathcal{L}}(c, d) < \aleph$ for all $c, d \in \mathcal{L}^{\sigma}$, then $\mathrm{card}\, H_{\mathcal{h}}(c, d) < \aleph$ for all $c, d \in \mathcal{h}^{\sigma}$.

<u>Lemma B</u>: Let $\mathcal{L}$ be a category with a system of null morphisms, let $a, b \in \mathcal{L}^{\sigma}$. For every $c \neq a$ and $\rho \in H_{\mathcal{L}}(a, c)$ let there be given a morphism ${}_{\mu}\rho \in H_{\mathcal{L}}(b, c)$ such that the statements a) b) from lemma A are satisfied. Then there exists a category $\mathcal{h}$ with a system of null morphisms such that $\mathcal{L}$ is a subcategory of $\mathcal{h}$, the inclusion-functor $\iota : \mathcal{L} \to \mathcal{h}$ is a null functor, conditions

1) and 3) from lemma A are satisfied, and

2') if $\Phi : \ell \to K$ is a null functor and if there exists $\mu' \in H_K((b)\Phi, (a)\Phi)$ such that $\mu' \cdot (\rho)\Phi = (\mu\rho)\Phi$ whenever $\rho \in H_\ell(a,d)$, $d \neq a$, then there exists exactly one null functor $\Psi : h \to K$ with $\mu' = (\mu)\Psi$, $\Phi = \iota\Psi$.

The proofs, rather lengthy and not particularly interesting, are given in the Appendix.

I.3. Notation. Denote by $\mathbb{K}$ the category of all small categories and all their functors. Denote by $\mathbb{K}_n$ its subcategory consisting of all small categories with a system of null morphisms and all their null functors. Denote by $\mathbb{M}$ the category of all sets and all their mappings. Denote by $\mathcal{M}$ the functor, $\mathcal{M} : \mathbb{K} \to \mathbb{M}$, which assigns to every small category k the set k^m of all its morphisms.

Lemma A: Every directed presheaf (i.e. a small functor with a directed set as domain)in $\mathbb{K}$ (or in $\mathbb{K}_n$.) has a direct limit in $\mathbb{K}$ (or in $\mathbb{K}_n$ respectively).

Proof: Let $\mathcal{H}$ be a presheaf in $\mathbb{K}$ (or in $\mathbb{K}_n$), let $L = |\overrightarrow{lim}_{\mathbb{M}} \mathcal{H}\mathcal{M}|$. If $\mathcal{H}$ is a directed presheaf, then one may define the composition in L in the natural manner (i.e. if $\mathcal{H} : \langle \mathcal{J}, \prec \rangle \to \mathbb{K}$, $\langle L, \{v_j ; j \in \mathcal{J}\} \rangle = \overrightarrow{lim}_{\mathbb{M}} \mathcal{H}\mathcal{M}$, then for $\alpha, \beta, \gamma \in L$ put $\alpha \cdot \beta = \gamma$ if and only if there exists a $j \in \mathcal{J}$ and $\alpha', \beta', \gamma' \in (j)\mathcal{H}$ such that $(\alpha')v_j = \alpha$, $(\beta')v_j = \beta$, $(\gamma')v_j = \gamma$ and $\alpha' \cdot \beta' = \gamma'$); and then L is the set of all morphisms of some category h for which evidently $h = |\overrightarrow{lim}_{\mathbb{K}} \mathcal{H}|$ (or $h = |\overrightarrow{lim}_{\mathbb{K}_n} \mathcal{H}|$ respectively).

Note 1: Evidently, if $\mathcal{H}$ is a directed presheaf in $\mathbb{K}_m$, then $\overrightarrow{lim}_{\mathbb{K}_m} \mathcal{H} = \overrightarrow{lim}_{\mathbb{K}} \mathcal{H} I$ where $I: \mathbb{K}_m \to \mathbb{K}$ is the inclusion functor.

Note 2: Let k be a small category, let $\mathcal{H}: \langle \mathcal{J}, \dashv \rangle \to \mathbb{K}$ be a directed presheaf such that k is a full subcategory of $(j)\mathcal{H}$ for every $j \in \mathcal{J}$ and that the functor $\mathcal{H}_j^{j'}$ is identical on k for every $j \dashv j'$. It is easily seen that then there exists a direct limit $\langle h; \{\mathcal{V}_j \; ; \; j \in \mathcal{J}\} \rangle$ of $\mathcal{H}$ in $\mathbb{K}$ (or in $\mathbb{K}_m$ respectively) such that k is the full subcategory of h and each $\mathcal{V}_j$ is identical on k. Moreover, if $\mathcal{A}$ is a subset of the set of all morphisms for every category $(j)\mathcal{H}$ and if every $\mathcal{H}_j^{j'}$ is identical on $\mathcal{A}$, then $\mathcal{A} \subset h^m$ and every $\mathcal{V}_j$ is identical on $\mathcal{A}$.

Lemma B: Let $\mathcal{H}: \langle \mathcal{J}, \dashv \rangle \to \mathbb{K}$ be a directed presheaf in $\mathbb{K}$, set $h_j = (j)\mathcal{H}$, $h = |\overrightarrow{lim}_{\mathbb{K}} \mathcal{H}|$. If $\varkappa$ is an infinite cardinal such that $card\, \mathcal{J} \leqq \varkappa$, $card\, h_j^m \leqq \varkappa$ for all $j \in \mathcal{J}$, then $card\, h^m \leqq \varkappa$. Moreover, if $\varkappa$ is regular, $card\, \mathcal{J} < \varkappa$, $card\, H_{h_j}(c,d) < \varkappa$ for every $c, d \in h_j^\sigma$, $j \in \mathcal{J}$, then $card\, H_h(c,d) < \varkappa$ for every $c, d \in h^\sigma$.

Proof: The first part of the lemma is evident. Thus let $\varkappa$ be a regular cardinal, and assume that $card\, \mathcal{J} < \varkappa$, $card\, H_{h_j}(c,d) < \varkappa$ for every $c, d \in h_j^\sigma$, $j \in \mathcal{J}$. Set $\langle h; \{v_j \,;\, j \in \mathcal{J}\} \rangle = \overrightarrow{lim}_{\mathbb{K}} \mathcal{H}$. Let there exist $a, b \in h^\sigma$ such that $card\, H_h(a,b) \geqq \varkappa$. For every $\alpha \in H_h(a,b)$ choose some $j_\alpha \in \mathcal{J}$, $a_\alpha, b_\alpha \in h_{j_\alpha}^\sigma$, $\widetilde{\alpha} \in H_{h_{j_\alpha}}(a_\alpha, b_\alpha)$ such that $(a_\alpha)v_{j_\alpha} = a$, $(b_\alpha)v_{j_\alpha} = b$, $(\widetilde{\alpha})v_{j_\alpha} = \alpha$. Since $card\, \mathcal{J} < \varkappa$, there exist $\bar{j} \in \mathcal{J}$ and $\overline{H} \subset H_h(a,b)$

such that $card\ \overline{H} \geq \aleph$ and $\bar{j}_\alpha = \bar{j}$ for every $\alpha \in H$.
Choose some $\alpha \in \overline{H}$, and for every $\beta \in \overline{H}$ choose some $\bar{j}_\beta \in \mathcal{J}$ such that $\bar{j} \rightrightarrows \bar{j}_\beta$, $(a_\alpha)\mathcal{H}_{\bar{j}}^{\bar{j}_\beta} = (a_\beta)\mathcal{H}_{\bar{j}}^{\bar{j}_\beta}$.
Since $card\ \mathcal{J} < \aleph$, there exist $\tilde{j} \in \mathcal{J}$ and $\widetilde{H} \subset \overline{H}$ such that $card\ \widetilde{H} \geq \aleph$, $\bar{j}_\beta = \tilde{j}$ for every $\beta \in \widetilde{H}$. For every $\beta \in \widetilde{H}$ choose some $\tilde{j}_\beta \in \mathcal{J}$ such that $\tilde{j} \rightrightarrows \tilde{j}_\beta$, $(b_\alpha)\mathcal{H}_{\tilde{j}}^{\tilde{j}_\beta} = (b_\beta)\mathcal{H}_{\tilde{j}}^{\tilde{j}_\beta}$. Then there exist $H^* \subset \widetilde{H}$ and $j^* \in \mathcal{J}$ such that $card\ H^* \geq \aleph$, and $\tilde{j}_\beta = j^*$ for every $\beta \in H^*$. For every $\beta, \beta' \in H^*$ there is $(a_\beta)\mathcal{H}_{j_\beta}^{j^*} = (a_{\beta'})\mathcal{H}_{j_{\beta'}}^{j^*} = a^*$, $(b_\beta)\mathcal{H}_{j_\beta}^{j^*} = (b_{\beta'})\mathcal{H}_{j_{\beta'}}^{j^*} = b^*$.
Since $card\ H_{k_{j^*}}(a^*, b^*) < \aleph$, there exist $\beta, \beta' \in H^*$, $\beta \neq \beta'$ such that $(\overline{\beta})\mathcal{H}_{j_\beta}^{j^*} = (\overline{\beta'})\mathcal{H}_{j_{\beta'}}^{j^*}$; however, this is impossible, since $\beta = (\overline{\beta})v_{j_\beta} = (\overline{\beta})(\mathcal{H}_{j_\beta}^{j^*} \cdot v_{j^*}) = (\overline{\beta'})(\mathcal{H}_{j_{\beta'}}^{j^*} \cdot v_{j^*}) = (\overline{\beta'})v_{j_{\beta'}} = \beta'$.

I.4. <u>Notation and definitions</u>: Let $\mathcal{F}: \mathcal{J} \to k$ be a diagram in a category k, let $\langle a; \{v_i ; i \in \mathcal{J}^\sigma\}\rangle$ be its direct limit in k. Denote by $P_{\mathcal{F}}$ the set of all v_i, $(i \in \mathcal{J}^\sigma)$, denote by $T_{\mathcal{F}}$ the s e t of all triples $\langle v_i, (\sigma)\mathcal{F}, v_{i'} \rangle$ where $i, i' \in \mathcal{J}^\sigma$, $\sigma \in H_{\mathcal{J}}(i, i')$.
The couple $\langle P_{\mathcal{F}}; T_{\mathcal{F}} \rangle$ will be called the d i r e c t s u b s t a n c e of the diagram $\mathcal{F}$ in the category k. Let k be a category. We shall say that two diagrams which both have a direct limit in k are directly equivalent if they have the same direct substance. Evidently direct equivalence is a reflexive, symmetric and transitive relation on the class x)

x) The class $\mathbf{D}$ is often a proper class (i.e. not a set) even for a small category k and therefore the notion of the direct substance was introduced

$\mathbb{D}$ of all diagrams in $\mathcal{k}$ which have a direct limit in $\mathcal{k}$. If $\mathbb{G}$ is a class of diagrams in $\mathcal{k}$, denote by $\mathbb{V}$ some choice-class of $\mathbb{G} \cap \mathbb{D}$ (i.e. no two distinct diagrams from $\mathbb{V}$ are directly equivalent, and for every diagram from $\mathbb{G} \cap \mathbb{D}$ there exists a diagram in $\mathbb{V}$ which is directly equivalent with it) and call it the directly substantial class of diagrams from $\mathbb{G}$. If $\mathcal{k}$ is small, then evidently $\mathbb{V}$ is a set. Now let a category $\mathcal{k}$ be a full subcategory of some category K , and denote by $\iota : \mathcal{k} \to K$ the inclusion-functor. Let $\mathcal{F} : \mathcal{J} \to \mathcal{k}$ be a diagram which has a direct limit in $\mathcal{k}$, denoted by $\langle a ; \{ v_i ; i \in \mathcal{J}^\sigma \} \rangle$. A direct bound $\langle b ; \{ \varphi_i ; i \in \mathcal{J}^\sigma \} \rangle$ of $\mathcal{F}\iota$ in K will be called the direct bound of the direct substance of $\mathcal{F}$ in $\mathcal{k}$ if $\varphi_i = \varphi_{i'}$ whenever $v_i = v_{i'}$. (If $\langle b ; \{ \varphi_i ; i \in \mathcal{J}^\sigma \} \rangle$ is a direct bound of $\mathcal{F}$ in $\mathcal{k}$, then it is evidently the direct bound of the direct substance of $\mathcal{F}$ in $\mathcal{k}$.)

I.5. <u>Lemma</u>: Let $\mathcal{k}$ be a full subcategory of a category K , and $\iota : \mathcal{k} \to K$ the inclusion-functor. Let diagrams $\mathcal{F}, \mathcal{G}$ be directly equivalent in $\mathcal{k}$. Let every direct bound in K of the diagram $\mathcal{G}\iota$ be the direct bound of the direct substance of $\mathcal{G}$ in $\mathcal{k}$. If ι preserves the direct limit of $\mathcal{F}$, then it also preserves the direct limit of $\mathcal{G}$.

<u>Proof</u>: Let $\langle a ; \{ v_i ; i \in \mathcal{J}^\sigma \} \rangle$ or $\langle b ; \{ w_j ; j \in \mathcal{J}^\sigma \} \rangle$ be direct limits in K of the diagram $\mathcal{F} : \mathcal{J} \to \mathcal{k}$ or $\mathcal{G} : \mathcal{J} \to \mathcal{k}$ respectively. Let $\langle c ; \{ \psi_j ; j \in \mathcal{J}^\sigma \} \rangle$ be a direct bound in K of $\mathcal{G}\iota$. We must prove that there exists exactly one morphism α in K such that

$w_j \cdot \alpha = \psi_j$ for every $j \in \mathcal{J}^\sigma$. Since $\psi_j = \psi_{j'}$ whenever $w_j = w_{j'}$, the mapping g such that $(w_j)g = \psi_j$ maps the set $P_{\mathcal{G}}$ onto the set of all ψ_j. But $P_{\mathcal{G}} = P_{\mathcal{F}}$, and therefore g maps the set $P_{\mathcal{F}}$ onto the set of all ψ_j. Now we shall show that $\langle c; \{(v_i)g; i \in \mathcal{J}^\sigma\}\rangle$ is the direct bound in K of $\mathcal{F}\iota$. If $i, i' \in \mathcal{J}^\sigma$, $\sigma \in H_y(i, i')$, then $\langle v_i, (\sigma)\mathcal{F}, v_{i'}\rangle \in T_{\mathcal{F}} = T_{\mathcal{G}}$; consequently $\langle v_i, (\sigma)\mathcal{F}, v_{i'}\rangle = \langle w_j, (\rho)\mathcal{G}, w_{j'}\rangle$ for some $j, j' \in \mathcal{J}^\sigma$, $\rho \in H_y(j, j')$. But then $\psi_j = (\rho)\mathcal{G} \cdot \psi_{j'}$, and therefore $(v_i)g = (\sigma)\mathcal{F} \cdot (v_{i'})g$. Now it is easy to see that a morphism α in K is the canonical morphism in K of the direct bound $\langle c; \{(v_i)g; i \in \mathcal{J}^\sigma\}\rangle$ of $\mathcal{F}_\iota$ if and only if $w_j \cdot \alpha = \psi_j$ for every $j \in \mathcal{J}^\sigma$.

I.6. __Lemma__: Let $\mathcal{k}$ be a small category, and $\mathcal{F}$ a diagram in $\mathcal{k}$. Then there exists a category $\mathcal{k}_o$ such that

1) $\mathcal{k}_o^\sigma - \mathcal{k}^\sigma = \{a\}$; $\mathcal{k}$ is a full subcategory of $\mathcal{k}_o$ (denote by $\bar{\iota}$ the inclusion-functor); there exists a direct bound $\langle a; \{v_i; i \in \mathcal{J}^\sigma\}\rangle$ of $\mathcal{F}\bar{\iota}$ in $\mathcal{k}_o$ such that
   - a) if $\langle b; \{\psi_i; i \in \mathcal{J}^\sigma\}\rangle$ is a direct bound of $\mathcal{F}$ in $\mathcal{k}$, then there exists an $f \in H_{\mathcal{k}_o}(a, b)$ such that $v_i \cdot f = \psi_i$ for all $i \in \mathcal{J}^\sigma$;
   - b) if $f, f' \in H_{\mathcal{k}_o}(a, c)$, $c \neq a$, $f \neq f'$, then $v_i \cdot f \neq v_i \cdot f'$ for some $i \in \mathcal{J}^\sigma$.
2) If $\Phi: \mathcal{k} \to K$ is a functor and $\mathcal{F}\Phi$ has a direct limit in K, then there exists a functor $\Psi: \mathcal{k}_o \to K$, unique up to natural equivalence, such that $\bar{\iota}\,\Psi = \Phi$ and that $\langle (a)\Psi; \{(v_i)\Psi; i \in \mathcal{J}^\sigma\}\rangle$ is a direct

limit of $\mathcal{F}\Phi$ in $\mathcal{k}$. If $\aleph$ is a strongly inaccessible cardinal with $card\ \mathcal{J}^m < \aleph$ and $card\ \mathcal{k}^m \leqq \aleph$, then $card\ \mathcal{k}_o^m \leqq \aleph$. Moreover, if $card\ H_{\mathcal{k}}(c, d) < \aleph$ for all $c, d \in \mathcal{k}^\sigma$, then $card\ H_{\mathcal{k}_o}(c,d) < \aleph$ for all $c, d \in \mathcal{k}_o^\sigma$.

Proof: I. Denote by $\mathbb{A}$ the set of all direct bounds $\alpha = \langle b_\alpha ; \{\psi_{i,\alpha} ; i \in \mathcal{J}^\sigma\} \rangle$ of the diagram $\mathcal{F} : \mathcal{J} \to \mathcal{k}$ in $\mathcal{k}$. ($\mathbb{A} = \emptyset$ is not excluded.) Let $\mathcal{A}$ be a set of elements f_α, where α varies over $\mathbb{A}$, such that $\mathcal{A} \cap \mathcal{k}^m = \emptyset$, $f_\alpha \neq f_{\alpha'}$, whenever $\alpha \neq \alpha'$. For $\mu \in H_{\mathcal{k}}(b_\alpha, c)$, put $f_\alpha \cdot \mu = f_{\alpha'}$, where $\alpha' = \langle c ; \{\psi_{i,\alpha} \cdot \mu ; i \in \mathcal{J}^\sigma\} \rangle \in \mathbb{A}$. Set $s_i = (i)\mathcal{F}$ for $i \in \mathcal{J}^\sigma$. Let $\mathbb{I}$ be a set of elements $\overline{v_i}$, where i varies over $\mathcal{J}^\sigma$, such that $\overline{v_i} \neq \overline{v_{i'}}$ whenever $i \neq i'$ and $[\mathbb{I} \cup (Z \times \mathbb{I}) \cup (\mathbb{I} \times Z)] \cap Z = \emptyset$ for $Z = \mathcal{k}^m \cup \mathcal{A}$. Put $\overline{v_i} \cdot f_\alpha = \psi_{i,\alpha}$. Denote by Q the set of all couples $\langle f_\alpha , \overline{v_i} \rangle$ such that $b_\alpha = s_i$. Put $\langle f_\alpha , \overline{v_i} \rangle \cdot \langle f_{\alpha'} , \overline{v_{i'}} \rangle = \langle f_\alpha \cdot (\overline{v_i} \cdot f_{\alpha'}), \overline{v_{i'}} \rangle$. It is easily shown that this composition on Q is associative. Let e be an element, $e \notin Z \cup (Z \times \mathbb{I}) \cup (\mathbb{I} \times Z)$, set $\Sigma = Q \cup \{e\}$ and $e \cdot \sigma = \sigma \cdot e = \sigma$ for every $\sigma \in Q$. Put $\sigma \cdot f_\beta = f_\beta$ whenever $\sigma = e$, $\beta \in \mathbb{A}$ and $\sigma \cdot f_\beta = f_\alpha \cdot (\overline{v_i} \cdot f_\beta)$ whenever $\sigma = \langle f_\alpha , \overline{v_i} \rangle \in Q$, $\beta \in \mathbb{A}$. Let a be an element such that $a \notin \mathcal{k}^\sigma$. Let $\mathcal{k}^*$ be a category with the following properties: $(\mathcal{k}^*)^\sigma = \mathcal{k}^\sigma \cup \{a\}$, $\mathcal{k}$ is a full subcategory of $\mathcal{k}^*$; if $b \in \mathcal{k}^\sigma$, then $H_{\mathcal{k}^*}(a, b)$ is the set of all f_α such that $b_\alpha = b$; $H_{\mathcal{k}^*}(b, a)$ is the set of all couples $\langle \mu , \overline{v_i} \rangle$ where $\mu \in H_{\mathcal{k}}(b, s_i)$ and $H_{\mathcal{k}^*}(a, a) = \Sigma$.

The definition of the composition in k^* is evident (of course, if $A = \emptyset$, then $\Sigma = \{e\}$, $H_{k^*}(a, b) = \emptyset$ for every $b \in k^\sigma$). Denote by $\iota^* : k \to k^*$ the inclusion-functor. Let now R be the following relation on $(k^*)^m$:
$\langle \mu, \overline{v}_i \rangle \; R \; \langle \mu . (\sigma)\mathcal{F}, \overline{v}_{i'} \rangle$ for every $i, i' \in \mathcal{J}^\sigma$, $\sigma \in H_{\mathcal{J}}(i, i')$. Then evidently $\langle \mu, \overline{v}_i \rangle f_\alpha = \langle \mu . (\sigma)\mathcal{F}, \overline{v}_{i'} \rangle . f_\alpha$ for all $\alpha \in A$, and lemma I.1 and note I.1 may be applied. Put $k_0 = k^*/R$. Set $e_a = (e)\,1/R$, $v_i = (\langle e_{s_i}, \overline{v}_i \rangle)\,1/R$, $\overline{\iota} = \overset{*}{\iota} . 1/R$.
Evidently k_0 satisfies 1) from lemma I.6.
II. Now let $\Phi : k \to K$ be a functor such that $\mathcal{F}\Phi$ has a direct limit in K; denote it by $\langle a' ; \{ v_i' ; i \in \mathcal{J}^\sigma \} \rangle$. We proceed to define $\Phi^* : k^* \to K$. Of course Φ^* is to be the extension of Φ; put $(a)\Phi^* = a'$ and $(\langle \mu, \overline{v}_i \rangle)\Phi^* = (\mu)\Phi . v_i'$, $(f_\alpha)\Phi^* = f_\alpha'$, where f_α' is the canonical morphism in K of the direct bound $\langle (b_\alpha)\Phi ; \{ (\psi_{i,\alpha})\Phi ; i \in \mathcal{J}^\sigma \} \rangle$. Evidently, if $x \, R \, x'$, then $(x)\Phi^* = (x')\Phi^*$; consequently, using lemma I.1, there exists exactly one $\Psi : k_0 \to K$ such that $\Phi^* = 1/R . \Psi$.
III. It is easy to see that $\mathrm{card}\, H_{k_0}(b, a) \leq \mathrm{card} \bigcup_{i \in \mathcal{J}^\sigma} H_k(b, s_i)$ for every $b \in k^\sigma$. For $b \in k^\sigma$ set $A_b = \{\alpha \in A ; b_\alpha = b\}$; then $A_b \leq \mathrm{card} \prod_{i \in \mathcal{J}^\sigma} H_k(s_i, b)$; of course $\mathrm{card}\, H_{k_0}(a, b) \leq \mathrm{card}\, A_b$ for all $b \in k^\sigma$, $\mathrm{card}\, H_{k_0}(a, a) \leq \mathrm{card} \bigcup_{i \in \mathcal{J}^\sigma} A_{s_i} + 1$. Consequently, if $\aleph$ is a strongly inaccessible cardinal number with $\mathrm{card}\, \mathcal{J}^\sigma < \aleph$ and $\mathrm{card}\, k^m \leq \aleph$, then evidently $\mathrm{card}\, k_0^m \leq \aleph$. Moreover, if $\mathrm{card}\, H_k(c, d) < \aleph$ for all $c, d \in k^\sigma$, then $\mathrm{card}\, H_{k_0}(c, d) < \aleph$ for all $c, d \in k_0^\sigma$.

Note: It is easy to see that the following lemma can be proved easily:
Let $\mathfrak{k}$ be a small category with a system of null morphisms, $\mathcal{F}$ a diagram in $\mathfrak{k}$. Then there exists a category $\mathfrak{k}_0$ with a system of null morphisms such that statement 1) from lemma I.6 holds, and that for every null functor $\Phi : \mathfrak{k} \to K$ such that $\mathcal{F}\Phi$ has a direct limit in K there exists a null functor $\Psi : \mathfrak{k}_0 \to K$ unique up to natural equivalence, such that $\tau \Psi = \Phi$ and that $\langle (a)\Psi ; \{(v_i)\Psi ; i \in \mathcal{J}^\sigma\} \rangle$ is a direct limit of $\mathcal{F}\Phi$ in K.
The proof of lemma I.6 should be modified as follows:
If $\alpha = \langle b_\alpha ; \{ \psi_{i,\alpha} ; i \in \mathcal{J}^\sigma \} \rangle$ is direct bound of $\mathcal{F}$ in $\mathfrak{k}$ such that all $\psi_{i,\alpha}$ are null morphisms of $\mathfrak{k}$, then denote f_α by ω_α. Evidently $\omega_\alpha \cdot \mu = \omega_{\alpha'}$ for every $\mu \in H_{\mathfrak{k}}(b_\alpha , b_{\alpha'})$. The category $\mathfrak{k}_0$ may be constructed as in the proof of lemma I.6, changing only the relation R: put $R = R_1 \cup R_2 \cup R_3$, where $\langle \mu, \overline{v}_i \rangle \; R_1 \langle \mu.(\sigma)\mathcal{F}, \overline{v}_{i'} \rangle$ for every $i, i' \in \mathcal{J}^\sigma$, $\sigma \in H_{\mathcal{J}}(i, i')$; $\langle \mu, \overline{v}_i \rangle \; R_2 \langle \nu, \overline{v}_{i'} \rangle$ for null morphisms μ, ν, $\mu \in H_{\mathfrak{k}}(c, s_i)$, $\nu \in H_{\mathfrak{k}}(c, s_{i'})$ and every $i, i' \in \mathcal{J}^\sigma$; $\langle \omega_\alpha , \overline{v}_i \rangle \; R_3 \langle \omega_{\alpha'}, \overline{v}_{i'} \rangle$ for every $i, i' \in \mathcal{J}^\sigma$.

I.7. **Lemma**: Let $\mathfrak{k}$ be a full subcategory of a category K, let $\iota : \mathfrak{k} \to K$ be the inclusion-functor. Let there exist a diagram $\mathcal{F}^a$ in $\mathfrak{k}$ for every $a \in K^\sigma$ such that $a = | \overrightarrow{\lim}_K \mathcal{F}^a \iota |$. Then ι is $\overleftarrow{\mathfrak{k}}^{z}$-preserving.
Proof: Let a diagram $\mathcal{G}: \mathcal{J} \to \mathfrak{k}$ have an inverse limit $\langle d ; \{\varphi_j ; j \in \mathcal{J}^\sigma\} \rangle$ in $\mathfrak{k}$. Let $\langle a ; \{\chi_j ; j \in \mathcal{J}^\sigma\} \rangle$ be an inverse bound of $\mathcal{G}\iota$ in K. We shall prove that

it has a canonical morphism in K. Let $\mathcal{F}: \mathcal{J} \to \mathcal{k}$ be a diagram such that the direct limit of $\mathcal{F}\iota$ in K is $\langle a; \{v_i;$ $i \in \mathcal{J}^\sigma\}\rangle$. Set $s_i = (i)\mathcal{F}$. Evidently the couple $\langle s_i;$ $\{v_i \cdot \chi_j;\ j \in \mathcal{J}^\sigma\}\rangle$ is an inverse bound of $\mathcal{Y}$ in $\mathcal{k}$ for every $i \in \mathcal{J}^\sigma$; denote by g_i its canonical morphism in $\mathcal{k}$. Now it is easy to see that the couple $\langle d; \{g_i; i \in \mathcal{J}^\sigma\}\rangle$ is the direct bound of $\mathcal{F}\iota$ in K; denote by f its canonical morphism in K. Then $v_i \cdot f \cdot \varphi_j = v_i \cdot \chi_j$ for every $i \in \mathcal{J}^\sigma$, $j \in \mathcal{J}^\sigma$, and therefore $f \cdot \varphi_j = \chi_j$ for every $j \in \mathcal{J}^\sigma$. If also $f' \cdot \varphi_j = \chi_j$ for some f', then necessarily $f' = f$, as can be shown easily.

I.8. <u>Lemma</u>: Let $\mathcal{k}$ be a small category, $\mathcal{F}$ a diagram in $\mathcal{k}$. Then there exists a small category K such that $\mathcal{k}$ is a full subcategory of K, the inclusion-functor $\iota: \mathcal{k} \to$ $\to K$ is $(\overrightarrow{\mathcal{k}^{z}}, \overleftarrow{\mathcal{k}^{z}})$-preserving and $\mathcal{F}\iota$ has a direct limit in K.

<u>Proof</u>: I. Let $\mathcal{F}: \mathcal{J} \to \mathcal{k}$ be a diagram in a small category $\mathcal{k}$. We may suppose that $\mathcal{F}$ has no direct limit in $\mathcal{k}$. We shall construct a category K with the required properties. Denote by A the set of all direct bounds $\alpha =$ $= \langle b_\alpha; \{\psi_{i,\alpha}; i \in \mathcal{J}^\sigma\}\rangle$ of $\mathcal{F}$ in $\mathcal{k}$. If $A = \emptyset$, then K with the required properties may be found easily. It is sufficient to adjoin one object a to the category $\mathcal{k}$, such that $H(a, b) = \emptyset$ and $H(b, a)$ contains exactly one morphism for every $b \in \mathcal{k}^\sigma$, $H(a, a) = \{e_a\}$. Consequently we may suppose that $A \neq \emptyset$.

II. Let $\mathcal{k}_0$ be a category satisfying the statements of lemma I.6. The notation from the proof of lemma I.6 will be used. Let S be the following relation on

$\bigcup_{c \in \mathcal{k}_0^\sigma} H_{\mathcal{k}_0}(c,a): \mu S \nu \Longleftrightarrow \mu \cdot f_\alpha = \nu \cdot f_\alpha$ for every $\alpha \in \mathcal{A}$. Put $K^o = \mathcal{k}_0/_S$, set $v_i^o = (v_i)^1/_S$ (cf. Note I.1). Set $\iota^o : \mathcal{k} \to K^o$, $\iota^o = \bar{\iota} \cdot {}^1/_S$.

III. Denote by $\mathbb{D}$ the class of all diagrams in $\mathcal{k}$ which have a direct limit in $\mathcal{k}$. In the present proof the following terminology will be used: if $\Gamma : K^o \to H$ is a functor, $m = \langle d; \{\chi_j ; j \in \mathcal{J}^\sigma\} \rangle$ is a direct bound of $\mathcal{Y}\,\iota^o\,\Gamma$ in H , where either $\mathcal{Y} \in \mathbb{D}$ or $\mathcal{Y} = \mathcal{F}$, and if $\langle b; \{\xi_j ; j \in \mathcal{J}^\sigma\} \rangle = \overrightarrow{lim}_{\mathcal{k}}\, \mathcal{Y}$ for $\mathcal{Y} \in \mathbb{D}$, $\langle b; \{\xi_j ; j \in \mathcal{J}^\sigma\} \rangle = \langle a; \{v_i^o ; i \in \mathcal{I}^\sigma\} \rangle$ for $\mathcal{Y} = \mathcal{F}$, then we shall call every morphism $\mu \in H_H((b)\Gamma, d)$ such that $(\xi_j)\Gamma \cdot \mu = \chi_j$ a canonical morphism of m (of $\mathcal{Y}\,\iota^o\,\Gamma$) in H .

IV. Let V be a directly substantial set of diagrams from $\mathbb{D}$ (cf.I.4) in $\mathcal{k}$. Let m be a regular cardinal, $m > card\ \mathcal{F}$, $m > card\ \mathcal{Y}$ for all $\mathcal{Y} \in V$. For ordinal s denote by $\mathbb{T}_s$ the set of all ordinals less than s . Let κ be the smallest ordinal such that $card\ \kappa = m$. A transfinite construction will be performed according to elements of the set $\mathbb{T}_\kappa$.

V. Let $s \in \mathbb{T}_\kappa$, and assume that ${}^s\mathcal{T} : \langle \mathbb{T}_s , < \rangle \to \mathbb{K}$ is an inductive presheaf x) in the category $\mathbb{K}$ of all small

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x) A presheaf $\mathcal{T}$ is called inductive if its domain is a directed set P and if $P' \subset P$, $p = sup\ P'$ imply that $(p)\mathcal{T}$ is a direct limit of $\mathcal{T}$ restricted to P' .

categories and all their functors, such that (writing $K^{u} = (u)^{s}\mathcal{T}$):

0) K^{o} is the category constructed in II of the present proof;

1) a) $(K^{u})^{\sigma} = k^{\sigma} \cup \{a\}$, k is a full subcategory of K^{u}; $H_{K^{u}}(a, c) = H_{K^{o}}(a, c)$ for every $c \in k^{\sigma}$;

   b) if $u < u'$ then the functor ${}^{s}\mathcal{T}_{u}^{u'}$ is identical on all of k, a, $\mathcal{A} = \bigcup_{c \in k} H_{K^{u}}(a, c) = \{f_{\alpha};\ \alpha \in \mathbb{A}\}$.

2) If $u' = u + 1$ then every direct bound $(m)^{s}\mathcal{T}_{u}^{u'}$, where m is a direct bound in K^{u} of $\mathcal{Y}\, \iota^{o}\, {}^{s}\mathcal{T}_{o}^{u}$ with either $\mathcal{Y} = \mathcal{F}$ or $\mathcal{Y} \in V$, has a canonical morphism in $K^{u'}$.

3) Every category K^{u} satisfies the following condition ($*$): if $\gamma \cdot f_{\alpha}$ is defined and if $\gamma \cdot f_{\alpha} = \delta \cdot f_{\alpha}$ for all $\alpha \in \mathbb{A}$, then $\gamma = \delta$.

VI. The properties 0) - 3) imply:

   a) $(v_{i}^{o})^{s}\mathcal{T}_{o}^{u} \cdot f_{\alpha} = \psi_{i,\alpha}$ for every $i \in \mathcal{J}^{\sigma}$, $\alpha \in \mathbb{A}$, $u \in T_{s}$.

   b) Every direct bound in K^{u} of every $\mathcal{Y}\, \iota^{o}\, {}^{s}\mathcal{T}_{o}^{u}$, where $\mathcal{Y} \in \mathcal{D}$, is the direct bound of the direct substance of $\mathcal{Y}$ in k. For, if $\langle \dot{a}; \{\chi_{j}; j \in \mathcal{J}^{\sigma}\}\rangle$ is a direct bound of $\mathcal{Y}\, \iota^{o}\, {}^{s}\mathcal{T}_{o}^{u}$ in K^{u}, then $\langle b_{\alpha}; \{\chi_{j} \cdot f_{\alpha}; j \in \mathcal{J}^{\sigma}\}\rangle$ is the direct bound of $\mathcal{Y}$ i n k, which must have the canonical morphism in k. Then use ($*$).

   c) Every direct bound of $\mathcal{Y}\, \iota^{o}\, {}^{s}\mathcal{T}_{o}^{u}$, where either $\mathcal{Y} = \mathcal{F}$ or $\mathcal{Y} \in \mathcal{D}$, has at most one canonical morphism in K^{u}. This also follows from ($*$).

VII. We shall construct $K^s, \varkappa_u^s, (u \in T_s)$ such that the presheaf ${}^{s+1}\mathcal{T}: \langle T_{s+1}, < \rangle \to \mathbb{K}$ which is an extension of ${}^s\mathcal{T}$ and $(s)^{s+1}\mathcal{T} = K^s$, $(\langle u, s \rangle)^{s+1}\mathcal{T} = \varkappa_u^s$, will be an inductive presheaf satisfying 0) - 3) from V. For s a non-isolated ordinal put $\langle K^s; \{\varkappa_u^s; u \in T_s\} \rangle = \overrightarrow{\lim}_{\mathbb{K}} {}^s\mathcal{T}$, where K^s is chosen so that it contains k, a, A and all $\varkappa_u^s$ are identical on of all k, a, A (cf. Note I.3). Then it is easy to see that 0) - 3) are satisfied. Thus, let $s = t + 1$. Then it is sufficient to construct K^s and $\varkappa_t^s$. Let P be the set of all direct bounds in K^t of $\mathcal{Y} \iota^o\, {}^s\mathcal{T}_o^t$, where either $\mathcal{Y} = \mathcal{F}$ or $\mathcal{Y} \in \in \mathbb{V}$, which have no canonical morphism in K^t. For $P = = \emptyset$ put $K^s = K^t$, $\varkappa_t^s$ identical. Now let $\mathbb{P} \neq \emptyset$. Let p be an ordinal such that there exists a one-to-one mapping $\mathfrak{p}$ of the set of all positive isolated ordinal numbers of the set T_p onto $\mathbb{P}$. We shall construct K^s by transfinite induction according to elements of T_p.

VIII. Let $q \in T_p$ and let an inductive presheaf ${}^q\mathcal{Y}$: $\langle \mathbb{T}_q, < \rangle \to \mathbb{K}$ be constructed such that (setting $H^w = = (w)\,{}^q\mathcal{Y}$):

0*) $H^o = K^t$;

1*) is analogous to 1), and 3*) to 3);

2*) if $w' = w + 1$, then the direct bound $(w')\mathfrak{p}\; {}^q\mathcal{Y}_o^{w'}$ has the canonical morphism in $H^{w'}$.

We shall construct H^q, λ_w^q for $w \in T_q$ so that the presheaf ${}^{q+1}\mathcal{Y}: \langle T_{q+1}, < \rangle \to \mathbb{K}$, where ${}^{q+1}\mathcal{Y}$ is an extension of ${}^q\mathcal{Y}$ and that $(q)^{q+1}\mathcal{Y} = H^q$, ${}^{q+1}\mathcal{Y}_w^q = \lambda_w^q$, will be an inductive presheaf satisfying 0*) - 3*).

If q is a non-isolated ordinal, the construction is evident.

IX. Let $q = x+1$. Then it is required to construct the category H^q and λ^q_x ($\lambda^q_u = {}^q\mathcal{S}^x_u \cdot \lambda^q_x$ for $u < x$, of course). Let $(q)p = m = \langle d; \{\bar{\chi}_j; j \in \mathcal{J}^\sigma\}\rangle$ be a direct bound in K^t of $\mathcal{G}\, \iota^\circ\, {}^s\mathcal{T}^t_o$, where either $\mathcal{G} = \mathcal{F}$ or $\mathcal{G} \in V$. Then m has no canonical morphism in K^t, and thus $d = a$. Set $(\bar{\chi}_j)\,{}^q\mathcal{S}^x_o = \chi_j$; denote by g_α the canonical morphism of the direct bound $\langle b_\alpha; \{\chi_j \cdot f_\alpha; j \in \mathcal{J}^\sigma\}\rangle$ in H^x (cf. part III of the present proof). Now use lemma I.2, writing H^x, H^* instead of ℓ, h and putting $b = |\overrightarrow{lim}_k\, \mathcal{G}|$ whenever $\mathcal{G} \in V$, $b = a$ whenever $\mathcal{G} = \mathcal{F}$, and putting $\mu(f_\alpha) = g_\alpha$ for every $\alpha \in A$. Denote by $\iota^*: H^x \to H^*$ the inclusion functor. Let Z be the following equivalence on $(H^*)^m$:

$\beta Z \gamma \Longleftrightarrow \beta \cdot f_\alpha, \gamma \cdot f_\alpha$ are defined and $\beta \cdot f_\alpha = \gamma \cdot f_\alpha$ for all $\alpha \in A$. Put $H^q = H^*/Z$, $\lambda^q_x = \iota^* \cdot 1/Z$ (cf. Note I.1). If $\mu \in H_{H^*}(b, a)$ is such that $\mu \cdot f_\alpha = g_\alpha$ then evidently $(\mu)\,1/Z$ is the canonical morphism of the direct bound $(m)\,{}^q\mathcal{S}^x_o\, \lambda^q_x$ in H^q. Indeed, putting $\langle b; \{\xi_j; j \in \mathcal{J}^\sigma\}\rangle = \overrightarrow{lim}_k\, \mathcal{G}$ whenever $\mathcal{G} \in V$, and $\langle b; \{\xi_j; j \in \mathcal{J}^\sigma\}\rangle = \langle a; \{(v_i^\circ)\,{}^s\mathcal{T}^t_o\, {}^q\mathcal{S}^x_o; i \in \mathcal{J}^\sigma\}\rangle$ whenever $\mathcal{G} = \mathcal{F}$, there is $\xi_j \cdot \mu \cdot f_\alpha = \xi_j \cdot g_\alpha = \chi_j \cdot f_\alpha$ for all $j \in \mathcal{J}^\sigma$, $\alpha \in A$ and thus $(\xi_j \cdot \mu)\,1/Z = (\chi_j)\,1/Z$. Evidently conditions 0*) - 3*) are satisfied.

X. Then, using transfinite induction, the inductive presheaf ${}^p\mathcal{S}: \langle T_p, <\rangle \to K$ satisfying the statements 0*) - 3*) is defined. Set $\langle K^s; \{\lambda^s_w; w \in T_p\}\rangle = \overrightarrow{lim}_{K}\, {}^p\mathcal{S}$, put $\varkappa^s_t = \lambda^s_o$ (where K^s is so chosen that it contains k, a, A and that all λ^s_w are identical on all of k, a, A). Then, evidently, conditions 0) - 3) are satisfied for the presheaf ${}^{s+1}\mathcal{T}$. Using transfinite induction, the inductive presheaf ${}^{r}\mathcal{T}: \langle T_r , < \rangle \to K$ satisfying conditions 0) - 3) is defined. Of course we put $\langle K ; \{ \varkappa_u ; u \in T_r \} \rangle = \overrightarrow{\lim}_K {}^r\mathcal{T}$, where K is so chosen that it contains k, a, A and that all $\varkappa_u$ are identical on k, a, A . Then evidently k is a full subcategory of K, $\iota = \iota^o \cdot \varkappa_o$ is the inclusion-functor and $\langle a ; \{ (v_i^o)\varkappa_o ; i \in \mathcal{J}^o \} \rangle$ is the direct bound of $\mathcal{F}\iota$ in K .

XI. Now we must prove that ι preserves direct limits of all diagrams in k and that $\mathcal{F}\iota$ has a direct limit in K. Evidently K satisfies condition (*), and therefore every direct bound of $\mathcal{Y}\iota$ in K , where $\mathcal{Y} \in D$, is the direct bound of the direct substance of $\mathcal{Y}$ in k (cf. part VI b) of the present proof). Using lemma I.5 it is sufficient to prove that ι preserves direct limits of all $\mathcal{Y} \in V$ and that $\mathcal{F}\iota$ has a direct limit in K . Evidently, every direct bound in K of $\mathcal{Y}\iota$, where either $\mathcal{Y} \in V$ or $\mathcal{Y} = \mathcal{F}$, has at most one canonical morphism in K ; this follows from (*). We must prove that it has at least one canonical morphism. Consequently let $\langle d ; \{ \chi_j ; j \in \mathcal{J}^\sigma \} \rangle = m$ be a direct bound of $\mathcal{Y}\iota$ in K , where either $\mathcal{Y} \in V$ or $\mathcal{Y} = \mathcal{F}$, $\mathcal{Y} : \mathcal{J} \to k$. It is sufficient to consider the case $d = a$. We shall find $\bar{u} \in T_r$ and a direct bound m' of $\mathcal{Y}\iota^{o} {}^{r}\mathcal{T}_o^{\bar{u}}$ in $K^{\bar{u}}$ such that $(m')\varkappa_{\bar{u}} = m$. If $j, j' \in \mathcal{J}^\sigma$, $\rho \in H_{\mathcal{J}}(j, j')$, then there exist $u_\rho \in T_r$, $\chi_j^\rho , \chi_{j'}^\rho \in (K^{u_\rho})^m$ such that $\chi_j^\rho = (\rho)\mathcal{Y} \cdot \chi_{j'}^\rho$, $(\chi_j^\rho)\varkappa_{u_\rho} = \chi_j$, $(\chi_{j'}^\rho)\varkappa_{u_\rho} = \chi_{j'}$. For $j \in \mathcal{J}^\sigma$ set $\tilde{M}_j = \bigcup_{j' \in \mathcal{J}^\sigma} H_{\mathcal{J}}(j, j')$, $\tilde{\tilde{M}}_j = \bigcup_{j' \in \mathcal{J}^\sigma} H_{\mathcal{J}}(j', j)$, $M_j = (\tilde{M}_j \cup \tilde{\tilde{M}}_j) \times (\tilde{M}_j \cup \tilde{\tilde{M}}_j)$.

If $\langle \sigma, \sigma' \rangle \in M_j$, then there exists a $v_{\sigma,\sigma'} \in T_n$ such that $(\chi_j^{\sigma})^{\varkappa}\mathcal{T}_{u_\sigma}^{v_{\sigma,\sigma'}} = (\chi_j^{\sigma'})^{\varkappa}\mathcal{T}_{u_{\sigma'}}^{v_{\sigma,\sigma'}}$. Put $v_j = \sup_{\langle\sigma,\sigma'\rangle \in M_j} v_{\sigma,\sigma'}$. Since $\operatorname{card} n > \operatorname{card} \mathcal{J}$, there is $v_j \in T_n$. Put $\bar{u} = \sup_{j \in \mathcal{J}^\sigma} v_j$ and $\bar{\chi}_j = (\chi^{e_j})^{\varkappa}\mathcal{T}_{u_{e_j}}^{\bar{u}}$.
Then $m' = \langle a, \{\bar{\chi}_j ; j \in \mathcal{J}^\sigma\}\rangle$ is a direct bound of $\mathcal{Y}\iota^{\circ}\,{}^{\varkappa}\mathcal{T}_0^{\bar{u}}$ in $K^{\bar{u}}$. Consequently $(m')^{\varkappa}\mathcal{T}_{\bar{u}}^{\bar{u}+1}$ has the canonical morphism in $K^{\bar{u}+1}$, and therefore m has the canonical morphism in K.

XII. ι preserves inverse limits of all diagrams in $\mathscr{k}$; this follows from lemma I.7.

Note: It is easy to see that if $\mathscr{k}$ has a system of null morphisms, then K also does (no change is necessary in the proof, only use Note I.6 and lemma I.2 B instead of lemma I.6 and I.2 A).

I.9. Lemma: Let $\mathscr{k}$ be a small category, let $\mathfrak{G}$ be a set of diagrams in $\mathscr{k}$ (or a c l a s s of collections in $\mathscr{k}$). Let $\mathcal{F}$ be a diagram in $\mathscr{k}$ (or a collection in $\mathscr{k}$, respectively). Then there exists a category K with the following properties:

1) $\mathscr{k}$ is a full subcategory of K, the inclusion-functor $\iota : \mathscr{k} \to K$ is $(\overrightarrow{\mathfrak{G}}, \overleftarrow{\mathscr{k}^2})$-preserving and $\mathcal{F}\iota$ has a direct limit in K; $K^\sigma - \mathscr{k}^\sigma$ contains at most one element.
2) If $\Phi : \mathscr{k} \to K'$ is a $\overrightarrow{\mathfrak{G}}$-preserving functor such that $\mathcal{F}\Phi$ has a direct limit in K', then there exists a $\Phi' : K \to K'$, unique up to natural equivalence, such that $\Phi = \iota \cdot \Phi'$ and Φ' is $\overrightarrow{\mathfrak{G}\iota}$-preserving and $\overrightarrow{(\mathcal{F})\iota}$-preserving.

Moreover, if $\varkappa$ is a strongly inaccessible cardinal such that $\operatorname{card} \mathscr{k}^m \leq \varkappa$, $\operatorname{card} \mathcal{F} < \varkappa$, $\operatorname{card} \mathcal{Y} < \varkappa$ for every $\mathcal{D} \in \mathfrak{G}$, $\operatorname{card} \mathfrak{G} \leqq \aleph$, then $\operatorname{card} K^m \leqq \aleph$. Moreover, if $\operatorname{card} \mathfrak{G} < \aleph$, $\operatorname{card} H_k(c,d) < \aleph$ for all $c, d \in k^\sigma$, then $\operatorname{card} H_K(c,d) < \aleph$ for all $c, d \in K^\sigma$.

Note 1 : The proof of lemma I.9 is similar to that of lemma I.8. But in the proof of lemma I.8 the identifications (such that condition ($\ast$) holds) are "too large" for the existence of a functor Φ' with the properties required in I.9.2). In the proof of lemma I.9 we identify morphisms only when it is necessary. But then some difficulties arise. The author does not know if it is possible to take an arbitrary c l a s s $\mathfrak{G}$ of diagrams in the lemma I.9.

Proof: I. Let there be given k, $\mathfrak{G}$, $\mathcal{F}$ with the properties described in lemma I.9 and suppose that $\mathcal{F} \notin \mathfrak{G}$;

we shall construct a category K with the required properties. Simultaneously let there be given K' and $\Phi : k \to K'$ with the properties from I.9.2; we shall construct simultaneously Φ' (of course, the construction of K is independent of Φ and K').

II. First apply lemma I.6. Denote by K^o the category, the existence of which follows from I.6, and by $\iota^o : k \to K^o$ the inclusion-functor. Denote by $\Phi^o : K^o \to K'$ the functor with the properties from I.6.2). Let $\langle a ; \{v_i ; i \in \mathcal{J}^\sigma\} \rangle$ be the direct bound of $\mathcal{F}\iota^o$ in K^o with the properties from lemma I.6; denote by $\mathcal{A}$ the set of all direct bounds $\alpha = \langle b_\alpha ; \{\psi_{i,\alpha} ; i \in \mathcal{J}^\sigma\} \rangle$ of $\mathcal{F}$ in k ; by $f_\alpha \in H_{K^o}(a, b_\alpha)$ the morphism such that $v_i \cdot f_\alpha = \psi_{i,\alpha}$; by A the set of all f_α .

III. The following terminology will be used in the proof: if $\Gamma : K^o \to H$ is a functor, if $m = \langle d ; \{\chi_j ; j \in \mathcal{J}^\sigma\} \rangle$

is a direct bound of $\mathcal{Y}\,\iota^\circ\,\Gamma$, where either $\mathcal{Y} \in \mathbb{G}$ or $\mathcal{Y} = \mathcal{F}$, and if $\langle b; \{\xi_j; j \in \mathcal{J}^\sigma\}\rangle = \overrightarrow{\lim}_k \mathcal{Y}$ for $\mathcal{Y} \in \mathbb{G}$, $\langle b; \{\xi_j; j \in \mathcal{J}^\sigma\}\rangle = \langle a; \{v_i; i \in \mathcal{J}^\sigma\}\rangle$ for $\mathcal{Y} = \mathcal{F}$, then every morphism $\mu \in H_H((b)\Gamma, d)$ such that $(\xi_j)\Gamma \cdot \mu = \chi_j$ will be called a canonical morphism of m (of $\mathcal{Y}\,\iota^\circ\,\Gamma$) in H.

IV. We may suppose that the category K' is small. If not, we replace K' by its full subcategory containing $(k)\Phi$ and some $|\overrightarrow{\lim}_{K'} \mathcal{F}\Phi|$.

V. We shall construct the category K with the required properties by transfinite induction. Put $V = \mathbb{G}$ whenever $\mathbb{G}$ is the s e t of diagrams in k. If $\mathbb{G}$ is the class of collections in k, denote by V some directly substantial set of collections from $\mathbb{G}$ (cf.I.4). Let m be a smallest regular cardinal with $m > \operatorname{card} \mathcal{F}$, $m > \operatorname{card} \mathcal{Y}$ for every $\mathcal{Y} \in V$. Let r be the smallest ordinal such that $\operatorname{card} r = m$. Let $s \in T_r$ and let there be constructed an inductive presheaf ${}^s\mathcal{T}: \langle T_s, < \rangle \to \mathbb{K}$ in the category $\mathbb{K}$ of all small categories and all their functors and its direct bound $\langle K'; \{\Phi^u; u \in T_s\}\rangle$, such that(setting $K^u = (u)\,{}^s\mathcal{T}$):

0) K^0 and Φ^0 are as constructed in part II of the present proof.

1) $(K^u)^\sigma = k^\sigma \cup \{a\}$, k is a full subcategory of K^u; $H_{K^u}(a, c) = H_{K^0}(a, c)$ for every $c \in k^\sigma$; if $u < u'$, then the functor ${}^s\mathcal{T}_u^{u'}$ is identical on all of k, a, $\mathcal{A}$.

2) If $u < u'$, u' is isolated, then the following condition holds: if m is a direct bound in K^u of some

$\mathcal{Y}\,\iota^{\circ}\ {}^{s}\mathcal{T}_{o}^{u}$, where either $\mathcal{Y} \in \mathbb{G}$ or $\mathcal{Y} = \mathcal{F}$, then the direct bound $(m)\,{}^{s}\mathcal{T}_{u}^{u'}$ of $\mathcal{Y}\,\iota^{\circ}\ {}^{s}\mathcal{T}_{o}^{u'}$ has exactly one canonical morphism in $K^{u'}$.

VI. We shall construct K^{s} , $\mathcal{H}_{u}^{s}$, Φ^{s} such that the presheaf ${}^{s+1}\mathcal{T} : \langle \mathbb{T}_{s+1}, < \rangle \to \mathbb{K}$ which is an extension of ${}^{s}\mathcal{T}$ and $(s)\,{}^{s+1}\mathcal{T} = K^{s}$, ${}^{s+1}\mathcal{T}_{u}^{s} = \mathcal{H}_{u}^{s}$, will be an inductive presheaf satisfying conditions 0) - 2), and $\langle K'; \{\Phi^{u}; u \in \mathbb{T}_{s+1}\}\rangle$ will be a direct bound of ${}^{s+1}\mathcal{T}$ in $\mathbb{K}$. If s is a non-isolated ordinal, the construction is evident. Let s be an isolated ordinal number, $s = t + 1$. Then K^{s} will be constructed by transfinite induction. Let $\mathbb{P}$ be the set of all direct bounds in K^{t} of all $\mathcal{Y}\,\iota^{\circ}\ {}^{s}\mathcal{T}_{o}^{t}$, where either $\mathcal{Y} \in \mathbb{V}$ or $\mathcal{Y} = \mathcal{F}$, which have no canonical morphism in K^{t} ; suppose $\mathbb{P} \neq \emptyset$. Let p be an ordinal such that there exists a one-to-one mapping $\mathfrak{p}$ of the set of all positive isolated ordinal numbers of $\mathbb{T}_{p}$ onto $\mathbb{P}$. Let $q \in \mathbb{T}_{p}$, and let there be constructed an inductive presheaf ${}^{q}\mathcal{S} : \langle \mathbb{T}_{q}, < \rangle \to \mathbb{K}$ and its direct bound $\langle K'; \{\Psi^{w}; w \in \mathbb{T}_{q}\}\rangle$, (setting $H^{w} = (w)\,{}^{q}\mathcal{S}$):

0*) $H^{o} = K^{t}$, $\Psi^{o} = \Phi^{t}$;

1*) is analogous to 1).

2*) If $w' = w + 1$, then the direct bound $(w')\ {}^{q}\mathcal{S}_{o}^{w'}$ has a canonical morphism in $H^{w'}$.

VII. We must construct H^{q}, ${}^{q+1}\mathcal{S}_{w}^{q} : H^{w} \to H^{q}$ and Φ^{q} . If q is a non-isolated ordinal, the construction is evident. Thus let $q = x + 1$. Let $(q)\mathfrak{p} = m = \langle d; \{\chi_{j}; j \in \mathcal{J}^{\sigma}\}\rangle \in \mathbb{P}$ be a direct bound in K^{t} of $\mathcal{Y}\,\iota^{\circ}\ {}^{s}\mathcal{T}_{o}^{t}$, where either $\mathcal{Y} \in \mathbb{V}$ or $\mathcal{Y} = \mathcal{F}$. Set $\langle b; \{\xi_{j}; j \in \mathcal{J}^{\sigma}\}\rangle = \overrightarrow{\lim}_{\mathfrak{k}}\,\mathcal{Y}$ whenever $\mathcal{Y} \in \mathbb{V}$ and $\langle b; \{\xi_{j}; j \in \mathcal{J}^{\sigma}\}\rangle =$

$= \langle a; \{(v_i)^{\flat} \mathcal{T}_0^t \, \& \mathcal{G}_0^\times ;\ i \in \mathcal{J}^\sigma\} \rangle$ whenever $\mathcal{U} = \mathcal{F}$.
Since m has no canonical morphism in K^t, so that $d = a$.
Denote by g_α the canonical morphism of the direct bound $\langle k_\alpha; \{\eta_j \cdot f_\alpha ; j \in \mathcal{J}^\sigma\}\rangle$ of $\mathcal{U}_{i^*}$. The following relation on $L = \bigcup_{c \in (H^\times)^\sigma} H_{H^\times}(c, a)$ will be defined:

$\rho S_1 \rho' \Longleftrightarrow \rho = \sigma \cdot g_\alpha \cdot \gamma, \rho' = \sigma \cdot g_{\alpha'} \cdot \gamma'$ and $f_\alpha \cdot \gamma = f_{\alpha'} \cdot \gamma'$;

$\rho S_{n+1} \rho' \Longleftrightarrow \rho = \sigma \cdot g_\alpha \cdot \gamma, \rho' = \sigma \cdot g_{\alpha'} \cdot \gamma'$ and $f_\alpha \cdot \gamma \,(\Delta_L \cup \bigcup_{y=1}^{n} S_y) f_{\alpha'} \cdot \gamma'$;

Set $S = \bigcup_{n=1}^{\infty} S_n$. Then it may be proved:

a) All S_n and S are symmetric;

b) if $\rho S \rho'$ then ρ, ρ' are from the same object to a ;

c) if $f_\alpha \cdot \gamma = f_{\alpha'} \cdot \gamma'$ and $\gamma, \gamma' \in k^m$, then $g_\alpha \cdot \gamma = g_{\alpha'} \cdot \gamma'$;

d) if $\rho S \rho'$ then $\rho \cdot f_\beta = \rho' \cdot f_\beta$ for all $\beta \in A$;

e) if $\rho S \rho'$ and $\sigma . \rho . \tau, \sigma . \rho' . \tau$ are defined, then either $\sigma \cdot \rho . \tau \; S \; \sigma . \rho' \cdot \tau$ or $\sigma \cdot \rho . \tau = \sigma . \rho' \cdot \tau$;

f) if $f_\alpha \cdot \gamma \; S \; f_{\alpha'} \cdot \gamma'$, then $g_\alpha \cdot \gamma \; S \; g_{\alpha'} \cdot \gamma'$.

Denote by S_n^* or S^* the smallest equivalence containing S_n or S respectively. Since $S_n \subset S_{n+1}$, there is $S^* = \bigcup_{n=1}^{\infty} S_n^*$. It is easy to see that b) d) e) remain true also if we replace S by S^*. Now we prove that f) also does: let $f_\alpha \cdot \gamma \; S^* f_{\alpha'} \cdot \gamma'$; then there exist $z_1, \ldots, z_n \in H_{H^\times}(a, a)$ such that $z_1 = f_\alpha \cdot \gamma$, $z_n = f_{\alpha'} \cdot \gamma'$ and $z_i S z_{i+1}$ for $i = 1, \ldots, n-1$. Consequently z_i, $i = 2, \ldots, n-1$, may be expressed $z_i = \sigma_i \cdot g_{\alpha_i} \cdot \gamma_i$. But $\sigma_i \cdot g_{\alpha_i} = f_{\beta_i}$ for some $\beta_i \in A$. Now use f).

Now it is easy to see that lemma I.1 and Note I.1 may be applied and if we set $H^* = H^\times / S^*$ then $f_\alpha \cdot \gamma = f_{\alpha'} \cdot \gamma'$ implies $g_\alpha \cdot \gamma = g_{\alpha'} \cdot \gamma'$ in H^*. Set $\eta_j^* = (\eta_j)^{\& \mathcal{G}_0^\times} 1/S^*$. Now lemma I.2.A may be used; we shall write only H^*, H^{**} instead of $\mathcal{l}, \mathcal{h}$ and put $\mathcal{b} = |\overrightarrow{\lim}_{\mathfrak{R}} \mathcal{Y}|$ whenever $\mathcal{Y} \in V$, $\mathcal{b} = \mathcal{a}$. whenever $\mathcal{Y} = \mathcal{F}$ and $\mu(f_\alpha) = g_\alpha$. Denote by $\iota^* : H^* \to H^{**}$ the inclusion-functor. Let T be the following relation on $(H^{**})^m$:

$[(\xi_j) 1/_{S^*}] \cdot \mu T \chi_j^*$ for every $j \in \mathcal{J}^\sigma$. Evidently lemma I.1 and Note I.1 may be used. Put $H^{\mathfrak{q}} = H^{**}/_T$ ${}^{\mathfrak{q}+1}\mathcal{S}_x^{\mathfrak{q}} = 1/_{S^*} \cdot \iota^* \cdot$ $\cdot 1/_T$, $\tilde{\mu} = (\mu) 1/_T$. Then, evidently, $\tilde{\mu}$ is the canonical morphism of the direct bound $(m)\, {}^{\mathfrak{q}+1}\mathcal{S}_o^{\mathfrak{q}}$ in $H^{\mathfrak{q}}$, and ${}^{\mathfrak{q}+1}\mathcal{S}$ is an inductive presheaf satisfying conditions $0^*) - 2^*)$.

VIII. Let $\langle K'; \{\Psi^w; w \in T_{\mathfrak{q}}\}\rangle$ be a direct bound in $\mathbb{K}$ of ${}^{\mathfrak{q}}\mathcal{S}$, let $\Psi^o = \Phi^t$; it is sufficient to find $\Psi^{\mathfrak{q}} : H^{\mathfrak{q}} \to K'$ such that ${}^{\mathfrak{q}+1}\mathcal{S}_x^{\mathfrak{q}} \cdot \Psi^{\mathfrak{q}} = \Psi^x$. Since $\langle (\mathcal{b})\Psi^x; \{(\xi_j)\Psi^x; j \in \mathcal{J}^\sigma\}\rangle$ is the direct limit of $\mathcal{Y}\Phi$, there is $(g_\alpha)\Psi^x = \mu' \cdot (f_\alpha)\Psi^x$, where by μ' is denoted the canonical morphism of the direct bound $\langle (\mathcal{a})\Phi^t; \{(\chi_j)\Phi^t; j \in \mathcal{J}^\sigma\}\rangle$. Consequently if $\rho\, S^* \rho'$, there is $(\rho)\Psi^x = (\rho')\Psi^x$. Using lemma I.1 there exists exactly one functor $\Psi^* : H^* \to K'$ such that $\Psi^x =$ $= 1/_{S^*}\Psi^*$. Then, using lemma I.2, there exists exactly one functor $\Psi^{**} : H^{**} \to K'$ such that $\Psi^* = \iota^* \cdot \Psi^{**}$ and $(\mu)\Psi^{**} = \mu'$. Now it is easy to see that $([(\xi_j) 1/_{S^*}] \cdot \mu)\Psi^{**} =$ $= (\chi_j^*)\Psi^{**}$ for all $j \in \mathcal{J}^\sigma$; consequently, using Note I.1, there exists exactly one functor $\Psi^{\mathfrak{q}} : H^{\mathfrak{q}} \to K'$ such that $\Psi^{**} = 1/_T \cdot \Psi^{\mathfrak{q}}$. Therefore ${}^{\mathfrak{q}+1}\mathcal{S}_x^{\mathfrak{q}} \cdot \Psi^{\mathfrak{q}} = \Psi^x$. If $\mu'' \in H_{K'}($ $(\mathcal{b})\Psi^x, (\mathcal{a})\Psi^x)$, $\mu'' \neq \mu'$ then $((\xi_j) 1/_{S^*})\Psi^* \cdot \mu'' \neq (\chi_j^*)\Psi^*$ for some $j \in \mathcal{J}^\sigma$. This implies the unicity $\Psi^{\mathfrak{q}}$.

IX. Using transfinite induction, one may construct an in-

ductive presheaf ${}^{r}\mathcal{Y}$ satisfying conditions 0*) - 2*).
Now set $\langle H ; \{ \mathcal{Y}_w ; w \in \mathbb{T}_p \} \rangle = \overrightarrow{\lim}_K {}^{r}\mathcal{Y}$, denote by Ψ the canonical morphism of the direct bound $\langle K' ; \{ \Psi^w ; w \in \mathbb{T}_p \} \rangle$ of ${}^{r}\mathcal{Y}$ in K . If $P = \emptyset$, put $H = K^t$, $\Psi = \Phi^t$, $\mathcal{Y}_o : K^t \to H$ is identical.
X. Next, define the relation R_y on $L = \bigcup_{c \in H^\sigma} H_H(c, a)$ for every $y \in \mathbb{T}_k$ as follows (denote by R^*_y the smallest equivalence containing R_y) : $\sigma . \beta \; R_o \sigma . \gamma$ whenever there exists a direct bound m in H of some $\mathcal{Y} \iota^\circ \,{}^{\wedge}\mathcal{T}_o^t \,\mathcal{Y}_o$ where either $\mathcal{Y} \in V$ or $\mathcal{Y} = \mathcal{F}$, such that β and γ are both canonical morphisms of m in H . If $y \in \mathbb{T}_k$, then $\sigma . \beta \; R_y \; \sigma . \gamma$ if and only if there exists a $\mathcal{Y} \in V \cup \{\mathcal{F}\}$ such that $\xi_j . \beta (\Delta_L \cup \bigcup_{x<y} R^*_x) \xi_j . \gamma$ for all $j \in \mathcal{J}^\sigma$, where $\langle b ; \{\xi_j ; j \in \mathcal{J}^\sigma\} \rangle = \overrightarrow{\lim}_k \mathcal{Y}$ whenever $\mathcal{Y} \in V$ and $\langle b ; \{\xi_j ; j \in \mathcal{J}^\sigma\} \rangle = \langle a ; \{(v_i)^{\wedge}\mathcal{T}_o^t \mathcal{Y}_o ; i \in \mathcal{J}^\sigma\} \rangle$ whenever $\mathcal{Y} = \mathcal{F}$. Put $R = \bigcup_{y \in \mathbb{T}_k} R^*_y$. Using transfinite induction, it may be proved easily that $\beta \; R \; \gamma$ implies $\beta \cdot f_\alpha = \gamma \cdot f_\alpha$ x)

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x) Thus the relation $\bar{R}$ on L such that $\beta \bar{R} \gamma$ if and only if $\beta \cdot f_\alpha = \gamma \cdot f_\alpha$ for all $\alpha \in A$ is greater than or equal to R. Compare with the proof of I.8.
In the present proof the following evident fact is used without any reference.
Let W be an equivalence on $L = \bigcup_{c \in C^\sigma} H_C(c, a)$ (where one substitutes C by H^x, H) such that if $\beta W \gamma$ then $\beta, \gamma \in H_C(c, a)$ for some $c \in C^\sigma$, $\beta . f_\alpha = \gamma . f_\alpha$ for all $\alpha \in A$ and if $\sigma . \beta . \tau$ is defined then either $\sigma . \beta . \tau \, W \sigma . \gamma . \tau$ or $\sigma . \beta . \tau = \sigma . \gamma . \tau$. Then if $\vartheta, \vartheta' \in C^m$, $(\vartheta)1/W = (\vartheta')1/W$, then necessarily $\vartheta W \cup \Delta \; \vartheta'$.

for all $\alpha \in A$, and $(\beta)\Psi = (\gamma)\Psi$. Put $K^{s} = H/_R$, ${}^{s+1}\mathcal{T}^{s} = \mathcal{S}_o \cdot 1/_R$, let Φ^{s} be the functor such that $\Psi = 1/_R \cdot \Phi^{s}$.

XI. Now we prove that ${}^{s+1}\mathcal{T}: \langle \mathbb{T}_{s+1}, < \rangle \to \mathbb{K}$ and $\langle K'; \{\Phi^{u}; u \in \mathbb{T}_{s+1}\} \rangle$ satisfy conditions 0) - 2). 0) and 1) are evident. In the proof of 2) it is sufficient to consider $u' = s$ only. Thus, let $u < s$, let m be a direct bound in K^{u} of some $\mathcal{Y} \iota^{o}$ ${}^{s+1}\mathcal{T}_o^{u} = \mathcal{Y}\iota^{o}\, {}^{s}\mathcal{T}_o^{u}$, where either $\mathcal{Y} \in V$ or $\mathcal{Y} = \mathcal{F}$. Then the direct bound $(m)\, {}^{s}\mathcal{T}_u^{t} \cdot \mathcal{S}_o$ has at least one canonical morphism in H, as follows from the construction of H; thus $(m)\, {}^{s+1}\mathcal{T}_u^{s}$ has at least one canonical morphism in K^{s}. We shall prove that it has exactly one canonical morphism. Set $\langle b; \{\xi_j;\ j \in \mathcal{J}^{\sigma}\} \rangle = \overrightarrow{\lim}_{k} \mathcal{Y}$ whenever $\mathcal{Y} \in V$ and $\langle b; \{\xi_j; j \in \mathcal{J}^{\sigma}\} \rangle = \langle a; \{(v_i)\, {}^{s}\mathcal{T}_o^{t} \mathcal{S}_o; i \in \mathcal{J}^{\sigma}\} \rangle$ whenever $\mathcal{Y} = \mathcal{F}$. Let β and γ be both the canonical morphisms of $(m)\, {}^{s+1}\mathcal{T}_u^{s}$ in K^{s}. Then there exist $\bar{\beta}, \bar{\gamma} \in H^{m}$ such that $(\bar{\beta})1/_R = \beta$, $(\bar{\gamma})1/_R = \gamma$, $\xi_j \cdot \bar{\beta}\ R\ \xi_j \cdot \bar{\gamma}$ for all $j \in \mathcal{J}^{\sigma}$. Since $R = \bigcup_{y \in \mathbb{T}_k} R^{*}_{y}$, one may choose $y_j \in \mathbb{T}_k$ for $j \in \mathcal{J}^{\sigma}$ such that $\xi_j \cdot \bar{\beta}\ R^{*}_{y_j}\ \xi_j \cdot \bar{\gamma}$. Put $\bar{y} = \sup_{j \in \mathcal{J}^{\sigma}} y_j$; then $\bar{y} \in \mathbb{T}_k$ and $\bar{\beta}\ R_{\bar{y}+1}\ \bar{\gamma}$, and therefore $\beta = (\bar{\beta})1/_R = (\bar{\gamma})1/_R = \gamma$.

XII. Using transfinite induction, one may define an inductive presheaf ${}^{k}\mathcal{T}$ in $\mathbb{K}$ satisfying conditions 0) - 2), and its direct bound $\langle K'; \{\Phi^{s};\ s \in \mathbb{T}_k\} \rangle$. Put $\langle K; \{\mathcal{T}_s;\ s \in \mathbb{T}_k\} \rangle = \overrightarrow{\lim}_{\mathbb{K}}\, {}^{k}\mathcal{T}$; let Φ' be the canonical morphism of the direct bound $\langle K'; \{\Phi^{s}; s \in \mathbb{T}_k\} \rangle$ in $\mathbb{K}$. Then, evidently, k is a full subcategory of K; put $\iota = \iota^{o} \cdot \mathcal{T}_o$. Obviously $\Phi = \iota \cdot \Phi'$

and $\langle (a)\Phi' ; \{(v_i)\mathcal{T}_0\Phi' ;\ i \in \mathcal{J}^\sigma\}\rangle = \overrightarrow{lim}_{K'}\ \mathcal{F}\Phi$.

XIII. Next prove that ι preserves direct limits of all diagrams from $\mathcal{V}$ and that $\langle a; \{(v_i)\mathcal{T}_0 ;\ i \in \mathcal{J}^\sigma\}\rangle$ is the direct limit of $\mathcal{F}\iota$ in K . Let m be a direct bound in K of some $\mathcal{G}\iota$, where either $\mathcal{G} \in \mathcal{V}$ or $\mathcal{G} = \mathcal{F}$. Then there exist $s \in T_k$ and a direct bound $\tilde{m}$ in K^s of $\mathcal{G}\ \iota^\circ\ {}^k\mathcal{T}_0^{s+1}$ such that $m = (\tilde{m})\mathcal{T}_s$ (the proof is analogous to part XI of the proof of lemma I.8). Then $(\tilde{m})\ {}^k\mathcal{T}_s^{s+1}$ has a canonical morphism μ in K^{s+1} . Obviously $(\mu)\ \mathcal{T}_{s+1}$ is the canonical morphism of m in K . If ν and ν' are both canonical morphisms of m in K , then there exist $t \in T_k$, $\bar{\nu}$, $\bar{\nu}' \in (K^t)^m$ and a direct bound $\bar{m}$ of $\mathcal{G}\ \iota^\circ\ {}^k\mathcal{T}_0^t$, such that $(\bar{m})\mathcal{T}_t = m$, $(\bar{\nu})\ \mathcal{T}_t = \nu$, $(\bar{\nu}')\ \mathcal{T}_t = \nu'$ and $\bar{\nu}$, $\bar{\nu}'$ are both canonical morphisms of $\bar{m}$. But then $(\bar{\nu})\ {}^k\mathcal{T}_t^{t+1} = (\bar{\nu}')\ {}^k\mathcal{T}_t^{t+1}$, and hence $\nu = \nu'$.

XIV. The proof is complete in the case that $\mathcal{V} = \mathbb{G}$ is a set of diagrams in k . If $\mathbb{G}$ is a class of collections in k and $\mathcal{V}$ is a directly substantial class from $\mathbb{G}$, then we must perform another identification. Let V be the following relation on $L = \bigcup_{a \in K^\sigma} H_K(c, a)$: $\sigma . \beta\ V_0\ \sigma . \gamma$ whenever there exists a direct bound $m = \langle a; \{\chi_j ; j \in \mathcal{J}^\sigma\}\rangle$ in K of some collection $\mathcal{G}\iota$,where $\mathcal{G} \in G$, such that $\beta = \chi_j$, $\gamma = \chi_{j'}$, and $\xi_j = \xi_{j'}$, where $\langle b ; \{\xi_j ; j \in \mathcal{J}^\sigma\}\rangle$ is a sum of $\mathcal{G}$ in k ; if $y \in \mathbb{T}_k$, $y > 0$, put $\sigma . \beta\ V_y\ \sigma . \gamma$ whenever there exists a collection $\mathcal{G} \in \mathcal{V} \cup \{\mathcal{F}\}$ such that $\xi_j \cdot \beta (\bigcup V_x^* \cup \Delta_L)\ \xi_j \cdot \gamma$ for all $j \in \mathcal{J}^\sigma$ where $\langle b ; \{\xi_j ; j \in \mathcal{J}^\sigma\}\rangle = \overrightarrow{lim}_k\ \mathcal{G}$ whenever $\mathcal{G} \in \mathcal{V}$, and $\langle b ; \{\xi_j ; j \in \mathcal{J}^\sigma\}\rangle = \langle a ; \{(v_i)\mathcal{T}_0 ; i \in \mathcal{J}^\sigma\}\rangle$

whenever $\mathcal{Y} = \mathcal{F}$ (where V^*_y denotes the smallest equivalence containing V_y). Put $V = \bigcup_{y \in T_n} V^*_y$. Then the category $K/_V$ has the properties required in lemma I.9. (If all the $\mathcal{Y} \in \mathbb{G}$ and $\mathcal{F}$ were not only collections, then this last identification could possibly give rise to further direct bounds.)

XV. Now we shall prove that if $\aleph$ is a strongly inaccessible cardinal with $card\ \mathfrak{k}^m \leqq \aleph$, $card\ \mathcal{F} < \aleph$, $card\ \mathcal{Y} < \aleph$ for all $\mathcal{Y} \in \mathbb{G}$ and $card\ \mathbb{G} \leqq \aleph$, then $card\ K^m \leqq \aleph$. Lemma I.6 implies $card\ (K^o)^m \leqq \aleph$. It is easy to see that $m \leqq \aleph$. Let an $s \in T_n$ be given, suppose $card\ (K^u)^m \leqq \aleph$ for all $u < s$; prove that $card\ (K^s)^m \leqq \aleph$. If s is a non-isolated ordinal, then this follows from lemma I.3 B. Let $s = t + 1$; since $card\ \mathbb{G} \leqq \aleph$, there is $card\ \mathbf{P} \leqq card \bigcup_{\substack{\mathcal{Y}:\mathcal{J} \to \mathfrak{k} \\ \mathcal{Y} \in \mathbb{G} \cup \{\mathcal{F}\}}} ((\prod_{j \in \mathcal{J}^o} H_{K^t}(\mathcal{J})\mathcal{Y}, a)) \leqq \aleph$. If $q \in T_n$ and $card (H^w)^m \leqq \aleph$ for all $w < q$ then evidently $card\ (H^q)^m \leqq \aleph$; this follows from either lemma I.3 or lemma I.2. Then, using lemma I.3, $card\ (K^s)^m \leqq \aleph$, and thus $card\ K^m \leqq \aleph$. Moreover, if $card\ \mathbb{G} < \aleph$ and $card\ H_{\mathfrak{k}}(c, d) < \aleph$ for all $c,\ d \in \mathfrak{k}^o$, then $m < \aleph$, $card\ \mathbb{P} < \aleph$, and it may be easily proved that $card\ H_K\ (c, d) < \aleph$ for all c, $d \in K^o$.

__Note 2__: It is easily seen that the following lemma also holds: Let $\mathfrak{k}$ be a small category with a system of null morphisms, $\mathbb{G}$ a set of diagrams in $\mathfrak{k}$ (or a class of collections in $\mathfrak{k}$). Let $\mathcal{F}$ be a diagram in $\mathfrak{k}$ (or a collection in $\mathfrak{k}$, respectively). Then there exists a category K with a system of null morphisms such that statement 1) from lemma I.9 holds, and 2') if $\Phi : \mathcal{k} \to K'$ is a null functor such that $\mathcal{F}\Phi$ has a direct limit in K' and Φ preserves direct limits of all diagrams from $\mathfrak{G}$, then there exists a null functor $\Phi' : K \to K'$, unique up to natural equivalence, such that $\Phi = \iota \cdot \Phi'$ and Φ' preserves direct limits of all $\mathcal{Y}\iota$, where either $\mathcal{Y} = \mathcal{F}$ or $\mathcal{Y} \in \mathfrak{G}$.

The proof of lemma I.9 need not be modified. It is sufficient to use Note I.6 in place of lemma I.6 in II, and lemma I.2 B in place of lemma I.2 A in VII.

Note 3: Obviously lemmas I.9 and I.1 imply the following proposition:

Let $\mathcal{k}$ be a small category, $\mathrm{card}\, H_{\mathcal{k}}(c,d) \leqq 1$ for all $c, d \in \mathcal{k}^{\sigma}$, let $\mathfrak{G}$ be a set of collections in $\mathcal{k}$. Let $\mathcal{F}$ be a collection in $\mathcal{k}$. Then there exists a category K such that $\mathrm{card}\, H_K(c,d) \leqq 1$ for all $c, d \in K^{\sigma}$ and statement 1) from lemma I.9 and also the following statement 2') hold.

2') If $\Phi : \mathcal{k} \to K'$ is a $\overrightarrow{\mathfrak{G}}$-preserving functor such that $\mathcal{F}\Phi$ has a direct limit in K' and if $\mathrm{card}\, H_{K'}(c,d) \leqq$ $\leqq 1$ for all $c, d \in (K')^{\sigma}$, then there exists a $\Phi' : K \to K'$, unique up to natural equivalence, such that $\Phi = \iota \cdot \Phi'$ and that Φ' is $\overrightarrow{\mathfrak{G}\iota}$-preserving and $\overrightarrow{(\mathcal{F}\iota)}$-preserving.

II. Embedding theorems for small categories.

In the present section there are given some theorems which follow from lemma I.8 and I.9, namely theorems II.3, II.5

and II.7.

II.1. **Definition**: Let S be a set which is ordered by $\prec$. Let $\{\mathfrak{k}_s ; s \in S\}$ be a system of small categories. We shall call it monotone if $\mathfrak{k}_s$ is a full subcategory of $\mathfrak{k}_{s'}$ whenever $s \prec s'$. Denote by $\bigcup_{s\in S} \mathfrak{k}_s$ the category K such that $K^\sigma = \bigcup_{s\in S} (\mathfrak{k}_s)^\sigma$ and that every $\mathfrak{k}_s$ is a full subcategory of K.

Note: Evidently, if $s_0 \in S$ and $\mathcal{Y}$ is a diagram in $\mathfrak{k}_{s_0}$, and if the inclusion-functor $\iota^{s}_{s_0} : \mathfrak{k}_{s_0} \to \mathfrak{k}_s$ preserves the direct limit of $\mathcal{Y}$ for every $s \in S$, $s \succ s_0$, then the inclusion-functor $\iota_{s_0} : \mathfrak{k}_{s_0} \to \bigcup_{s\in S} \mathfrak{k}_s$ also preserves the direct limit of $\mathcal{Y}$.

II.2. **Lemma**: Let $\mathfrak{k}$ be a small category, $\mathfrak{G}$ a set of diagrams in $\mathfrak{k}$ (or a class of collections in $\mathfrak{k}$). Let V be a set of diagram schema (or a set of discrete categories, respectively). Then there exists a small category K such that:

1) $\mathfrak{k}$ is a full subcategory of K, the inclusion-functor $\iota : \mathfrak{k} \to K$ is $(\overrightarrow{\mathfrak{G}}, \overleftarrow{\mathfrak{k}^{Z}})$-preserving. If $\mathcal{Y}$ is a V-diagram in $\mathfrak{k}$, then $\mathcal{Y}\iota$ has a direct limit in K. Every $a \in K^\sigma$ is a direct limit of a $\mathcal{Y}\iota$, where $\mathcal{Y}$ is a V-diagram in $\mathfrak{k}$.
2) If K' is a $\overrightarrow{V}$-complete category and if $\Phi : \mathfrak{k} \to K'$ is a $\overrightarrow{\mathfrak{G}}$-preserving functor, then there exists a functor $\Phi' : K \to K'$, unique up to natural equivalence, such that $\Phi = \iota \cdot \Phi'$ and Φ' preserves direct limits of all diagrams $\mathcal{Y}\iota$, where $\mathcal{Y}$ is a V-diagram in $\mathfrak{k}$.

Moreover, if $\aleph$ is a strongly inaccessible cardinal such that

(a) $card\ \mathfrak{k}^m \leqq \aleph$ and $card\ \mathfrak{G} \leqq \aleph$;

(b) $h \in V \Rightarrow card\ h^m < \aleph$; $\mathcal{G} \in \mathfrak{G} \Rightarrow card\ \mathcal{G} < \aleph$,

then $card\ K^m \leqq \aleph$.

Moreover, if

(c) $card\ \mathfrak{G} < \aleph$, $card\ H_{\mathfrak{k}}(c,d) < \aleph$ for every $c, d \in \mathfrak{k}^{\sigma}$,

then $card\ H_K(c,d) < \aleph$ for every $c, d \in K^{\sigma}$.

Proof: Let $\overline{\mathfrak{G}}$ be the set of all diagrams from $\mathfrak{G}$ which have a direct limit in $\mathfrak{k}$. The isomorphism of small categories is an equivalence-relation on V , denote by $\overline{V}$ some choice-set. Let $\mathbb{V}$ be the set of all $\overline{V}$-diagrams in $\mathfrak{k}$. Let $\dashv$ be a well-ordering of $\mathbb{V}$; denote by $\mathcal{U}_o$ the smallest element of $\mathbb{V}$. Denote by $\mathbb{V}_{\mathcal{U}}$ the set of all $\mathcal{V} \in \mathbb{V}$, $\mathcal{V} \dashv \mathcal{U}$. Using lemma I.9 one may construct (by transfinite induction) the monotone system $\{K^{\mathcal{U}}; \mathcal{U} \in \mathbb{V}\}$ of small categories (denote by $\iota^{\mathcal{U}}_{\mathcal{V}} : K^{\mathcal{V}} \to K^{\mathcal{U}}$ the inclusion-functor) and the system $\{\mathfrak{G}^{\mathcal{U}} ; \mathcal{U} \in \mathbb{V}\}$ and, if some K', $\Phi : \mathfrak{k} \to K'$ satisfying statement 2) are given, also the system $\{\Phi^{\mathcal{U}} ; \mathcal{U} \in \mathbb{V}\}$ such that $K^{\mathcal{U}_o} = \mathfrak{k}$; $\Phi^{\mathcal{U}_o} = \Phi$; $\mathfrak{G}^{\mathcal{U}_o} = \overline{\mathfrak{G}}$; $\mathfrak{G}^{\mathcal{U}}$ is the set of all diagrams $\mathcal{G}\,\iota^{\mathcal{U}}_{\mathcal{U}_o}$, where $\mathcal{G} \in \overline{\mathfrak{G}} \cup \mathbb{V}_{\mathcal{U}}$; $\Phi^{\mathcal{U}} : K^{\mathcal{U}} \to K'$ is a functor which preserves direct limits of all diagrams from $\mathfrak{G}^{\mathcal{U}}$ and if $\mathcal{V} \dashv \mathcal{U}$, then $\Phi^{\mathcal{V}} = \iota^{\mathcal{U}}_{\mathcal{V}} \cdot \Phi^{\mathcal{U}}$, $\iota^{\mathcal{U}}_{\mathcal{V}}$ preserves direct limits of all diagrams from $\mathfrak{G}^{\mathcal{V}}$ and inverse limits of all diagrams, $\mathcal{V}\,\iota^{\mathcal{U}}_{\mathcal{U}_o}$ has a direct limit in $K^{\mathcal{U}}$. Evidently, $K = \bigcup_{\mathcal{U} \in \mathbb{V}} K^{\mathcal{U}}$ has the properties required in the lemma. If $\aleph$ is a strongly inaccessible cardinal satisfying (a),(b) (or (a),(b),(c)), then evidently $card\ V \leq \aleph$; consequently $card\ K^m \leq \aleph$ (or, moreover, $card\ H_K(c,d) < \aleph$ for every $c, d \in K^\sigma$, respectively).

<u>Note</u>: a) It is easy to see,[3], that if $\mathscr{k}$ is a full subcategory of K, every $a \in K^\sigma$ is a sum of a collection in $\mathscr{k}$ and every collection in $\mathscr{k}$ has a sum in K, then every collection in K has a sum in K.

b) It is easy to see (using I.9, Note 3) that lemma II.2 remains valid if $\mathscr{k}, K, K'$ are partially ordered sets (if K is only a quasi-ordered set, take some of its skeletons containing $\mathscr{k}$).

c) Thus using a) b) we obtain the following proposition (choose V to be the set of all discrete subcategories of $\mathscr{k}$):

Let $(\mathscr{k}, \dashv)$ be a partially ordered set, $\mathfrak{G} \subset exp\ \mathscr{k}$. Then there exists a complete lattice $(K, \dashv)$ such that:

1) $\mathscr{k} \subset K$, the inclusion mapping $\iota : \mathscr{k} \to K$ is strongly order-preserving (i.e. if $a, b \in \mathscr{k}$ then $a \dashv b \Longleftrightarrow a \dashv b$); if $h \in \mathfrak{G}$ then $sup_{\mathscr{k}}\, h = sup_K\, h$ whenever $sup_{\mathscr{k}}\, h$ exists, if $h \subset \mathscr{k}$ then $inf_{\mathscr{k}}\, h = inf_K\, h$ whenever $inf_{\mathscr{k}}\, h$ exists.

2) If $(K', \dashv)$ is a complete lattice and $\Phi : \mathscr{k} \to K'$ is an order-preserving mapping (i.e. if $a, b \in \mathscr{k}$, then $a \dashv b \Longrightarrow (a)\Phi \dashv (b)\Phi$) which preserves least upper bounds of all elements of $\mathfrak{G}$, then there exists exactly one mapping $\Phi' : K \to K'$ such that Φ' preserves least upper bounds of all $H \subset K$ and $\iota\Phi' = \Phi$.

II.3. <u>Theorem</u>: Let $\mathscr{k}$ be a small category. Let $\mathfrak{G}$ be a set of diagrams (or a class of collections) in $\mathscr{k}$, let

V be a class of diagram schema (or a class of discrete categories, respectively). Then there exists a $\overrightarrow{V}$-complete category K such that:

1) $\mathcal{k}$ is a full subcategory of K, the inclusion-functor $\iota : \mathcal{k} \to K$ is $(\overrightarrow{\mathbb{G}}, \overleftarrow{\mathcal{k}^{Z}})$-preserving.

2) If K' is a $\overrightarrow{V}$-complete category and if $\Phi : \mathcal{k} \to K'$ is a $\overrightarrow{\mathbb{G}}$-preserving functor, then there exists a $\overrightarrow{K^{V}}$-preserving functor $\Phi' : K \to K'$, unique up to natural equivalence, such that $\Phi = \iota \cdot \Phi'$. If V is a set, then K is a small category. Moreover, if $\aleph$ is a strongly inaccessible cardinal satisfying (a),(b) from lemma II.2, then $\mathrm{card}\, K^{m} \leqq \aleph$. Moreover, if (c) is satisfied, then $\mathrm{card}\, H_K(c, d) < \aleph$ for every $c, d \in K^{\sigma}$.

Proof: I. First suppose that V is a set. Let m be the smallest regular cardinal such that $m > \mathrm{card}\, h^{m}$ whenever $h \in V$. Let n be the smallest ordinal with $\mathrm{card}\, n = m$. Using lemma II.2 one may construct (by transfinite induction) the monotone system $\{ K^{u} ; u \in T_n \}$ of small categories (denote by $\iota^{u}_{u'} : K^{u'} \to K^{u}$ the inclusion functor), the system $\{ \mathbb{G}^{u} ; u \in T_n \}$, and, if K' and $\Phi : \mathcal{k} \to K'$ satisfying 2) are given, also the system $\{ \Phi^{u} ; u \in T_n \}$ such that $K^{\circ} = \mathcal{k}$, $\mathbb{G}^{\circ} = \mathbb{G}$, $\Phi^{\circ} = \Phi$; $\mathbb{G}^{u}$ is the set of all diagrams $\mathcal{D}\, \iota^{u}_{o}$, where $\mathcal{D} \in \mathbb{G}$, and of all $\mathcal{D}\, \iota^{u}_{u'}$, where $u' < u$, and $\mathcal{D}$ is a V-diagram in K^{u}; $\iota^{u}_{u'}$ is $(\overrightarrow{\mathbb{G}^{u'}}, \overleftarrow{K^{u' Z}})$-preserving; $\Phi^{u} : K^{u} \to K'$ is a $\overrightarrow{\mathbb{G}^{u}}$-preserving functor and $\Phi^{u'} = \iota^{u}_{u'} \Phi^{u}$; if $\mathcal{D}$ is a V-diagram in K^{u}, then $\mathcal{D}\, \iota^{u+1}_{u}$ has a direct limit in K^{u+1}. Then evidently

$K = \bigcup_{u \in T_n} K^u$ has the properties required in the theorem.

II. Now let V be a class. Isomorphism is an equivalence relation on V, denote by $\overline{V}$ some choice-class. For every cardinal m denote by V_m the set of all $h \in \overline{V}$ such that $\mathrm{card}\, h^m < m$. Using part I of the present proof one may construct (by transfinite induction) the monotone system $\{K_m ; m \in \mathbb{N}\}$ of small categories ($\mathbb{N}$ denotes the class of all cardinals), the system $\{\mathfrak{G}_m ; m \in \mathbb{N}\}$, and, if there are given K' and $\Phi : k \to K'$ satisfying II.3.2), also the system $\{\Phi_m ; m \in \mathbb{N}\}$ such that: $k = K^o$, $\mathfrak{G}_m$ is the system of all V_m-diagrams in K_m; if $n < m$, then the inclusion-functor $\iota^m_n : K_n \to K_m$ is $(\overrightarrow{(\mathfrak{G}\,\iota^n_o) \cup \mathfrak{G}_n}, \overleftarrow{K^Z_n})$-preserving; every diagram from $\mathfrak{G}_m$ has a direct limit in K_m; $\Phi_o = \Phi$, $\Phi_m : K_m \to K'$ is a functor such that $\Phi_n = \iota^m_n \cdot \Phi_m$ for $n < m$ and Φ_m is $\overrightarrow{\mathfrak{G}_m \cup (\mathfrak{G}\,\iota^m_o)}$-preserving. Then evidently $K = \bigcup_{m \in \mathbb{N}} K_m$ has the properties required in the theorem.

II.4. <u>Note</u>: If V is a set of diagram schema and if every $\mathcal{Y} \in \mathfrak{G}$ is a V-diagram, then it follows from theorem II.3 that there exists "une solution du problème d'application universelle pour E relativement à la donnée de Σ, σ et α" (cf.[2],p.43), where "ensembles munis d'une structure d'espèce Σ" are skeletons of small $\overrightarrow{V}$-complete categories, "σ-morphismes" are functors preserving direct limits of all V-diagrams, "E" is a skeleton of a small category and "α-applications" are functors preserving direct limits of diagrams from $\mathfrak{G}$. Further "solutions du problème d'application universelle" are given in the following theorems.

II.5. **Theorem**: Let k be a small category, let $\mathbb{G}_d$, $\mathbb{G}_i$ be sets of diagrams in k (or classes of collections in k). Let V_d, V_i be classes of diagram schema (or classes of discrete categories, respectively).
Then there exists a $(\overrightarrow{V_d}, \overleftarrow{V_i})$-complete category K such that:

1) k is a full subcategory of K, the inclusion-functor $\iota : k \to K$ is $(\overrightarrow{\mathbb{G}_d}, \overleftarrow{\mathbb{G}_i})$-preserving.
2) If K' is a $(\overrightarrow{V_d}, \overleftarrow{V_i})$-complete category and $\Phi : k \to K'$ is a $(\overrightarrow{\mathbb{G}_d}, \overleftarrow{\mathbb{G}_i})$-preserving functor, then there exists a $(\overrightarrow{K^{V_d}}, \overleftarrow{K^{V_i}})$-preserving functor $\Phi : K \to K'$, unique up to natural equivalence, such that $\Phi = \iota \cdot \Phi'$. If V_d, V_i are sets, then K is a small category.
Moreover, if $\aleph$ is a strongly inaccessible cardinal satisfying (a),(b) from lemma II.2 with $\mathbb{G} = \mathbb{G}_d \cup \mathbb{G}_i$, $V = V_d \cup V_i$, then $card\, K^m \leqq \aleph$. Moreover, if (c) is satisfied, then $card\, H_K(c, d) < \aleph$ for every $c, d \in K^\sigma$.

Proof: I. First suppose that V_d and V_i are sets. Let m be the smallest regular cardinal such that $m > card\, h^m, h^m \in V_d \cup V_i$; let κ be the smallest ordinal such that $card\, \kappa = m$.
To given $u_0 \in \mathbb{T}_\kappa, u_0 > 0$ let there be constructed the systems $\{K^u ; u \in \mathbb{T}_{u_0}\}$, $\{\mathbb{G}_d^u ; u \in \mathbb{T}_{u_0}\}$, $\{\mathbb{G}_i^u ; u \in \mathbb{T}_{u_0}\}$, if some K' and Φ with the properties required in the theorem are given, then also the system $\{\Phi^u ; u \in \mathbb{T}_{u_0}\}$ such that:
1) $K^0 = k$, $\mathbb{G}_d^0 = \mathbb{G}_d$, $\mathbb{G}_i^0 = \mathbb{G}_i$, $\Phi^0 = \Phi$; $\{K^u ; u \in \mathbb{T}_{u_0}\}$ is

a monotone system of small categories; for $v < u$ denote by $\iota_v^u : K^v \longrightarrow K^u$ the inclusion-functor; $\mathbb{G}_d^u$ and $\mathbb{G}_i^u$ are sets of diagrams in K^u; $\Phi^u : K^u \longrightarrow K'$ is a functor.

2) For $v < u$, ι_v^u is $(\overrightarrow{\mathbb{G}_d^v}, \overleftarrow{\mathbb{G}_i^v})$-preserving; $\Phi^v = \iota_v^u \Phi^u$ and Φ^u is $(\overrightarrow{\mathbb{G}_d^u}, \overleftarrow{\mathbb{G}_i^u})$-preserving.

3) For isolated $u > 0$, if u is odd, then: $\mathbb{G}_d^u = \mathbb{G}_d^{u-1} \cdot \iota_{u-1}^u$, $\mathbb{G}_i^u = (\mathbb{G}_i^{u-1} \cdot \iota_{u-1}^u) \cup \{ \mathcal{Y} \iota_{u-1}^u ;$ $\mathcal{Y}$ is a V_i-diagram$\}$;

if $\mathcal{Y}$ is a V_i-diagram in K^{u-1} then $\mathcal{Y} \iota_{u-1}^u$ has an inverse limit in K^u;

if u even, then: $\mathbb{G}_d^u = \mathbb{G}_d^{u-1} \cdot \iota_{u-1}^u \cup \{ \mathcal{Y} \iota_{u-1}^u ;$ $\mathcal{Y}$ is a V_d-diagram$\}$, $\mathbb{G}_i^u = \mathbb{G}_i^{u-1} \cdot \iota_{u-1}^u$;

if $\mathcal{Y}$ is a V_d-diagram in K^{u-1} then $\mathcal{Y} \iota_{u-1}^u$ has a direct limit in K^u.

4) If u is non-isolated, then $K^u = \bigcup_{v<u} K^v$, $\mathbb{G}_d^u = \bigcup_{v<u} \mathbb{G}_d^v \iota_v^u$, $\mathbb{G}_i^u = \bigcup_{v<u} \mathbb{G}_i^v \iota_v^u$, $\Phi^u = \bigcup_{v<u} \Phi^v$.

We are to construct K^{u_0}, $\mathbb{G}_d^{u_0}$, $\mathbb{G}_i^{u_0}$, Φ^{u_0}; however, this is simple. If u_0 is non-isolated, then the construction is evident. If u_0 is isolated, use lemma II.2 whenever u_0 is even, and the dual to lemma II.2 (i.e. replace "direct" by "inverse" and conversely) whenever u_0 is odd.

Then put $K = \bigcup_{u \in T_n} K^u$; this has the required properties.

II. If V_d and V_i are classes, then the proof is analoguous to that of theorem II.5.

II.6. Using lemma I.8, the following lemma may be proved easily:

Lemma: Let $\mathscr{k}$ be a small category, $\mathscr{V}$ a set of diagrams in $\mathscr{k}$. Then there exists a small category K such that $\mathscr{k}$ is a full subcategory of K, the inclusion-functor $\iota : \mathscr{k} \to K$ is $(\overrightarrow{\mathscr{k}^{Z}}, \overleftarrow{\mathscr{k}^{Z}})$-preserving and every $\mathscr{Y}\iota$, where $\mathscr{Y} \in \mathscr{V}$, has a direct limit in K.

II.7. Using lemma II.6 and its dual (i.e. replace "direct" by "inverse" and conversely) the following theorems are proved easily:

Theorem A: Let $\mathscr{k}$ be a small category, V a set of diagram schema. Then there exists a small $(\overrightarrow{V}, \overleftarrow{V})$-complete category K such that $\mathscr{k}$ is a full subcategory of K, and the inclusion-functor $\iota : \mathscr{k} \to K$ is $(\overrightarrow{\mathscr{k}^{Z}}, \overleftarrow{\mathscr{k}^{Z}})$-preserving.

Theorem B: Let $\mathscr{k}$ be a small category. Then there exists a complete category K such that $\mathscr{k}$ is a full subcategory of K and the inclusion-functor is $(\overrightarrow{\mathscr{k}^{Z}}, \overleftarrow{\mathscr{k}^{Z}})$-preserving.

II.8. Note: If a small category $\mathscr{k}$ has a system of null morphisms, then the category K from theorems II.7 A), II.7 B) also does. The theorems II.3 and II.5 may be modified for categories with a system of null morphisms as follows (use Note I.8 and Note I.9.2) in the proofs):

Definition. Every couple $\langle \mathcal{J}, p \rangle$ where $\mathcal{J}$ is a small category and $p \subset \mathcal{J}^{m}$, will be called a diagram schema with a fixity. If $\mathcal{J}$ is a discrete category and $p = \emptyset$, then $\langle \mathcal{J}, p \rangle$ will be called a discrete diagram schema with a fixity. Let V be a class of diagram schema with a fixity, let $\mathscr{k}$ be a category with a system of null morphisms. Let $\langle \mathcal{I}, p \rangle \in V$, and $\mathcal{G}: \mathcal{I} \to k$ a functor such that $(\alpha)\mathcal{G}$ is a null morphism for every $\alpha \in p$. Then $\mathcal{G}$ will be called a V-diagram in k. If V (or W) is a class of diagram schema with a fixity, then every category k with a system of null morphisms, in which every V-diagram (or W-diagram) has a direct (or an inverse) limit, will be called $\overrightarrow{V}$-complete (or $\overleftarrow{W}$-complete, respectively). If it is both $\overrightarrow{V}$-complete and $\overleftarrow{W}$-complete, then it will be called $(\overrightarrow{V}, \overleftarrow{W})$-complete.

Theorem (II.3)$'$: Let k be a small category with a system of null morphisms, $\mathbb{G}$ a set of diagrams in k (or a class of collections in k), V a class of diagram schema with a fixity (or a class of discrete diagram schema with a fixity, respectively). Then there exists a $\overrightarrow{V}$-complete category K such that:

1) k is a full subcategory of K, the inclusion-functor $\iota: k \to K$ is $(\overrightarrow{\mathbb{G}}, \overleftarrow{k^{z}})$-preserving.
2) If K' is a $\overrightarrow{V}$-complete category and if $\Phi: k \to K'$ is a $\overrightarrow{\mathbb{G}}$-preserving null functor, then there exists a $\overrightarrow{K^{V}}$-preserving null functor $\Phi': K \to K'$, unique up to natural equivalence, such that $\Phi = \iota \cdot \Phi'$.

If V is a set, then K is a small category.
Moreover, if $\aleph$ is a strongly inaccessible cardinal satisfying (a),(b) from lemma II.2, then $\operatorname{card} K^{m} \leqq \aleph$. Moreover, if (c) is satisfied, then $\operatorname{card} H_K(c, d) < \aleph$ for all $c, d \in K^{\sigma}$.

Theorem (II.5)$'$: Let k be a small category with a system of null morphisms. Let $\mathfrak{G}_d$, $\mathfrak{G}_i$ be sets of diagrams in $\mathfrak{k}$ (or classes of collections in $\mathfrak{k}$). Let V_d, V_i be classes of diagram schema with a fixity (or classes of discrete diagram schema with a fixity, respectively). Then there exists a $(\overrightarrow{V_d}, \overleftarrow{V_i})$-complete category K with a system of null morphisms such that:

1) $\mathfrak{k}$ is a full subcategory of K, the inclusion-functor $\iota : \mathfrak{k} \to K$ is $(\overrightarrow{\mathfrak{G}_d}, \overleftarrow{\mathfrak{G}_i})$-preserving.
2) If K' is a $(\overrightarrow{V_d}, \overleftarrow{V_i})$-complete category and $\Phi : \mathfrak{k} \longrightarrow K'$ is a $(\overrightarrow{\mathfrak{G}_d}, \overleftarrow{\mathfrak{G}_i})$-preserving null functor, then there exists a $(\overrightarrow{K^{V_d}}, \overleftarrow{K^{V_i}})$-preserving null functor $\Phi' : K \to K'$, unique up to natural equivalence, such that $\Phi = \iota \cdot \Phi'$.

If V_d, V_i are sets, then K is a small category. Furthermore, for a strongly inaccessible cardinal $\aleph$ the same conclusions obtain as in theorem II.5.

II.9. Corollary to theorem (II.5)$'$:

Let $\mathfrak{k}$ be a category with a system of null morphisms; let $\mathfrak{h}$ be a category such that $\mathfrak{h}^o = \{a, b\}$, $a \neq b$, $\mathfrak{h}^m = \{e_a, e_b, \alpha, \beta\}$, where $\alpha, \beta \in H_{\mathfrak{h}}(a, b)$, $\alpha \neq \beta$. Let $\mathcal{G} : \mathfrak{h} \to \mathfrak{k}$ be a V-diagram, where $V = \{\langle \mathfrak{h}, \{\beta\}\rangle\}$. Let $\langle a; \{v_\alpha, v_\beta\}\rangle$ be its direct limit in $\mathfrak{k}$, let $\langle b; \{\pi_\alpha, \pi_\beta\}\rangle$ be its inverse limit in $\mathfrak{k}$. Then it is well-known that

π_α is a kernel of $(\alpha)\mathcal{G}$, v_α is a cokernel of $(\alpha)\mathcal{G}$. Consequently the following theorem follows immediately from theorem (II.5)$'$:

Let $\mathfrak{k}$ be a small category with a system of null morphisms. Then there exists a small category K with a system of null morphisms such that

1) $\mathcal{k}$ is a full subcategory of K, the inclusion-functor $\iota : \mathcal{k} \to K$ preserves kernels and cokernels of all morphisms existing in $\mathcal{k}$. Every morphism of K has a kernel and a cokernel in K.

2) If K' is a category with a system of null morphisms, in which every morphism has a kernel and a cokernel, and if $\Phi : \mathcal{k} \to K'$ is a null functor which preserves kernels and cokernels existing in $\mathcal{k}$, then there exists a null functor $\Phi' : K \to K'$, unique up to natural equivalence, such that $\Phi = \iota . \Phi'$ and Φ' preserves kernels and cokernels.

Moreover, if $\aleph$ is a strongly inaccessible cardinal such that $card\, \mathcal{k}^m \leqq \aleph$, then $card\, K^m \leqq \aleph$. Moreover, if $card\, H_{\mathcal{k}}(c,d) < \aleph$ for all $c, d \in \mathcal{k}^\sigma$, then $card\, H_K(c,d) < \aleph$ for all $c, d \in K^\sigma$.

II.10. The theorem for partially ordered sets, analogous to Theorem II.5, may be proved in the same manner, only using II.2 Note b). If we choose $V_d = V_i = \{\mathcal{k}_0\}$, where $\mathcal{k}_0$ is a discrete category such that $\mathcal{k}_0^\sigma$ is a two-point set, and if every element of $\mathfrak{G}_d \cup \mathfrak{G}_i$ is V_d-diagram, we obtain the following theorem:

Let $(\mathcal{k}, \prec)$ be a partially ordered set, let $\mathfrak{G}_d \subset \mathcal{k} \times \mathcal{k}$, $\mathfrak{G}_i \subset \mathcal{k} \times \mathcal{k}$. Then there exists a lattice $(K, \precsim)$ such that

1) $\mathcal{k} \subset K$, the inclusion mapping $\iota : \mathcal{k} \to K$ is strongly order-preserving (i.e. if $a, b \in \mathcal{k}$, then $a \prec b \Longleftrightarrow a \precsim b$), if $\langle a, b \rangle \in \mathfrak{G}_d$ then $\sup_{\mathcal{k}} \{a, b\} =$

$= sup_K \{a, b\}$ whenever $sup_k \{a, b\}$ exists, if $\langle a, b\rangle \in \mathfrak{G}_i$ then $inf_k \{a, b\} = inf_K \{a, b\}$ whenever $inf_k \{a, b\}$ exists.

2) If $(K', \preccurlyeq)$ is a lattice and $\Phi : k \to K'$ is an order-preserving mapping (i.e. if $a, b \in k$ then $a \prec b \Rightarrow (a)\Phi \preccurlyeq (b)\Phi$) which preserves least upper bounds of all elements of $\mathfrak{G}_d$ and greatest lower bounds of all elements of $\mathfrak{G}_i$, then there exists exactly one lattice-homomorphism $\Phi' : K \to K'$ which extends Φ.

III. Embedding theorems for arbitrary categories.

The present section treats the same problems as section II; however, it is not assumed that k is a small category. The situation is then rather different.

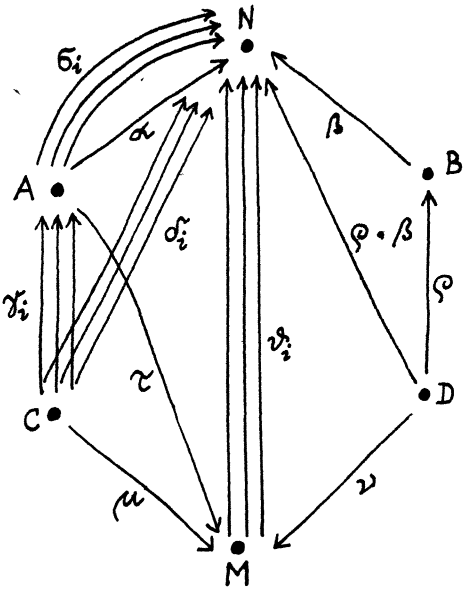
III.1. The theorems II.7 A) and II.7 B) are incorrect if we do not suppose that the category k is small. Moreover, the following proposition is not true:

If k is an arbitrary category, then there exists a category K such that k is isomorphic with a full subcategory of K, the embedding functor $\iota : k \to K$ preserves sums of all two-point collections in k and $\mathcal{C}\iota$, where $\mathcal{C}$ is a given two-point collection in k, has a sum in K.

The corresponding example will be given now:

Let m be a positive cardinal, $\mathcal{J}_m$ a set, $card\, \mathcal{J}_m = m$, denote by k_m the category as in the diagram (identities are not indicated), where i varies over $\mathcal{J}_m$, and for $i, j \in \mathcal{J}_m$ put

$$\gamma_i \cdot \alpha = \gamma_j \cdot \sigma_i = \tilde{\sigma}_i \ ;$$
$$\gamma_i \cdot \tau = \mu \ ;$$
$$\tau \cdot \vartheta_i = \sigma_i \ ;$$
$$\nu \cdot \vartheta_i = \varphi \cdot \beta \ ;$$



It is easy to see that $\langle M ; \{\mu, \nu\} \rangle$ is the sum of the collection $\{C, D\}$ in k_m and that $\{A, B\}$ has no sum in k_m. Let K be a category which contains k_m as a full subcategory and such that $\{A, B\}$ has a sum in K (denote it by $\langle S ; \{v_A , v_B\} \rangle$) and that $\langle M ; \{\mu, \nu\} \rangle$ is the sum of $\{C, D\}$ in K. We shall prove that necessarily $card\, H_K(A, S) \geq m$. Denote by φ the canonical morphism of the direct bound $\langle N ; \{\alpha, \beta\} \rangle$ in K. Since $v_A \cdot \varphi = \alpha$, $\gamma_i \cdot \alpha \neq \gamma_j \cdot \alpha$ for $i \neq j$, there is $\gamma_i \cdot v_A \neq \gamma_j \cdot v_A$. Now $\langle S ; \{\gamma_i \cdot v_A ; \varphi \cdot v_B\} \rangle$ is the direct bound of $\{C, D\}$ in K, denote by ψ_i its canonical morphism. Choose some $i_0 \in \mathcal{I}_m$. Then $\gamma_{i_0} \cdot \tau \cdot \psi_i \cdot \varphi = \mu \cdot \psi_i \cdot \varphi = \gamma_i \cdot v_A \cdot \varphi = \gamma_i \cdot \alpha$ for every $i \in \mathcal{I}_m$. Thus $\tau \cdot \psi_i \neq \tau \cdot \psi_j$

whenever $j \neq i$. Let now $\mathfrak{k}$ be the category obtained by binding together all $\mathfrak{k}_{\mathfrak{m}}$ (for all positive cardinals $\mathfrak{m}$) at the objects A, B . Equip all symbols for elements of $(\mathfrak{k}_{\mathfrak{m}})^{\sigma} \cup (\mathfrak{k}_{\mathfrak{m}})^{m}$, except for the objects A, B , by a suffix $\mathfrak{m}$, and extend the definition of $\mathfrak{k}$ as follows: If $\mathfrak{m} \neq \mathfrak{n}$, then the set $H_{\mathfrak{k}}(C^{\mathfrak{m}}, N^{\mathfrak{n}})$, or $H_{\mathfrak{k}}(D^{\mathfrak{m}}, N^{\mathfrak{n}})$ or $H_{\mathfrak{k}}(M^{\mathfrak{m}}, N^{\mathfrak{n}})$ contains exactly one morphism, denoted $c^{\mathfrak{n}}_{\mathfrak{m}}$ or $d^{\mathfrak{n}}_{\mathfrak{m}}$ or $m^{\mathfrak{n}}_{\mathfrak{m}}$ respectively. Put $H_{\mathfrak{k}}(C^{\mathfrak{m}}, M^{\mathfrak{n}}) = \{{}_i s^{\mathfrak{n}}_{\mathfrak{m}} ; i \in \mathcal{I}_{\mathfrak{m}}\}$ for $\mathfrak{m} \neq \mathfrak{n}$. If $\mathfrak{m} \neq \mathfrak{n} \neq \mathfrak{p}$,put $\gamma_i^{\mathfrak{m}} . \alpha^{\mathfrak{n}} = \gamma_i^{\mathfrak{m}} . \sigma_j^{\mathfrak{n}} = c^{\mathfrak{n}}_{\mathfrak{m}}$; $\rho^{\mathfrak{m}} . \beta^{\mathfrak{n}} = d^{\mathfrak{n}}_{\mathfrak{m}}$; $\mu^{\mathfrak{m}} . m^{\mathfrak{n}}_{\mathfrak{m}} = c^{\mathfrak{n}}_{\mathfrak{m}}$; $\nu^{\mathfrak{m}} . m^{\mathfrak{n}}_{\mathfrak{m}} = d^{\mathfrak{n}}_{\mathfrak{m}}$; $\gamma_i^{\mathfrak{m}} . \tau^{\mathfrak{n}} = {}_i s^{\mathfrak{n}}_{\mathfrak{m}}$; $\tau^{\mathfrak{n}} . m^{\mathfrak{m}}_{\mathfrak{n}} = \alpha^{\mathfrak{m}}$; ${}_i s^{\mathfrak{n}}_{\mathfrak{m}} . m^{\mathfrak{p}}_{\mathfrak{n}} = \gamma_i^{\mathfrak{m}} . \alpha^{\mathfrak{p}}$.
Then it is easy to see that every $\mathfrak{k}_{\mathfrak{m}}$ is a full subcategory of $\mathfrak{k}$, $\langle M^{\mathfrak{m}}; \{\mu^{\mathfrak{m}}, \nu^{\mathfrak{m}}\}\rangle$ is the sum of $\{C^{\mathfrak{m}}, D^{\mathfrak{m}}\}$ in $\mathfrak{k}$ and $\{A, B\}$ has no sum in $\mathfrak{k}$. If now K is a category such that $\mathfrak{k}$ is a full subcategory of K , $\langle M^{\mathfrak{m}}; \{\mu^{\mathfrak{m}}, \nu^{\mathfrak{m}}\}\rangle$ is the sum of $\{C^{\mathfrak{m}}, D^{\mathfrak{m}}\}$ in K and $\{A, B\}$ has a sum in K , denoted by $\langle S; \{v_A, v_B\}\rangle$, then necessarily $\operatorname{card} H_K(A, S) \geq \mathfrak{m}$ for every cardinal $\mathfrak{m}$, which is impossible.

III. 2. The proof of theorem II.3 is based on a construction by transfinite induction. However, within the Bernays-Gödel set-theory this cannot be carried out for categories which are not small. On the other hand, this is possible in any model of set theory where classes are modelled by sets. The existence of such a model follows from the existence of a strongly inaccessible cardinal number.
But it is well-known that the existence of a strongly inac-

cessible cardinal is not provable (from the axioms of the set-theory), and neither consistency nor inconsistency of the existence of such a cardinal number with the axioms of the set-theory has been proved. Now we shall suppose that there exists such a cardinal number μ x), and we will sketch the construction in this case. Denote by T the set of all cardinals smaller than μ. Put $U_0 = \{\emptyset\}$. For $\alpha \in T$ put $V_\alpha = \bigcup_{\beta<\alpha} U_\beta$, $\overline{V_\alpha} = V_\alpha \cup (V_\alpha \times V_\alpha)$, $U_\alpha = \overline{V_\alpha} \cup \exp V_\alpha$; $U = \bigcup_{\alpha \in T} U_\alpha$. Then the sets of the model are all subsets of U, the power of which is smaller than μ, and the classes of the model are all subsets of U. The relation $\in^*$ of belonging to in this model is a partialisation of $\in$. The construction of this model is given in detail in [1].

Using theorems II.3 and II.5 for categories of the model, we obtain the following result:

If the existence of a strongly inaccessible cardinal number is consistent with the axioms of the set-theory, then the following theorems are also consistent with the axioms of the set-theory xx):

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x) The assumption of existence of a strongly inaccessible cardinal is weaker, of course, than axiom A 5 given in [11]. The construction of a set U to follow is the construction of universe.

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xx) These theorems are true in the model.

Theorem A: Let $\mathcal{k}$ be an almost-category x), let $\mathfrak{G}$ be a class of diagrams in $\mathcal{k}$, let V be a class of diagram schema. Then there exists a $\overrightarrow{V}$-complete almost-category K such that:

1) $\mathcal{k}$ is a full sub-almost-category of K, the inclusion-functor $\iota : \mathcal{k} \to K$ is $(\overrightarrow{\mathfrak{G}}, \overleftarrow{\mathcal{k}^{z}})$-preserving;
2) If K' is a $\overrightarrow{V}$-complete almost-category and $\Phi : \mathcal{k} \to$ $\to K'$ is a $\overrightarrow{\mathfrak{G}}$-preserving functor, then there exists a $\overrightarrow{K^{V}}$-preserving functor $\Phi' : K \to K'$, unique up to natural equivalence, such that $\Phi = \iota \cdot \Phi'$.

Theorem B: Let $\mathcal{k}$ be a category, let $\mathfrak{G}$ be a s e t of diagrams in $\mathcal{k}$, V a class of diagram schema. Then there exists a $\overrightarrow{V}$-complete category K such that:

1) $\mathcal{k}$ is a full subcategory of K, the inclusion-functor $\iota : \mathcal{k} \to K$ is $(\overrightarrow{\mathfrak{G}}, \overleftarrow{\mathcal{k}^{z}})$-preserving.
2) If K' is a $\overrightarrow{V}$-complete category, and if $\Phi : \mathcal{k} \to K'$ is a $\overrightarrow{\mathfrak{G}}$-preserving functor, then there exists a $\overrightarrow{K^{V}}$-preserving functor $\Phi' : K \to K'$, unique up to natural equivalence, such that $\iota\, \Phi' = \Phi$.

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x) The notion of the almost-category is obtained if, in the definition of the notion of the category, we omit the axiom that all morphisms from one object to another are to form a set. Notions such as functors, their direct limits and so on may be introduced for almost-categories without any change. The diagram in an almost-category is a functor, the domain of which is a small category. A diagram schema is a small category.

Theorem C: Let $\mathfrak{k}$ be a category (or almost- category). Let $\mathfrak{G}_d$, $\mathfrak{G}_i$ be sets (or classes) of diagrams in $\mathfrak{k}$, let V_d, V_i be classes of diagram schema. Then there exists a category (or almost-category) K such that the statements 1) 2) from theorem II.5 are satisfied (or the statements 1) 2) from theorem II.5, where one replaces "category", "subcategory" by "almost-category", "sub-almost-category", respectively).

The theorems (II.3)$'$ and (II.5)$'$ given in II.9 for categories with a system of null morphisms, may be reformulated analogously.

III.3. Now we shall prove that if $\mathfrak{G} = \emptyset$, then theorem III.2 B may be proved for some classes V not only in the model, but in the theory (cf. III.5 - III.7).

Conventions and notation : Let $\mathfrak{k}$ be a category, $a \in \mathfrak{k}^{\sigma}$, let $\mathcal{G} : \mathcal{J} \to \mathfrak{k}$ be a diagram in $\mathfrak{k}$, $\alpha \in H_{\mathfrak{k}}(a, (j)\mathcal{G})$, $\beta \in H_{\mathfrak{k}}(a, (j')\mathcal{G})$, $j, j' \in \mathcal{J}^{\sigma}$. We shall say that $\langle \alpha, j \rangle$ and $\langle \beta, j' \rangle$ are $\mathcal{G}$-chainable if there exist $j_1, \dots, j_n \in \mathcal{J}^{\sigma}$ and $\gamma_\ell \in H_{\mathfrak{k}}(a, (j_\ell)\mathcal{G})$ for $\ell = 1, \dots, n$ such that $j_1 = j$, $j_n = j'$, $\gamma_1 = \alpha$, $\gamma_n = \beta$ and for every $\ell = 1, 2, \dots, n-1$ either there exists a $\sigma \in H_{\mathcal{J}}(j_\ell, j_{\ell+1})$ such that $\gamma_\ell \cdot (\sigma)\mathcal{G} = \gamma_{\ell+1}$, or there exists a $\sigma \in H_{\mathcal{J}}(j_{\ell+1}, j_\ell)$ such that $\gamma_{\ell+1} \cdot (\sigma)\mathcal{G} = \gamma_\ell$. Let $\mathcal{F} : \mathcal{I} \to \mathfrak{k}$, $\mathcal{G} : \mathcal{J} \to \mathfrak{k}$ be diagrams in $\mathfrak{k}$, denote by $P_{\mathcal{F}}^{\mathcal{G}}$ the set of all systems $v = \{ V_i ; i \in \mathcal{I}^{\sigma} \}$ such that:

a) $V_i \neq \emptyset$ for all $i \in \mathcal{I}^{\sigma}$.

$s \in V_i \Longrightarrow s = \langle \alpha, j \rangle$, where $j \in \mathcal{J}^{\sigma}$, $\alpha \in H_{\mathfrak{k}}((i)\mathcal{F}, (j)\mathcal{G})$.

b) $\langle \alpha, j \rangle \in V_i$, $\beta \in \mathfrak{k}^m$, $j' \in \mathcal{J}^\sigma \Rightarrow (\langle \beta, j' \rangle \in V_i \Leftrightarrow \langle \alpha, j \rangle$ and $\langle \beta, j' \rangle$ are $\mathcal{G}$-chainable).

c) $i, i' \in \mathcal{J}^\sigma$, $\sigma \in H_{\mathcal{J}}(i, i')$, $\langle \alpha, j \rangle \in V_i$, $\langle \alpha', j' \rangle \in V_{i'} \Rightarrow \langle \alpha, j \rangle$ and $\langle (\sigma)\mathcal{F} \cdot \alpha', j' \rangle$ are $\mathcal{G}$-chainable.

III.4. <u>Lemma</u>: Let $\mathcal{F}: \mathcal{J} \to \mathfrak{k}$, $\mathcal{G}: \mathcal{J} \to \mathfrak{k}$ be diagrams in $\mathfrak{k}$, let $v = \{V_i ; i \in \mathcal{J}^\sigma\} \in P_{\mathcal{F}}^{\mathcal{G}}$. Let $a \in \mathfrak{k}^\sigma$, $i, i' \in \mathcal{J}^\sigma$, $\mu \in H_{\mathfrak{k}}(a, (i)\mathcal{F})$, $\mu' \in H_{\mathfrak{k}}(a, (i')\mathcal{F})$, $\langle \alpha, j \rangle \in V_i$, $\langle \alpha', j' \rangle \in V_{i'}$.

If $\langle \mu, i \rangle$ and $\langle \mu', i' \rangle$ are $\mathcal{F}$-chainable, then $\langle \mu \cdot \alpha, j \rangle$ and $\langle \mu' \cdot \alpha', j' \rangle$ are $\mathcal{G}$-chainable.

<u>Proof</u>: If $\langle \mu, i \rangle$ and $\langle \mu', i' \rangle$ are $\mathcal{F}$-chainable, then there exist $i_1, \dots, i_n \in \mathcal{J}^\sigma$ and $\gamma_\ell \in H_{\mathfrak{k}}(a, (i_\ell)\mathcal{F})$ with the properties from III.3. The $\mathcal{G}$-chainability of $\langle \mu \cdot \alpha, j \rangle$ and $\langle \mu' \cdot \alpha', j' \rangle$ is proved easily by induction according to n.

III.5. <u>Theorem</u>: Let $\mathfrak{k}$ be a category, V a class of diagram schema.

Then there exists a category K with the following properties:

1) $\mathfrak{k}$ is isomorphic with a full subcategory of K, the embedding-functor $\iota : \mathfrak{k} \to K$ is $\overleftarrow{\mathfrak{k}}^{\mathfrak{z}}$-preserving. If $\mathcal{G}$ is a V-diagram in $\mathfrak{k}$, then $\mathcal{G}\iota$ has a direct limit in K.

2) If K' is a $\overrightarrow{V}$-complete category and $\Phi : \mathfrak{k} \to K'$ a functor, then there exists a functor $\Phi' : K \to K'$, unique up to natural equivalence, such that $\Phi = \iota \cdot \Phi'$ and Φ' preserves direct limits of all $\mathcal{G}\iota$, where $\mathcal{G}$ is a V-diagram in $\mathfrak{k}$.

Proof: I. The notation of III.3 will be used. Denote by k_0 a category such that k_0^m is a one-point set; put $\overline{V} = V \cup \{k_0\}$. Let k be a category, denote by $\mathbb{P}$ the class of all $\overline{V}$-diagrams in k. Let K be the category defined as follows: $K^\sigma = \mathbb{P}$; $H_K(\mathcal{F}, \mathcal{G}) = P_{\mathcal{F}}^{\mathcal{G}}$ for every $\mathcal{F}, \mathcal{G} \in \mathbb{P}$; the composition of morphisms in K is defined as follows: if $\mathcal{F}, \mathcal{G}, \mathcal{H} \in \mathbb{P}$, $v = \{V_i ; i \in \mathcal{I}^\sigma\} \in P_{\mathcal{F}}^{\mathcal{G}}$, $w = \{W_j ; j \in \mathcal{J}^\sigma\} \in P_{\mathcal{G}}^{\mathcal{H}}$, then $v \cdot w = \{\mathcal{U}_i ; i \in \mathcal{I}^\sigma\} \in P_{\mathcal{F}}^{\mathcal{H}}$ such that $\langle \alpha \cdot \beta, \ell \rangle \in \mathcal{U}_i$, where $\langle \alpha, j \rangle \in V_i$, $\langle \beta, \ell \rangle \in W_j$. That this definition of the composition is correct follows from lemma III.4.

II. Evidently, k may be embedded into K as a full subcategory; denote by ι the embedding-functor (i.e. $(a)\iota$ is the diagram $\mathcal{F} : k_0 \to k$ such that $i \in k_0^\sigma \Rightarrow (i)\mathcal{F} = a$).

III. Let now $\mathcal{G} : \mathcal{J} \to k$ be a diagram, $\mathcal{J} \in V$. We shall prove that $\mathcal{G} = |\overrightarrow{\lim}_K \mathcal{G}\iota|$. If $j \in \mathcal{J}^\sigma$, put $s_j = (j)\mathcal{G}$; denote by e_j the identity $e_{s_j} \in H_k(s_j, s_j)$. Take $v_j = \{V_j\} \in P_{(s_j)\iota}^{\mathcal{G}}$ such that $\langle e_j, j \rangle \in V_j$. Evidently, if $j, j' \in \mathcal{J}^\sigma$, $\sigma \in H(j, j')$, then $\langle e_j, j \rangle$ and $\langle (\sigma)\mathcal{G} \cdot e_{j'}, j' \rangle$ are $\mathcal{G}$-chainable, consequently $v_j = (\sigma)\mathcal{G} \cdot v_{j'}$. Thus $\langle \mathcal{G}; \{v_j ; j \in \mathcal{J}^\sigma\} \rangle$ is a direct bound of $\mathcal{G}\iota$ in K. If $\langle \mathcal{H}; \{h_j ; j \in \mathcal{J}^\sigma\} \rangle$ is a direct bound of $\mathcal{G}\iota$ in K, $h_j = \{H_j\} \in P_{(s_j)\iota}^{\mathcal{H}}$, then there exists a canonical morphism $h \in P_{\mathcal{G}}^{\mathcal{H}}$, namely $h = \{H_j ; j \in \mathcal{J}^\sigma\}$.

IV. Now it is easy to see that the embedding functor preserves inverse limits of all diagrams in $\mathfrak{k}$; this follows from the part III of the present proof and from lemma I.7.

V. Now let K' be a category in which every V-diagram has a direct limit, let $\Phi : \mathfrak{k} \to K'$ be a functor. We are to construct Φ' . Choose some $\overrightarrow{lim}_{K'} \mathcal{G}\Phi$ for every $\mathcal{G} \in K^{\sigma}$ (such that $|\overrightarrow{lim}_{K'} \mathcal{F}\Phi| = (c)\Phi$ whenever $c \in \mathfrak{k}^{\sigma}$, $(c)\iota = \mathcal{F}$), and put $(\mathcal{G})\Phi' = |\overrightarrow{lim}_{K'} \mathcal{G}\Phi|$.
Let $\mathcal{F}, \mathcal{G} \in K^{\sigma}$, $\mathcal{F} : \mathcal{I} \to \mathfrak{k}$, $\mathcal{G} : \mathcal{J} \to \mathfrak{k}$, $v = \{V_i ; i \in \mathcal{I}^{\sigma}\} \in H_K(\mathcal{F}, \mathcal{G})$.
Set $\langle a ; \{\varphi_i ; i \in \mathcal{I}^{\sigma}\}\rangle = \overrightarrow{lim}_{K'} \mathcal{F}\Phi$, $\langle b ; \{\psi_j ; j \in \mathcal{J}^{\sigma}\}\rangle = \overrightarrow{lim}_{K'} \mathcal{G}\Phi$.
Choose $\langle \alpha_i , j_i \rangle \in V_i$ for every $i \in \mathcal{I}^{\sigma}$.
Then $\langle b ; \{(\alpha_i)\Phi \cdot \psi_{j_i} ; i \in \mathcal{I}^{\sigma}\}\rangle$ is the direct bound in K' of $\mathcal{F}\Phi$. Denote by v' its canonical morphism and put $(v)\Phi = v'$ (if we choose another $\langle \alpha'_i , j'_i \rangle$, then, since $\langle \alpha_i , j_i \rangle$ and $\langle \alpha'_i , j'_i \rangle$ are $\mathcal{G}$-chainable, there is $(\alpha_i)\Phi \cdot \psi_{j_i} = (\alpha'_i)\Phi \cdot \psi_{j'_i}$). Now it is easy to see that if $\mathcal{F}, \mathcal{G}, \mathcal{H} \in K^{\sigma}$, $v \in P_{\mathcal{F}}^{\mathcal{G}}$, $w \in P_{\mathcal{G}}^{\mathcal{H}}$, then $(v)\Phi' \cdot (w)\Phi' = (v \cdot w)\Phi'$ and $(e_{\mathcal{F}})\Phi' = e_{(\mathcal{F})\Phi'}$; consequently Φ' is a functor. Obviously $\iota\Phi' = \Phi$.

VI. Now we should prove that Φ' preserves direct limits of all $\mathcal{G}\iota$, where $\mathcal{G}$ is a V-diagram in $\mathfrak{k}$. Set $\mathcal{G} : \mathcal{J} \to \mathfrak{k}$, $\overrightarrow{lim}_K \mathcal{G}\iota = \langle \mathcal{G} ; \{v_j ; j \in \mathcal{J}^{\sigma}\}\rangle$, $\overrightarrow{lim}_{K'} \mathcal{G}\iota\Phi' = \overrightarrow{lim}_{K'} \mathcal{G}\Phi = \langle b ; \{\psi_j ; j \in \mathcal{J}^{\sigma}\}\rangle$.

Then $(\mathcal{G})\Phi' = b$ follows immediately from the definition of Φ' . Set $(j)\mathcal{G}\iota = \mathcal{F}_j \in K^{\sigma}$; then $v_j = \{V_j\} \in P_{\mathcal{F}_j}^{\mathcal{G}}$.
Since ψ_j is the canonical morphism of the direct bound $\langle b ; \{\psi_j\}\rangle$ of $\mathcal{F}_j\Phi$ in K', there is $(v_j)\Phi' = \psi_j$.

VII. It remains to prove the unicity of Φ''. Let Φ'' be another functor $\Phi'' : K \to K'$ which also preserves direct limits of all $\mathcal{G}\iota$ with $\mathcal{G} \in K^{\sigma}$ and such that $\iota \cdot \Phi'' = \Phi$. We shall prove that Φ' and Φ'' are naturally equivalent. Let S be a skeleton of the category K', let $c : K' \to S$ be the natural functor of K' onto S. It is sufficient to prove $\Phi' c = \Phi'' c$. Evidently $(\alpha)\Phi' = (\alpha)\Phi''$ whenever $\alpha = (\beta)\iota$, $\beta \in k^{\sigma} \cup k^{m}$; consequently $\mathcal{G}\iota\,\Phi' = \mathcal{G}\iota\,\Phi''$ for every diagram $\mathcal{G}$ in k. Let $\mathcal{F}, \mathcal{G} \in K^{\sigma}$. Since $\langle \mathcal{F}; \{\varphi_i ; i \in \mathcal{I}^{\sigma}\} \rangle = \overrightarrow{\lim}_K \mathcal{F}\iota$, $\langle \mathcal{G}; \{\gamma_j ; j \in \mathcal{J}^{\sigma}\} \rangle = \overrightarrow{\lim}_K \mathcal{G}\iota$, necessarily $(\mathcal{F})\Phi' c = (\mathcal{F})\Phi'' c$, $(\varphi_i)\Phi' c = (\varphi_i)\Phi'' c$, and analogously for $\mathcal{G}$ and γ_j. Let $v = \{V_i ; i \in \mathcal{I}^{\sigma}\} \in P_{\mathcal{F}}^{\mathcal{G}}$ be given. Choose $\rho_i \in V_i$, $\rho_i \in H_k((i)\mathcal{F}, (j_i)\mathcal{G})$. Then $m = \langle \mathcal{G}; \{[(\rho_i)\iota] \cdot \gamma_{j_i} ; i \in \mathcal{I}^{\sigma}\} \rangle$ is the direct bound of $\mathcal{F}\iota$ in K and v is its canonical morphism. Since $(\mathcal{G})\Phi' c = (\mathcal{G})\Phi'' c$, $([(\rho_i)\iota] \cdot \gamma_{j_i})\Phi' c =$
$= ((\rho_i)\iota)\Phi' c \cdot (\gamma_{j_i})\Phi' c = ((\rho_i)\iota)\Phi'' c \cdot (\gamma_{j_i})\Phi'' c =$
$= ([(\rho_i)\iota] \cdot \gamma_{j_i})\Phi'' c$, necessarily $(v)\Phi' c = (v)\Phi'' c$.

III.6. Note and definition: For some classes V of diagram schema it may happen that the category K constructed in III.5 is V-complete and Φ' (cf. III.5.2)) preserves direct limits of all V-diagrams in K. Such classes V will be called c l o s e d. It is easy to see that the class of all small discrete categories and the class of all finite discrete categories are clos d. (If V is the class of all small discrete categories, then the construction given here and that given in [3] are the same.)

Now it will be proved that the class of all quasi-ordered sets is closed. By a modification of the proof it may be shown that the class of all small categories is closed (cf. [10]).

III.7. <u>Theorem</u>: The class of all quasi-ordered sets is closed.

<u>Proof</u>: The notation of the proof of theorem III.5 is preserved. Of course, $\mathbb{P}$ is now the class of all presheaves in $\mathcal{k}$.

I. Let $\mathbb{H} : \langle I, \rightarrow \rangle \to K$ be a presheaf in K ; we shall prove that it has a direct limit in K. Then $(i)\mathbb{H}$ is a presheaf in $\mathcal{k}$, denote it by ${}^i\mathcal{H} : \langle {}^i\mathcal{L}, {}^i\rho \rangle \to \mathcal{k}$ and set $\mathbb{H}_i^{i'} = \{ S_{i,\ell}^{i'} ; \ell \in {}^i\mathcal{L} \}$. We may suppose that *all* the sets ${}^i\mathcal{L}$, $\{\ell\} \times S_{i,\ell}^{i'}$ ($i \in I$, $\ell \in {}^i\mathcal{L}$) are disjoint. For every $i \in I$ let I_i be the set of all $i' \in I$, $i' \neq i$, $\langle i, i' \rangle \in \rightarrow$. Put $A_{i,\ell} = \bigcup_{i' \in I_i} \{\ell\} \times S_{i,\ell}^{i'}$, where $i \in I$, $\ell \in {}^i\mathcal{L}$; put $A_i = \bigcup_{\ell \in {}^i\mathcal{L}} A_{i,\ell}$.
Now we shall construct a presheaf $\mathcal{H} : \langle \mathcal{L}, \rho \rangle \to \mathcal{k}$ such that $\mathcal{H} = |\overrightarrow{lim}_K \ \mathbb{H}|$. Put $\mathcal{L} = (\bigcup_{i \in I} {}^i\mathcal{L}) \cup (\bigcup_{i \in I} A_i)$, $(\ell)\mathcal{H} = (\ell)\,{}^i\mathcal{H}$ for $\ell \in {}^i\mathcal{L}$, $(\langle \ell, \alpha, j \rangle)\mathcal{H} = (\ell)\,{}^i\mathcal{H}$ for $\langle \ell, \alpha, j \rangle \in A_{i,\ell}$.
Now define relations ${}^1\sigma$, ${}^2\sigma$ on $\mathcal{L}$ as follows:

$x \, {}^1\sigma \, y \iff x = \langle \ell, \alpha, j \rangle \in A_{i,\ell}$ and either $y = \ell$ (then put ${}^1\sigma_x^y = e_{(\ell)\,{}^i\mathcal{H}}$) or $y \in {}^{i'}\mathcal{L}$, $i' \in I_i$, $\alpha \in H_{\mathcal{k}}((\ell)^i\mathcal{H}, (y)^{i'}\mathcal{H})$ (then put ${}^1\sigma_x^y = \alpha$).

$x \, {}^2\sigma \, y \iff x, y \in {}^i\mathcal{L}$ for some $i \in I$ and $x \, {}^i\rho \, y$ (then put ${}^2\sigma_x^y = {}^i\mathcal{H}_x^y$).

Put $\bar{\rho} = {}^1\sigma \circ {}^2\sigma$, i.e. $\langle x, y \rangle \in \bar{\rho}$ if and only if there exists a z such that $\langle x, z \rangle \in {}^1\sigma$, $\langle z, y \rangle \in {}^2\sigma$.

Put $\rho = \overline{\rho} \cup {}^2\sigma \cup \Delta$, where Δ is the diagonal of $\mathcal{L} \times \mathcal{L}$. It is easy to see that ρ is reflexive and transitive. Put $\mathcal{H}^y_x = e_{(x)\mathcal{H}}$ for $\langle x, y \rangle \in \Delta$, $\mathcal{H}^y_x = {}^2\sigma^y_x$ for $\langle x, y \rangle \in {}^2\sigma$, $\mathcal{H}^y_x = {}^1\sigma^z_x \cdot {}^2\sigma^y_z$ for $\langle x, z \rangle \in {}^1\sigma$, $\langle z, y \rangle \in {}^2\sigma$. Evidently $\mathcal{H} : \langle \mathcal{L}, \rho \rangle \to \mathcal{k}$ as defined above is a presheaf in $\mathcal{k}$.

II. Now we shall define $v_i : {}^i\mathcal{H} \to \mathcal{H}$ such that $\langle \mathcal{H} ; \{ v_i ; i \in I \} \rangle$ will be the direct limit of H in K . Put $v_i = \{ V_{i,\ell} ; \ell \in {}^i\mathcal{L} \}$ where $V_{i,\ell}$ is the set of all $\langle \alpha, j \rangle$ $\mathcal{H}$ -chainable with $\langle e_{(\ell)^i\mathcal{H}}, \ell \rangle$ (of course $e_{(\ell)^i\mathcal{H}} \in H_{\mathcal{k}}((\ell)^i\mathcal{H}, (\ell)\mathcal{H})$).

a) First prove that $v_i = \mathsf{H}^{i'}_i \cdot v_{i'}$ for every $\langle i, i' \rangle \in \mathcal{A}$; it is sufficient to prove this for $i \neq i'$ only. Let $\ell \in {}^i\mathcal{L}$, $\langle \beta, \ell' \rangle \in S^{i'}_{i,\ell}$ be given, let $\beta \in H_{\mathcal{k}}((\ell)^i\mathcal{H}, (\ell')^{i'}\mathcal{H})$; it is sufficient to find $\langle \sigma, \overline{\ell} \rangle \in V_{i,\ell}$, $\langle \sigma', \overline{\ell}' \rangle \in V_{i',\ell'}$ such that $\langle \sigma, \overline{\ell} \rangle$, $\langle \beta \cdot \sigma', \overline{\ell}' \rangle$ are $\mathcal{H}$ - chainable .

$$\begin{tikzcd}
 & ((\langle \ell, \beta, \ell' \rangle))\mathcal{H} \arrow[dl, "e_{(\ell)\mathcal{H}}"'] \arrow[dr, "\beta"] & \\
(\ell)\mathcal{H}\ (\ell \in \mathcal{L}) & & (\ell')\mathcal{H}\ (\ell' \in \mathcal{L}) \\
(\ell)^i\mathcal{H}\ (\ell \in {}^i\mathcal{L}) \arrow[u, "e_{(\ell)^i\mathcal{H}}"] \arrow[uur, dashed, "e_{(\ell)^i\mathcal{H}}"'] \arrow[rr, "\beta"] & & (\ell')^{i'}\mathcal{H}\ (\ell' \in {}^{i'}\mathcal{L}) \arrow[u, "e_{(\ell')^{i'}\mathcal{H}}"']
\end{tikzcd}$$

Put $\langle \sigma, \overline{\ell} \rangle = \langle e_{(\ell)^i\mathcal{H}}, \ell \rangle$, $\langle \sigma', \overline{\ell}' \rangle = \langle e_{(\ell')^{i'}\mathcal{H}}, \ell' \rangle$. We shall show that $\langle e_{(\ell)^i\mathcal{H}}, \ell \rangle$ and $\langle \beta, \ell' \rangle$ are $\mathcal{H}$ -chainable..

Take $\ell_1 = \ell$, $\ell_2 = \langle \ell, \beta, \ell' \rangle$, $\ell_3 = \ell'$, $\gamma_1 = e_{(\ell)\mathcal{H}}$, $\gamma_2 = e_{(\ell)^i\mathcal{H}}$, $\gamma_3 = \beta$. Then $\ell_2 \,\rho\, \ell_1$, $\gamma_2 \cdot \mathcal{H}^{\ell_1}_{\ell_2} = \gamma_1$ and $\ell_2 \,\rho\, \ell_3$, $\gamma_2 \cdot \mathcal{H}^{\ell_3}_{\ell_2} = \beta$.

b) Let now $\mathsf{H}' = \langle \mathcal{H}' ; \{ v'_i ; i \in I \} \rangle$ be a direct bound of H in K , let $v'_i = \{ V'_{i,\ell} ; \ell \in {}^i\mathcal{L} \}$. It is easy to see that there exists exactly one canonical morphism of H' in K, namely $\mathfrak{v} = \{ V'_{i,(x)\pi_i} ; x \in {}^i\mathcal{L} \cup A_i, i \in \mathbb{I} \}$, where $\pi_i : {}^i\mathcal{L} \cup A_i \to {}^i\mathcal{L}$ is a mapping such that $(\ell)\pi_i = \ell$ for $\ell \in {}^i\mathcal{L}$, $(\langle \ell, \beta, \ell' \rangle)\pi_i = \ell$ for $\langle \ell, \beta, \ell' \rangle \in A_i$.

III. Now we shall prove that Φ' preserves direct limits of all presheaves in K. Let $\mathbb{H} : \langle \mathbb{I} ; \prec \rangle \to K$ be a presheaf in K. The notation from parts I and II of the present proof will be used. Set $\langle {}^i a ; \{ {}^i\varphi_\ell ; \ell \in {}^i\mathcal{L} \} \rangle = \overrightarrow{\lim}_{K'} {}^i\mathcal{H}\Phi$, $\langle a ; \{ v_x ; x \in \mathcal{L} \} \rangle = \overrightarrow{\lim}_{K'} \mathcal{H}\Phi$.
If $\rho^{i'}_{i,\ell} \in S^{i'}_{i,\ell}$, $\rho^{i'}_{i,\ell} \in H_k((\ell)^i\mathcal{H}, (\ell')^{i'}\mathcal{H})$, then ${}^i\varphi_\ell \cdot (\mathbb{H}^{i'}_i)\Phi' = (\rho^{i'}_{i,\ell})\Phi \cdot {}^{i'}\varphi_{\ell'}$; this follows directly from the definition of Φ'. $\langle (\mathcal{H})\Phi' ; \{ (\mathfrak{v}_i)\Phi' ; i \in \mathcal{I} \} \rangle$ is direct bound of $\mathbb{H}\Phi'$ in K', since $\langle \mathcal{H} ; \{ \mathfrak{v}_i ; i \in \mathcal{I} \} \rangle$ is the direct bound of $\mathbb{H}$ in K; ${}^i\varphi_\ell \cdot (\mathfrak{v}_i)\Phi' = v_x$ whenever $x = \ell \in {}^i\mathcal{L}$. Let now $\langle b ; \{ \psi_i ; i \in \mathbb{I} \} \rangle$ be a direct bound of $\mathbb{H}\Phi'$ in K', $\psi_i \in H_{K'}({}^i a, b)$.
Set ${}^i\alpha_\ell = {}^i\varphi_\ell \cdot \psi_i$. Then $\langle b ; \{ {}^i\alpha_\ell ; \ell \in {}^i\mathcal{L} \} \rangle$ is the direct bound of ${}^i\mathcal{H}\Phi$ in K', and if $\langle i, i' \rangle \in \prec$, $\rho^{i'}_{i,\ell} \in S^{i'}_{i,\ell}$, $\rho^{i'}_{i,\ell} \in H_k((\ell)^i\mathcal{H}, (\ell')^{i'}\mathcal{H})$, then ${}^i\alpha_\ell = (\rho^{i'}_{i,\ell})\Phi \cdot {}^{i'}\alpha_\ell$. Put $\alpha_x = {}^i\alpha_\ell$ whenever either $x = \ell \in {}^i\mathcal{L}$ or $x \in \langle \ell, \alpha, j \rangle \in A_i$. Then $\langle b ; \{ \alpha_x ; x \in \mathcal{L} \} \rangle$ is a direct bound in K' of $\mathcal{H}\Phi$; denote by f its canonical morphism in K', i.e. $v_x \cdot f = \alpha_x$. Then ${}^i\varphi_\ell \cdot (\mathfrak{v}_i)\Phi' \cdot f = {}^i\varphi_\ell \cdot \psi_i$ whenever $i \in \mathbb{I}$, $\ell \in {}^i\mathcal{L}$, and thus $(\mathfrak{v}_i)\Phi' \cdot f = \psi_i$ for all $i \in \mathbb{I}$. The unicity of such a morphism need be proved. Let also

$(v_i)\Phi' . f' = \psi_i$ for some f' and all $i \in I$. Then $v_x . f' = \alpha_x$ for all $x \in \ell \in {}^i\mathcal{L}$; if $x' = \langle \ell, \alpha, j \rangle \in \in A_i$, $x = \ell \in {}^i\mathcal{L}$, then $\alpha_x = \alpha_{x'}$, $(x)\mathcal{H} = (x')\mathcal{H}$, $\langle x', x \rangle \in \rho$, $\mathcal{H}^x_{x'} = e_{(x)\mathcal{H}}$, $v_{x'} = (\mathcal{H}^x_{x'})\Phi . v_x = v_x$, and consequently $v_{x'} . f' = \alpha_{x'}$. Since f' is also the canonical morphism of the direct bound $\langle b ; \{\alpha_x ; x \in \mathcal{L}\} \rangle$ of $\mathcal{H}\Phi$ in K' , necessarily $f' = f$.

APPENDIX.

The proof of lemmas I.2 will be given now. The proof of lemma I.2 A is given explicitly; the modifications necessary for the proof of lemma I.2 B are indicated at the appropriate places in parentheses ⊂ ⊃ .

Let there be given a category ℓ , let $a, b \in \ell^\sigma$. For **every $\rho \in H_\ell(a, c)$ with $c \neq a$, let there be given a morphism ${}_\mu\rho \in H_\ell(b, c)$ such that statements a) b) from the lemma are satisfied.**

A) <u>Suppose that</u> $e_a = \alpha . \beta$, <u>where</u> $d \neq a$, $\alpha \in H_\ell(a,d)$, $\beta \in H_\ell(d, a)$:

Put $\mu = {}_\mu\alpha . \beta$. **Then evidently** $\mu . \rho = {}_\mu\rho$ **for every** $\rho \in H_\ell(a, c)$, $c \neq a$. **If** $\Phi : \ell \to K$ **is a functor and there exists a** $\mu' \in H_K((b)\Phi, (a)\Phi)$ **such that** $\mu' . (\rho)\Phi = ({}_\mu\rho)\Phi$ **whenever** $\rho \in H_\ell(a, d)$, $d \neq a$, **then necessarily** $(\mu)\Phi = \mu'$. **Consequently we may put** $h = \ell$, $\Psi = \Phi$.

<u>Convention</u>: **Let n be arbitrary positive integer, let M be a non-empty set. An n-tuple $\langle m_1, \dots, m_n \rangle$ of elements of M is an element $m_1 \in M$ for $n = 1$,**

and for $n \geq 2$ it is a mapping of the set $\{1, \dots, n\}$ into the set M such that the image of each $i \in \{1, \dots, n\}$ is $m_i \in M$; then m_i is called i-th member of the n - tuple. Now let m be an element, let M be a set, let n be a positive integer. Denote by M_n the set of all n - tuples of elements from the set $M \cup \{m\}$, at least one member of which is m. The fact that $M \cap \cap (\bigcup_{n=1}^{\infty} M_n) = \emptyset$ will be denoted by $m \notin\notin M$. The fact that for every set M there exists an m such that $m \notin\notin M$ will often be used.

B) Suppose that A) does not hold and $a \neq b$.

We shall describe a category h with the required properties. Take some $\bar{u} \notin\notin \ell^m$. Put $h^\sigma = \ell^\sigma$, $H_h(c, d) = H_\ell(c, d)$ for every $c, d \in h^\sigma$, $d \neq a$. Let Σ be the set of all $\sigma \in H_\ell(a, a)$ such that $\sigma = \alpha . \beta$, where $\alpha \in H_\ell(a, c)$, $\beta \in H_\ell(c, a)$ for some $c \neq a$. Set $\Sigma' = H_\ell(a, a) - \Sigma$. Put $H_h(d, a) = H_\ell(d, a) \cup H$ for every $d \in \ell^\sigma$, where $H = H_\ell(d, b) \times \{\bar{u}\} \times \Sigma'$. $\langle$ Moreover it is necessary to identify $\langle \omega_{d,b}, \bar{u}, \sigma \rangle$ with $\omega_{d,a}$ where ω denotes the null morphisms. $\rangle$

It is sufficient to define the composition only when some factor is an element of H (for other cases the composition law is given by the requirement that ℓ is to be a subcategory of h). Let ν be a morphism of ℓ into b, let $\sigma \in H_\ell(a, a)$. Set $\langle \nu, \bar{u}, \sigma \rangle^* = \langle \nu, \bar{u}, \sigma \rangle$ if $\sigma \in \Sigma'$, and $\langle \nu, \bar{u}, \sigma \rangle^* = \nu ._\mu \alpha . \beta$ if $\sigma \in \Sigma$, $\sigma = \alpha . \beta$, $\alpha \notin H_\ell(a, a)$. Put $\rho . \langle \nu, \bar{u}, \sigma \rangle = \langle \rho . \nu, \bar{u}, \sigma \rangle$ for $\rho \in \ell^m$;

$\langle \nu, \bar{u}, \sigma \rangle . \tau = \langle \nu, \bar{u}, \sigma . \tau \rangle^*$ for $\tau \in H_\ell(a, a)$;

$\langle \nu, \bar{\mu}, \sigma \rangle . \tau = \nu \cdot_{\mu} (\sigma . \tau)$ for $\tau \in H_{\ell}(a, c)$, $c \neq a$;
$\langle \nu, \bar{\mu}, \sigma \rangle . \langle \nu', \bar{\mu}, \sigma' \rangle = \langle \nu \cdot_{\mu} (\sigma \cdot \nu'), \bar{\mu}, \sigma' \rangle$ for
$\nu' \in H_{\ell}(a, b)$.

The associative law for this composition will now be proved.

a) Evidently $\rho . (\rho' . \langle \nu, \bar{\mu}, \sigma \rangle) = (\rho . \rho') . \langle \nu, \bar{\mu}, \sigma \rangle$ for $\rho, \rho' \in \ell^m$;

b) $(\langle \nu, \bar{\mu}, \sigma \rangle . \tau) . \tau' = \langle \nu, \bar{\mu}, \sigma \rangle . (\tau . \tau')$ for $\tau \in$ $\in H_{\ell}(a, a)$, $\tau' \in \ell^m$;

c) $(\langle \nu, \bar{\mu}, \sigma \rangle . \tau) . \tau' = \nu \cdot_{\mu} (\sigma . \tau) . \tau' = \nu \cdot_{\mu} (\sigma . \tau . \tau') = \langle \nu, \bar{\mu}, \sigma \rangle . (\tau . \tau')$ for $\tau \in H_{\ell}(a, c)$, $\tau' \in H_{\ell}(c, d)$, $c \neq a \neq d$;

d) $(\langle \nu, \bar{\mu}, \sigma \rangle . \tau) . \tau' = \nu \cdot_{\mu} (\sigma . \tau) . \tau' = \langle \nu, \bar{\mu}, (\sigma . \tau) . \tau' \rangle^* = \langle \nu,$ $\bar{\mu}, \sigma \rangle . (\tau . \tau')$ for $\tau \in H_{\ell}(a, c)$, $\tau' \in H_{\ell}(c, a)$;

e) $\rho . (\langle \nu, \bar{\mu}, \sigma \rangle . \tau) = (\rho . \langle \nu, \bar{\mu}, \sigma \rangle) . \tau$ for $\rho, \tau \in$ $\in \ell^m$;

f) $(\langle \nu, \bar{\mu}, \sigma \rangle . \langle \nu', \bar{\mu}, \sigma' \rangle) . \tau = \langle \nu \cdot_{\mu} (\sigma . \nu'), \bar{\mu}, \sigma' . \tau \rangle^* =$ $= \langle \nu, \bar{\mu}, \sigma \rangle . (\langle \nu', \bar{\mu}, \sigma' \rangle . \tau)$ for $\tau \in H_{\ell}(a, a)$;

g) $(\langle \nu, \bar{\mu}, \sigma \rangle . \langle \nu', \bar{\mu}, \sigma' \rangle) . \tau = \nu \cdot_{\mu} (\sigma . \nu') \cdot_{\mu} (\sigma' . \tau) = \nu \cdot_{\mu} (\sigma .$ $. \nu' \cdot_{\mu} (\sigma' . \tau)) = \langle \nu, \bar{\mu}, \sigma \rangle . (\langle \nu', \bar{\mu}, \sigma' \rangle . \tau)$ for $\tau \in H_{\ell}(a, c), c \neq a$;

h) $\rho . (\langle \nu, \bar{\mu}, \sigma \rangle . \langle \nu', \bar{\mu}, \sigma' \rangle) = \langle \rho . \nu \cdot_{\mu} (\sigma . \nu'), \bar{\mu}, \sigma' \rangle =$ $= (\rho . \langle \nu, \bar{\mu}, \sigma \rangle) . \langle \nu', \bar{\mu}, \sigma' \rangle$ for $\rho \in \ell^m$;

i) $\langle \nu, \bar{\mu}, \sigma \rangle . (\rho . \langle \nu', \bar{\mu}, \sigma' \rangle) = \langle \nu \cdot_{\mu} (\sigma . \rho . \nu'), \bar{\mu}, \sigma' \rangle =$ $= (\langle \nu, \bar{\mu}, \sigma \rangle . \rho) . \langle \nu', \bar{\mu}, \sigma' \rangle$ for $\rho \in \ell^m$;

j) $(\langle \nu, \bar{\mu}, \sigma \rangle . \langle \nu', \bar{\mu}, \sigma' \rangle) . \langle \nu'', \bar{\mu}, \sigma'' \rangle = \langle \nu \cdot_{\mu} (\sigma . \nu'), \bar{\mu},$ $\sigma' \rangle . \langle \nu'', \bar{\mu}, \sigma'' \rangle = \langle \nu \cdot_{\mu} (\sigma . \nu') \cdot_{\mu} (\sigma' . \nu''), \bar{\mu}, \sigma'' \rangle = \langle \nu \cdot_{\mu} (\sigma .$ $. \nu' \cdot_{\mu} (\sigma' . \nu'')), \bar{\mu}, \sigma'' \rangle = \langle \nu, \bar{\mu}, \sigma \rangle . \langle \nu' \cdot_{\mu} (\sigma' . \nu''),$ $\bar{\mu}, \sigma'' \rangle = \langle \nu, \bar{\mu}, \sigma \rangle . (\langle \nu', \bar{\mu}, \sigma' \rangle . \langle \nu'', \bar{\mu}, \sigma'' \rangle)$.

Set $\mu = \langle e_b, \bar{\mu}, e_a \rangle$. Then evidently $\langle \nu, \bar{\mu}, \sigma \rangle =$ $= \nu . \mu . \sigma$ and $\mu . \rho = {}_{\mu}\rho$ for every

$\rho \in H_\ell(a,c)$, $c \neq a$.

Now let $\Phi : \ell \to K$ be a functor with the properties from the lemma. To show that Φ can be extended to the whole $\mathcal{h}$ it suffices to prove the following implication:

$\rho . \mu . \sigma = \rho' . \mu . \sigma' \Rightarrow (\rho)\Phi . \mu' . (\sigma)\Phi = (\rho')\Phi . \mu' . (\sigma')\Phi$.

The implication is trivial $\langle$ if $\rho . \mu . \sigma \neq \omega$ and $\rangle$ if $\sigma, \sigma' \in \Sigma'$; then necessarily $\rho = \rho'$, $\sigma = \sigma'$. If $\sigma \in \Sigma$ and $\rho . \mu . \sigma = \rho' . \mu . \sigma' \langle \neq \omega \rangle$ then $\sigma' \in \Sigma$. Let $\sigma = \alpha . \beta$, $\sigma' = \alpha' . \beta'$, $\alpha \notin H_\ell(a,a)$, $\alpha' \notin H_\ell(a,a)$. Then $\rho ._\mu \alpha . \beta = \rho . \mu . \sigma = \rho' . \mu . \sigma' = \rho' ._\mu \alpha' . \beta'$, and thus $(\rho)\Phi . \mu' . (\sigma)\Phi = (\rho')\Phi . \mu' . (\sigma')\Phi$. The proof is analogous if $\sigma, \sigma' \in H_\ell(a,c)$, $c \neq a$. The unicity of the extension of Φ such that μ' is the image of μ is evident. $\langle$If $\rho.\mu.\sigma = \omega$ then also $(\rho)\Phi . \mu' . (\sigma)\Phi = (\rho')\Phi . \mu' . (\sigma')\Phi$.$\rangle$

C) <u>Suppose that A) does not hold and</u> $a = \ell$:

First the following sublemma will be proved:

<u>Sublemma</u>: Let G be a semigroup with the unit e $\langle$ and with the zero $0 \rangle$. Let $H \subset G$, let $e \notin H$, let $\rho . \sigma . . \rho' \in H$ for every $\rho, \rho' \in G$, $\sigma \in H$. For every $\sigma \in H$ let there be given some ${}_\tau\sigma \in H$ with ${}_\tau\sigma . \rho = {}_\tau(\sigma . \rho)$. Then there exists a semigroup P such that:

1) G is a subsemigroup of P, e is the unit of P $\langle$ 0 is the zero of $P \rangle$, and there exists a $\mu \in P$ such that $\mu . \sigma = {}_\tau\sigma$ for all $\sigma \in H$;

2) if Φ is a homomorphism of G into some semigroup G' with unit e $\langle$and zero $0' \rangle$, $(e)\Phi = e'$ $\langle (0)\Phi = 0' \rangle$, and if there exists a $\mu' \in G'$ such that $\mu' . (\sigma)\Phi = ({}_\tau\sigma)\Phi$ for every $\sigma \in H$,

then there exists exactly one homomorphism $\Psi : P \to G'$ such that $(\mu)\Psi = \mu'$ and that $\Phi = \iota \cdot \Psi$ for the inclusion $\iota : G \to P$.

3) Every $\alpha \in P$ may be written in the form $\alpha = \alpha_1 \cdot \ldots \cdot \alpha_n$, where $\alpha_1 \in G$, $\alpha_i \in \{\mu\} \cup [G - (\{e\} \cup H)]$ for $i = 2, \ldots, n$ and if $\alpha_i \neq \mu$, then $\alpha_{i+1} = \mu$ for $i = 1, \ldots, n-1$. This expression is unique (for $\alpha \neq 0$).

Proof of the sublemma: Take some $\tau \notin G$. Let P be the set of all n-tuples $(n = 1, 2, \ldots) \langle \alpha_1, \ldots, \alpha_n \rangle$ of elements of the set $G \cup \{\tau\}$ such that $\alpha_1 \in G$ (if $\alpha_1 = 0$ then it $= 0$), if $n \geq 2$ then $\alpha_2 = \tau$, $\alpha_i \notin H \cup \{e\}$ for $i \in \{2, 3, \ldots, n\}$, and either $\alpha_i \notin G$ or $\alpha_{i+1} \notin G$ for $i \in \{2, 3, \ldots, n\}$. Evidently (cf. the convention) $G \subset P$ and $\langle \alpha_1, \ldots, \alpha_n \rangle = \langle \beta_1, \ldots, \beta_m \rangle$ if and only if $n = m$ and $\alpha_i = \beta_i$ for $i = 1, \ldots, n$. Set $\overline{G} = G - (\{e\} \cup H)$, and ${}_{\tau^n}\sigma = {}_{\tau}({}_{\tau^{n-1}}\sigma)$ for every $\sigma \in H$ and positive integer n. Evidently ${}_{\tau^m}\sigma \cdot \rho = {}_{\tau^m}(\sigma \cdot \rho)$. Now we shall define the composition in P. Let $\langle \alpha_1, \ldots, \alpha_n \rangle, \langle \beta_1, \ldots, \beta_m \rangle \in P$. Put $\langle \alpha_1, \ldots, \alpha_n \rangle \cdot \langle \beta_1, \ldots, \beta_m \rangle =$ (0 whenever $\langle \alpha_1, \ldots, \alpha_n \rangle = 0$ or $\langle \beta_1, \ldots, \beta_m \rangle = 0$; in the remaining cases let)

$= \langle \alpha_n \cdot \beta_1, \beta_2, \ldots, \beta_m \rangle$ for $n = 1$;

$= \langle \alpha_1, \ldots, \alpha_n, \beta_2, \ldots, \beta_m \rangle$ for $\beta_1 = e$;

$= \langle \alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_m \rangle$ for $\alpha_n = \tau$, $\beta_1 \in \overline{G}$;

$= \langle \alpha_1, \ldots, \alpha_n \cdot \beta_1, \beta_2, \ldots, \beta_m \rangle$ for $n \geq 2$, $\alpha_n \in \overline{G}$, $\alpha_n \cdot \beta_1 \in \overline{G}$;

$= \langle \alpha_1, \ldots, \alpha_{n-1}, \beta_2, \ldots, \beta_m \rangle$ for $n \geq 2$, $\alpha_n \in \overline{G}$, $\alpha_n \cdot \beta_1 = e$;

$= \langle \sigma, \beta_2, \ldots, \beta_m \rangle$ for $\alpha_n \in \overline{G}$, $\alpha_n \cdot \beta_1 \in H$, $n \geq 2$;

$= \langle \rho, \beta_2, \ldots, \beta_m \rangle$ for $\alpha_n = \tau$, $\beta_1 \in H$;

here σ and ϱ are the elements of G defined as follows:
Let $i_1 < i_2 < \dots < i_k$, $i_\ell \in \{1, \dots, n\}$, $(\ell = 1, \dots, k)$ and let $\alpha_i \neq \tau$ for $i \in \{1, \dots, n-1\}$ if and only if $i = i_\ell$ for some $\ell \in \{1, \dots, k\}$ (evidently $i_1 = 1$). Then
$\sigma = \alpha_{i_1} \cdot \tau^{i_2-1-i_1} \{\alpha_{i_2} \cdot \dots \cdot \tau^{i_{k-1}-1-i_{k-2}} [\alpha_{i_{k-1}} \cdot \tau^{i_k-1-i_{k-1}} ((\alpha_{i_k} \cdot \tau^{n-1-i_k} (\alpha_n \cdot \beta_1))] \dots \}$, $\varrho = \alpha_{i_1} \cdot \tau^{i_2-1-i_1} \{\alpha_{i_2} \cdot \dots \cdot \tau^{i_{k-1}-1-i_{k-2}} [\alpha_{i_{k-1}} \cdot \tau^{i_k-1-i_{k-1}} (\alpha_{i_k} \cdot \tau^{n-i_k} \beta_1)] \dots \}$.
Now we shall prove that the composition is associative:
Let $\langle \alpha_1, \dots, \alpha_n \rangle, \langle \beta_1, \dots, \beta_m \rangle, \langle \gamma_1, \dots, \gamma_z \rangle \in P$. The equality $(*)$:
$(\langle \alpha_1, \dots, \alpha_n \rangle \cdot \langle \beta_1, \dots, \beta_m \rangle) \cdot \langle \gamma_1, \dots, \gamma_z \rangle = \langle \alpha_1, \dots, \alpha_n \rangle \cdot (\langle \beta_1, \dots, \beta_m \rangle \langle \gamma_1, \dots, \gamma_z \rangle)$
(holds trivially if $\langle \alpha_1, \dots, \alpha_n \rangle = 0$ or $\langle \beta_1, \dots, \beta_m \rangle = 0$ or $\langle \gamma_1, \dots, \gamma_z \rangle = 0$; in the remaining cases it) is obvious if $m \geq 2$ and neither $(\beta_m \in \overline{G}$ and $\beta_m \cdot \gamma_1 \in H)$ nor $(\beta_m = \tau$ and $\gamma_1 \in H)$.
Indeed, $\langle \alpha_1, \dots, \alpha_n \rangle \cdot \langle \beta_1, \dots, \beta_m \rangle = \langle .^{+}., \beta_2, \dots, \beta_m \rangle$, $\langle \beta_1, \dots, \beta_m \rangle \cdot \langle \gamma_1, \dots, \gamma_z \rangle = \langle \beta_1, \dots, \beta_{m-1}, .^{++}. \rangle$, and therefore $(\langle \alpha_1, \dots, \alpha_n \rangle \cdot \langle \beta_1, \dots, \beta_m \rangle) \cdot \langle \gamma_1, \dots, \gamma_z \rangle = \langle .^{+}., \beta_2, \beta_3, \dots, \beta_{m-1}, .^{++}. \rangle = \langle \alpha_1, \dots, \alpha_n \rangle \cdot (\langle \beta_1, \dots, \beta_m \rangle \cdot \langle \gamma_1, \dots, \gamma_z \rangle)$, where $.^{+}.$ and $.^{++}.$ are to be replaced by the appropriate expressions. Now we will treat the following cases.

a) $m = 1$:

1) if $\beta_1 = e$ then $(*)$ is trivial.

2) Let $\beta_1 \in G - \{e\}$, $\beta_1 \cdot \gamma_1 \notin H$; then evidently $\beta_1 \in \overline{G}$ and the following cases may occur:

I. $\alpha_n = \tau$. Then $(\langle \alpha_1, \dots, \alpha_n \rangle \cdot \beta_1) \cdot \langle \gamma_1, \dots, \gamma_z \rangle = \langle \alpha_1, \dots, \alpha_n, \beta_1 \cdot \gamma_1, \gamma_2, \dots, \gamma_z \rangle$ for $\beta_1 \cdot \gamma_1 \neq e$ or $(\langle \alpha_1, \dots, \alpha_n \rangle \cdot \beta_1) \cdot \langle \gamma_1, \dots, \gamma_z \rangle = \langle \alpha_1, \dots, \alpha_n, \gamma_2, \dots, \gamma_z \rangle$ for $\beta_1 \cdot \gamma_1 = e$; then $(*)$ evidently holds.

II. $\alpha_n \in G$ and $\alpha_n \cdot \beta_1 \cdot \gamma_1 \notin H$; then evidently $\alpha_n \cdot \beta_1 \notin H$ and it is easy to see that $(*)$ holds.

III. $\alpha_n \in G$ and $\alpha_n \cdot \beta_1 \cdot \gamma_1 \in H$; then necessarily $\alpha_n \cdot \beta_1 \neq e$.

Let $i_1 < i_2 < \ldots < i_k$, $i_\ell \in \{1, \ldots, n\}$ for $\ell \in \{1, \ldots, k\}$, and let $\alpha_i \neq \tau$ if and only if $\alpha_i = \alpha_{i_\ell}$ for some $\ell \in \{1, \ldots, k\}$. The following cases may occur:

α) $\alpha_n \cdot \beta_1 \notin H$. Then $(\langle \alpha_1, \ldots, \alpha_n \rangle \cdot \beta_1) \cdot \langle \gamma_1, \ldots, \gamma_z \rangle =$
$= \langle \alpha_1, \ldots, \alpha_n \cdot \beta_1 \rangle \cdot \langle \gamma_1, \ldots, \gamma_z \rangle = \langle \mu, \gamma_2, \ldots, \gamma_z \rangle$, $\langle \alpha_1, \ldots, \alpha_n \rangle \cdot$
$\cdot (\beta_1 \cdot \langle \gamma_1, \ldots, \gamma_z \rangle) = \langle \alpha_1, \ldots, \alpha_n \rangle \cdot \langle \beta_1 \cdot \gamma_1, \gamma_2, \ldots, \gamma_z \rangle = \langle \mu, \gamma_2, \ldots, \gamma_z \rangle$

where $\mu = \alpha_{i_1} \cdot \tau^{i_2-1-i_1} (\alpha_{i_2} \cdot \ldots \tau^{i_k-1-i_{k-1}} (\alpha_{i_k} \cdot$
$\cdot \tau^{n-1-i_k} (\alpha_n \cdot \beta_1 \cdot \gamma_1)) \ldots)$.

β) $\alpha_n \cdot \beta_1 \in H$. Then $\beta_1 \cdot \langle \gamma_1, \ldots, \gamma_z \rangle = \langle \beta_1 \cdot \gamma_1, \ldots, \gamma_z \rangle$ and $(\langle \alpha_1, \ldots, \alpha_n \rangle \cdot \beta_1) \cdot \gamma_1 = \{ \alpha_{i_1} \cdot \tau^{i_2-1-i_1} [\alpha_{i_2} \cdot \ldots \tau^{n-1-i_k} ($
$(\alpha_n \cdot \beta_1) \ldots] \} \cdot \gamma_1 = \alpha_{i_1} \cdot \tau^{i_2-1-i_1} [\alpha_{i_2} \cdot \ldots \cdot \tau^{n-1-i_k} (\alpha_n \cdot \beta_1 \cdot \gamma_1) \ldots]$.

Now it is easy to see that $(*)$ holds in this case.

3) The case $\beta_1 \in G - \{e\}$, $\beta_1 \cdot \gamma_1 \in H$ is a special case of b) and d).

b) $\beta_m \in G - \{e\}$, $\beta_m \cdot \gamma_1 \in H$ <u>and neither</u> $(\alpha_n \in G$ <u>and</u> $\alpha_n \cdot \beta_1 \in H)$ <u>nor</u> $(\alpha_n = \tau$ <u>and</u> $\beta_1 \in H)$ <u>holds</u>:

Let $i_1 < i_2 < \ldots < i_k$, $i_\ell \in \{1, \ldots, n\}$ for $\ell \in \{1, \ldots, k\}$ and let $\alpha_i \neq \tau \iff \alpha_i = \alpha_{i_\ell}$ for some ℓ . Let $j_1 < j_2 < \ldots < j_s$, $j_t \in \{1, \ldots, m\}$ for $t \in \{1, \ldots, s\}$ and let $\beta_j \neq \tau \iff \beta_j = \beta_{j_t}$ for some t .

Then $\langle \beta_1, \ldots, \beta_m \rangle \cdot \langle \gamma_1, \ldots, \gamma_z \rangle = \langle \mu, \gamma_2, \ldots, \gamma_z \rangle$ where
$\mu = \beta_{j_1} \cdot \tau^{j_2-1-j_1} [\beta_{j_2} \cdot \ldots \cdot \tau^{j_s-1-j_{s-1}} \{\beta_{j_s} \cdot \tau^{m-1-j_s} (\beta_m \cdot \gamma_1)\} \ldots]$.

1) Let $\alpha_n = \tau$. Then $\langle \alpha_1, .., \alpha_n \rangle \cdot \langle \mu, \gamma_2, .., \gamma_z \rangle = \langle \nu, \gamma_2, .., \gamma_z \rangle$, where $\nu = \alpha_{i_1} \cdot \tau^{i_2 - 1 - i_1} [\alpha_{i_2} \cdot \ldots \cdot \tau^{i_k - 1 - i_{k-1}} ((\alpha_{i_k} \cdot \tau^{n - i_k} (\mu) \ldots]$.

But $\langle \alpha_1, .., \alpha_n \rangle \cdot \langle \beta_1, .., \beta_m \rangle$ is either $\langle \alpha_1, .., \alpha_n, \beta_1, .., \beta_m \rangle$ or $\langle \alpha_1, .., \alpha_n, \beta_2, .., \beta_m \rangle$ (and then $\beta_1 = e$); on substituting the expression for μ into that for ν one sees easily that $(*)$ holds.

2) Let $\alpha_n \in G$. Then $\langle \alpha_1, .., \alpha_n \rangle \cdot \langle \mu, \gamma_2, .., \gamma_z \rangle = \langle \nu, \gamma_2, .., \gamma_z \rangle$, where $\nu = \alpha_{i_1} \cdot \tau^{i_2 - 1 - i_1} [\alpha_{i_2} \cdot \ldots \cdot \tau^{i_k - 1 - i_{k-1}} \{ \{\alpha_{i_k} \cdot \tau^{n - 1 - i_k} (\alpha_n \cdot \mu)\} \ldots]$.

But $\langle \alpha_1, .., \alpha_n \rangle \cdot \langle \beta_1, .., \beta_m \rangle$ is either $\langle \alpha_1, .., \alpha_{n-1}, \alpha_n \cdot \beta_1, \beta_2, .., \beta_m \rangle$ or $\langle \alpha_1, .., \alpha_{n-1}, \beta_2, .., \beta_m \rangle$ (and then $\alpha_n \cdot \beta_1 = e$); on substituting the expression for μ in the expression for ν one sees easily that $(*)$ holds.

c) $\beta_m = \tau$, $\gamma_1 \in H$ <u>and neither</u> ($\alpha_n \in G$ <u>and</u> $\alpha_n \cdot \beta_1 \in H$) <u>nor</u> ($\alpha = \tau$ <u>and</u> $\beta_1 \in H$) <u>holds</u>.
This case is analogous to b), with another expression for μ. There is $\mu = \beta_{j_1} \cdot \tau^{j_2 - 1 - j_1} [\beta_{j_2} \cdot \ldots \cdot \tau^{j_s - 1 - j_{s-1}} (\beta_{j_s} \cdot \tau^{m - j_s} \gamma_1) \ldots]$.

d) <u>Let</u> $[(\beta_m \in G - \{e\}$ <u>and</u> $\beta_m \cdot \gamma_1 \in H)$ <u>or</u> $(\beta_m = \tau$ <u>and</u> $\gamma_1 \in H)]$ <u>and</u> $[(\alpha_n \in G$ <u>and</u> $\alpha_n \cdot \beta_1 \in H)$ <u>or</u> $(\alpha_n = \tau$ <u>and</u> $\beta_1 \in H)]$.

We shall prove $(*)$ by induction. First prove $(*)$ for $n \leq 2$, $m \leq 2$, $z \leq 2$. Let $\alpha, \beta, \gamma \in G$ be such that for all the cases 1) – 8) which follow the requirement d) is satisfied. Then

1) $\alpha \cdot (\beta \cdot \gamma) = (\alpha \cdot \beta) \cdot \gamma$;

2) $(\langle \alpha, \tau \rangle \cdot \beta) \cdot \gamma = (\alpha \cdot_\tau \beta) \cdot \gamma = \alpha \cdot_\tau (\beta \cdot \gamma) = \langle \alpha, \tau \rangle \cdot (\beta \cdot \gamma)$;

3) $(\alpha\cdot\langle\beta\cdot\tau\rangle)\cdot\gamma=\langle\alpha\cdot\beta,\tau\rangle\cdot\gamma=\alpha\cdot\beta\cdot_\tau\gamma=\alpha\cdot(\langle\beta,\tau\rangle\cdot\gamma)$;

4) $(\alpha\cdot\beta)\cdot\langle\gamma,\tau\rangle=\langle\alpha\cdot\beta\cdot\gamma,\tau\rangle=\alpha\cdot(\beta\cdot\langle\gamma,\tau\rangle)$;

5) $(\langle\alpha,\tau\rangle\cdot\langle\beta,\tau\rangle)\cdot\gamma=\langle\alpha\cdot_\tau\beta,\tau\rangle\cdot\gamma=\alpha\cdot_\tau\beta\cdot_\tau\gamma=$
$=\alpha\cdot_\tau(\beta\cdot_\tau\gamma)=\langle\alpha,\tau\rangle\cdot(\langle\beta,\tau\rangle\cdot\gamma)$;

6) $(\langle\alpha,\tau\rangle\cdot\beta)\cdot\langle\gamma,\tau\rangle=(\alpha\cdot_\tau\beta)\cdot\langle\gamma,\tau\rangle=$
$=\langle\alpha\cdot_\tau(\beta\cdot\gamma),\tau\rangle=\langle\alpha,\tau\rangle\cdot(\beta\cdot\langle\gamma,\tau\rangle)$;

7) $(\alpha\cdot\langle\beta,\tau\rangle)\cdot\langle\gamma,\tau\rangle=\langle\alpha\cdot\beta\cdot_\tau\gamma,\tau\rangle=\alpha\cdot(\langle\beta,\tau\rangle\cdot\langle\gamma,\tau\rangle)$;

8) $(\langle\alpha,\tau\rangle\cdot\langle\beta,\tau\rangle)\cdot\langle\gamma,\tau\rangle=\langle\alpha\cdot_\tau\beta\cdot_\tau\gamma,\tau\rangle=$
$=\langle\alpha\cdot_\tau(\beta\cdot_\tau\gamma),\tau\rangle=\langle\alpha,\tau\rangle\cdot(\langle\beta,\tau\rangle\cdot\langle\gamma,\tau\rangle)$.

Now let $\alpha=\langle\alpha_1,\dots,\alpha_n\rangle$, $\beta=\langle\beta_1,\dots,\beta_m\rangle$, $\gamma=\langle\gamma_1,\dots,\gamma_z\rangle$. We have proved that (*) holds for $n\leqq 2$, $m\leqq 2$, $z\leqq 2$ in all the possible cases.

I. Let $k\geqq 2$ be an integer, and let (*) hold for all cases a) – d) whenever $n\leqq 2$, $m\leqq 2$, $z\leqq k$. We shall prove that (*) holds for all cases a) – d) for $n\leqq 2$, $m\leqq 2$, $z\leqq k+1$. It is sufficient to prove this for d) and $z=$ $=k+1$ only.

It is easy to see that either $\langle\gamma_1,\gamma_2,\dots,\gamma_{k+1}\rangle=\langle\gamma_1,\gamma_2\rangle\cdot\langle\gamma_3,\dots,\gamma_{k+1}\rangle$ (for $\gamma_3\in\overline{G}$) or $\langle\gamma_1,\gamma_2,\dots,\gamma_{k+1}\rangle=\langle\gamma_1,\gamma_2\rangle\cdot\langle e,\gamma_3,\dots,\gamma_{k+1}\rangle$ (for $\gamma_3=\tau$).

Only the second case will be written out explicitly. Then
$\alpha\cdot(\beta\cdot\langle\gamma_1,\dots,\gamma_{k+1}\rangle)=\alpha\cdot[\beta\cdot(\langle\gamma_1,\gamma_2\rangle\cdot\langle e,\gamma_3,\dots,\gamma_{k+1}\rangle)]=\alpha\cdot[(\beta\cdot\langle\gamma_1,\gamma_2\rangle)\cdot\langle e,\gamma_3,\dots,\gamma_z\rangle]$.

But having d) $\beta\cdot\langle\gamma_1,\gamma_2\rangle$ is at most a couple, and therefore $\alpha\cdot[(\beta\cdot\langle\gamma_1,\gamma_2\rangle)\cdot\langle e,\gamma_3,\dots,\gamma_{k+1}\rangle]=[\alpha\cdot(\beta\cdot\langle\gamma_1,\gamma_2\rangle)]\cdot\langle e,\gamma_3,\dots\gamma_{k+1}\rangle=[(\alpha\cdot\beta)\cdot\langle\gamma_1,\gamma_2\rangle]\cdot\langle e,\gamma_3,\dots,\gamma_z\rangle$. But having d) $\alpha\cdot\beta$ is at most a couple and therefore $[(\alpha\cdot\beta)\cdot\langle\gamma_1,\gamma_2\rangle]\cdot\langle e,\gamma_3,\dots,\gamma_z\rangle=$ $=(\alpha\cdot\beta)\cdot\langle\gamma_1,\dots,\gamma_z\rangle$.

II. Let $k \geq 2$ be an integer, and let $(*)$ hold for all cases a) – d) whenever $n \leq 2$, $m \leq k$. Now we shall prove that $(*)$ holds for all cases a) – d) whenever $n \leq 2$, $m \leq k+1$. It is sufficient to prove this for $m = k+1$ and d) only. It is easy to see that either $\langle \beta_1, \ldots, \beta_{k+1} \rangle = \langle \beta_1, \beta_2 \rangle \cdot \langle \beta_3, \ldots, \beta_{k+1} \rangle$ or $\langle \beta_1, \ldots, \beta_{k+1} \rangle = \langle \beta_1, \beta_2 \rangle \cdot \langle e, \beta_3, \ldots, \beta_{k+1} \rangle$. Only the second case will be written out explicitly. Then $(\alpha \cdot \langle \beta_1, \ldots, \beta_{k+1} \rangle) \cdot \gamma = [\alpha \cdot (\langle \beta_1, \beta_2 \rangle \cdot \langle e, \beta_3, \ldots, \beta_{k+1} \rangle)] \cdot \gamma = [(\alpha \cdot \langle \beta_1, \beta_2 \rangle) \cdot \langle e, \beta_3, \ldots, \beta_{k+1} \rangle] \cdot \gamma = (\alpha \cdot \langle \beta_1, \beta_2 \rangle) \cdot (\langle e, \beta_3, \ldots, \beta_{k+1} \rangle \cdot \gamma) = \alpha \cdot [\langle \beta_1, \beta_2 \rangle \cdot (\langle e, \beta_3, \ldots, \beta_{k+1} \rangle \cdot \gamma)] = \alpha \cdot [\langle \beta_1, \beta_2, \beta_3, \ldots, \beta_{k+1} \rangle \cdot \gamma]$.

III. Let $k \geq 2$ be an integer, and let $(*)$ hold for all cases a) – d) whenever $n \leq k$. We shall prove that $(*)$ holds for all cases a) – d) for $n \leq k+1$. It is sufficient to prove this for d) and $n = k+1$ only. It is easy to see that $\langle \alpha_1, \ldots, \alpha_{k+1} \rangle$ is either $\langle \alpha_1, \ldots, \alpha_k \rangle \cdot \alpha_{k+1}$ or $\langle \alpha_1, \ldots, \alpha_k \rangle \cdot \langle e, \alpha_{k+1} \rangle$. Only the second case will be written out explicitly. Then $(\langle \alpha_1, \ldots, \alpha_{k+1} \rangle \cdot \beta) \cdot \gamma = [(\langle \alpha_1, \ldots, \alpha_k \rangle \cdot \langle e, \alpha_{k+1} \rangle) \cdot \beta] \cdot \gamma = [\langle \alpha_1, \ldots, \alpha_k \rangle \cdot (\langle e, \alpha_{k+1} \rangle \cdot \beta)] \cdot \gamma = \langle \alpha_1, \ldots, \alpha_k \rangle \cdot [(\langle e, \alpha_{k+1} \rangle \cdot \beta) \cdot \gamma] = \langle \alpha_1, \ldots, \alpha_{k+1} \rangle \cdot (\beta \cdot \gamma)$.

The proof of the associativity of the composition is finished. Now it is easy to see that G is a subsemigroup of P. Set $\mu = \langle e, \tau \rangle$. Then evidently $\mu \cdot \sigma = {}_{\tau}\sigma$ for every $\sigma \in H$. If Φ is a homomorphism of G into some semigroup G' with the properties from the sublemma, put $(\langle \alpha_1, \ldots, \alpha_n \rangle) \Psi = \bar{\alpha}_1 \cdot \ldots \cdot \bar{\alpha}_n$ for $\langle \alpha_1, \ldots, \alpha_n \rangle \in P$, where $\bar{\alpha}_i =$

$= (\alpha_i)\Phi$ for $\alpha_i \in G$, $\bar{\alpha}_i = \mu'$ for $\alpha_i = \tau$. Then Ψ is a homomorphism of P into G', and is an extension of Φ such that $(\mu)\Psi = \mu'$. The unicity is evident. If $\langle \alpha_1, \ldots, \alpha_n \rangle \in P$, set $\tilde{\alpha}_i = \alpha_i$ for $\alpha_i \in G$, $\tilde{\alpha}_i = \mu$ for $\alpha_i = \tau$. Then evidently $\langle \alpha_1, \ldots, \alpha_n \rangle = \tilde{\alpha}_1 \cdot \ldots \cdot \tilde{\alpha}_n$, and this decomposition has the properties from statement 3) of the sublemma. This concludes the proof of the sublemma.

Now the proof of the lemma for the case C) will be given:

Set $G = H_\ell(a, a)$, $H = \{\sigma \in G; \sigma = \alpha \cdot \beta, \alpha \notin G\}$, $e_a = e$. Then evidently $H \subset G$, $e \notin H$, $\rho \cdot \sigma \cdot \rho' \in H$ for every $\sigma \in H$ $\rho, \rho' \in G$. If $\sigma \in H$, $\sigma = \alpha \cdot \beta$, $\alpha \notin G$, put ${}_\tau\sigma = {}_\mu\alpha \cdot \beta$. The assumptions of the sublemma are satisfied; let P be a semigroup with the properties from the sublemma. The form $\alpha_1 \cdot \ldots \cdot \alpha_n$ described in 3) of the sublemma will be called the standard decomposition of α.

Now we describe the category h : $h^\sigma = \ell^\sigma$, $H_h(c, d) = H_\ell(c, d)$ for all $c, d \in h^\sigma$, $d \neq a$; put $H_h(a, a) = P$. If $c \in h^\sigma$, $c \neq a$, then $H_h(c, a)$ is the set of all n-tuples $\langle \nu, \alpha_2, \ldots, \alpha_n \rangle$ where $\nu \in H_\ell(c, a)$, $\sigma \in P$ and $e \cdot \alpha_2 \cdot \ldots \cdot \alpha_n$ is the standard decomposition of σ (and if $\nu = \omega_{c,a}$, then put $\langle \nu, \alpha_2, \ldots, \alpha_n \rangle = \langle \nu, \beta_2, \ldots, \beta_m \rangle$ for every $\beta_2 \cdot \ldots \cdot \beta_m \in P$). Now we define the composition in h. If ν, $\varkappa \in \ell^m$, then the composition of ν and $\varkappa$ is the same in h as in ℓ. If $\sigma, \sigma' \in H_h(a, a)$, then the composition in h is the same as in P. We define the composition for the remaining cases. Let $c \in h^{\sigma} - \{a\}$:

1) if $a \neq d$, $\nu' \in H_h(d, c)$, then $\nu' \cdot \langle \nu, \alpha_2, \dots, \alpha_m \rangle = \langle \nu' \cdot \nu, \alpha_2, \dots, \alpha_m \rangle$;

2) if $\nu' \in H_h(a, c)$, then $\nu' \cdot \langle \nu, \alpha_2, \dots, \alpha_m \rangle = (\nu' \cdot \nu) \cdot \alpha_2 \cdot \dots \cdot \alpha_m$;

3) if $\langle \nu, \alpha_2, \dots, \alpha_m \rangle \in H_h(c, a)$, $\rho \in H_h(a, a)$ and $\beta_1 \cdot \dots \cdot \beta_m$ is the standard decomposition of $e \cdot \alpha_2 \cdot \dots \cdot \alpha_m \cdot \rho$, then $\langle \nu, \alpha_2, \dots, \alpha_n \rangle \cdot \rho = \langle \nu \cdot \beta_1, \beta_2, \dots, \beta_m \rangle$.

4) if $\rho \in H_h(a, c)$, then $\mu \cdot \rho = {}_{\mu}\rho$;

5) if $\rho \in H_h(a, c)$, and $\alpha_1 \cdot \dots \cdot \alpha_m$ is the standard decomposition of $\sigma \in H_h(a, a)$, then $\sigma \cdot \rho = \alpha_1 \cdot \{ \dots [\alpha_{m-1} \cdot (\alpha_m \cdot \rho)] \dots \}$.

6) if $\langle \nu, \alpha_2, \dots, \alpha_m \rangle \in H_h(c, a)$, $\rho \in H_h(a, d)$, $d \neq a$ put $\langle \nu, \alpha_2, \dots, \alpha_m \rangle \cdot \rho = \nu \cdot [(\alpha_2 \cdot \dots \cdot \alpha_m) \cdot \rho]$.

If $\langle \nu, \alpha_2, \dots, \alpha_m \rangle \in H_h(c, a)$, $c \neq a$, then as $e \cdot \alpha_2 \cdot \dots \cdot \alpha_n$ is the standard decomposition of $\sigma = \alpha_2 \cdot \dots \cdot \alpha_m$, there is $\langle \nu, \alpha_2, \dots, \alpha_m \rangle = \nu \cdot \sigma$. Now we prove that the composition in h is associative. Let $\alpha, \beta, \gamma \in h^m$; we are to prove

$$(*) \qquad (\alpha \cdot \beta) \cdot \gamma = \alpha \cdot (\beta \cdot \gamma)$$

whenever all the compositions are defined.

a) $(*)$ is easily verified if $\alpha, \beta \in \ell^m$.

b) Let $\alpha \in H_\ell(c, a)$, $c \neq a$, $\beta, \gamma \in H_h(a, a)$. Let $\beta_1 \cdot \dots \cdot \beta_m$ and $\rho_1 \cdot \dots \cdot \rho_s$ be the standard decomposition of β and $\beta_2 \cdot \dots \cdot \beta_m \cdot \gamma$ respectively. Then $(\alpha \cdot \beta) \cdot \gamma = \langle \alpha \cdot \beta_1, \beta_2, \dots, \beta_m \rangle \cdot \gamma = \langle \alpha \cdot \beta_1 \cdot \rho_1, \rho_2, \dots, \rho_s \rangle = \alpha \cdot [(\beta_1 \cdot \rho_1) \cdot \rho_2 \cdot \dots \cdot \rho_s] = \alpha \cdot (\beta \cdot \gamma)$
$((\beta_1 \cdot \rho_1) \cdot \rho_2 \cdot \dots \cdot \rho_s$

is the standard decomposition of $\beta \cdot \gamma$) .

c) Let $\alpha \in H_\ell(c,a), \beta \in H_h(a,a), \gamma \in H_\ell(a,d)$, $c \neq a \neq d$.
If $\beta_1 \cdot \ldots \cdot \beta_m$ is the standard decomposition of β, then $(\alpha \cdot \beta) \cdot \gamma = \langle \alpha \cdot \beta_1, \beta_2, \ldots, \beta_m \rangle \cdot \gamma = \alpha \cdot \beta_1 \cdot \{\beta_2 \cdot [\ldots \cdot$ $\cdot (\beta_m \cdot \gamma) \ldots]\} = \alpha \cdot (\beta \cdot \gamma)$.

d) Let $\alpha \in H_h(a,a)$, let $\alpha_1 \cdot \alpha_2 \cdot \ldots \cdot \alpha_n$ be its standard decomposition, let $\beta \in H_h(a,c)$, $c \neq a$, $\gamma \in H_\ell(c,d)$, $d \in \ell^\sigma$. Then (*) is evident if either $n = 1$ or $\alpha = \mu$. We shall prove it for all cases. Then $\alpha \cdot (\beta \cdot \gamma) = \alpha_1 \cdot$ $\cdot \{\ldots \cdot \alpha_{n-1} \cdot [\alpha_n \cdot (\beta \cdot \gamma)] \ldots \}$, $(\alpha \cdot \beta) \cdot \gamma = \{\alpha_1 \cdot [\ldots \cdot \alpha_{n-1} \cdot (\alpha_n \cdot \beta) \ldots]\} \cdot \gamma$.
Using induction according to n , (*) is proved for $n =$ $= 1$ and the inductive step is trivial; indeed, $\alpha \cdot (\beta \cdot \gamma) =$ $= \alpha_1 \cdot \{\ldots \cdot \alpha_{n-1} \cdot [(\alpha_n \cdot \beta) \cdot \gamma] \ldots \}$ and the supposition of the induction apply to $(\alpha_n \cdot \beta)$.

e) Let $\alpha, \beta \in H_h(a,a)$, and let $\alpha_1 \cdot \ldots \cdot \alpha_n$ and $\beta_1 \cdot \ldots \cdot \beta_m$ be the standard decompositions of α and β , respectively. Let $\rho_1 \cdot \ldots \cdot \rho_s$ be the standard decomposition of $\alpha \cdot \beta$. Let $\gamma \in H_\ell(a,c)$, $c \neq a$.
It is easy to see that $\langle \rho_1, \ldots, \rho_s \rangle$ is either $\langle \alpha_1, \ldots$ $\ldots, \alpha_n, \beta_1, \ldots, \beta_m \rangle$ or $\langle \alpha_1, \ldots, \alpha_{n-1}, \alpha_n \cdot \beta_1, \beta_2, \ldots, \beta_m \rangle$ or $\langle \alpha_1, \ldots, \alpha_{n-1}, \beta_2, \ldots, \beta_m \rangle$ (and then $\alpha_n \cdot \beta_1 = e$) or $\langle \alpha_1, \ldots, \alpha_{n-1}, \alpha_n, \beta_2, \ldots, \beta_m \rangle$ (and then $\beta_1 = e$) or $\langle \alpha_1 \cdot \alpha_2 \cdot \ldots \cdot \alpha_n \cdot \beta_1, \beta_2, \ldots, \beta_m \rangle$.

In all cases except the last, (*) is proved easily. Indeed, setting $\delta = \beta_2 \cdot [\beta_3 \cdot \ldots \cdot (\beta_m \cdot \gamma) \ldots]$ it suffices to prove $(\alpha_n \cdot \beta_1) \cdot \delta = \alpha_n \cdot (\beta_1 \cdot \delta)$. Since $\beta_1 \in H_\ell(a,a) = G$, the case $\alpha_n \in G$ is trivial. If $\alpha_n = \mu$ and $\beta_1 \in G - H$, then $e \cdot \alpha_n \cdot \beta_1$ is the standard decomposition of $\alpha_n \cdot$ $\cdot \beta_1$ and the equality follows immediately from the definition of the composition in $\mathcal{h}$. Let $\alpha_n = \mu$, $\beta_1 = \eta \cdot \vartheta$, where $\eta \in H_\ell(a,d)$, $d \neq a$. Then $\alpha_n \cdot \beta_1 =_\mu \eta \cdot \vartheta$, $(\alpha_n \cdot \beta_1)\cdot \sigma =_\mu \eta \cdot (\vartheta \cdot \sigma) =_\mu [\eta \cdot (\vartheta \cdot \sigma)] = \mu \cdot [\eta \cdot \vartheta \cdot \sigma] = \alpha_n \cdot (\beta_1 \cdot \sigma)$.

The last case (i.e. $\langle \varphi_1, \dots, \varphi_s \rangle = \langle \alpha_1 \cdot \ldots \cdot \alpha_n \cdot \beta_1, \beta_2, \dots, \beta_m \rangle$) will now be proved by induction according to n. If $n = 1$, then (*) holds; consider the inductive step. Let $n > 1$, and set $\sigma = \beta_2 \cdot \{\beta_3 \cdot [\ldots \cdot (\beta_m \cdot \gamma)\ldots]\}$; evidently $\sigma \in H_\ell(a,c)$. Now $\alpha_n \cdot \beta_1 \in H$ follows from the definition of the composition in $H_h(a,a)$. Then there exist $\eta \in H_\ell(a,d)$, $d \neq a$, $\vartheta \in H_\ell(d,a)$ such that $\alpha_n \cdot \beta_1 = \eta \cdot \vartheta$. Then, using d), $\alpha_1 \cdot \ldots \cdot \alpha_n \cdot \beta_1 = \alpha_1 \cdot \ldots \cdot \alpha_{n-1} \cdot (\eta \cdot \vartheta) = [(\alpha_1 \cdot \ldots \cdot \alpha_{n-1})\cdot \eta]\cdot \vartheta$ and therefore

$$(\alpha \cdot \beta)\cdot \gamma = (\alpha_1 \cdot \ldots \cdot \alpha_n \cdot \beta_1)\cdot \sigma = \{[(\alpha_1 \cdot \ldots \cdot \alpha_{n-1})\cdot \eta]\cdot \vartheta\}\cdot \sigma = \\ = [(\alpha_1 \cdot \ldots \cdot \alpha_{n-1})\cdot \eta]\cdot(\vartheta \cdot \sigma)$$

using the associativity of the composition in ℓ, and then from d),

$$= (\alpha_1 \cdot \ldots \cdot \alpha_{n-1})\cdot[\eta \cdot (\vartheta \cdot \sigma)] = (\alpha_1 \cdot \ldots \cdot \alpha_{n-1})\cdot[(\eta \cdot \vartheta)\cdot \sigma] = \\ = (\alpha_1 \cdot \ldots \cdot \alpha_{n-1})\cdot[(\alpha_n \cdot \beta_1)\cdot \sigma] = (\alpha_1 \cdot \ldots \cdot \alpha_{n-1})\cdot[\alpha_n \cdot (\beta \cdot \gamma)];$$

now the supposition on induction may be used.

In a) - e), (*) is thus proved whenever $\alpha, \beta, \gamma \in \ell^m \cup H_h(a,a)$. For other cases (*) is easily proved using the fact that $\nu \in h^m - \ell^m$ implies $\nu = \nu' \cdot \sigma$ for some $\nu' \in \ell^m$, $\sigma \in H_h(a,a)$.

Not it is easy to see that the category h has the required properties. Property 1) follows immediately from the construction. Let $\Phi : \ell \to K$ be a functor into a category K with the properties from the lemma. Then, using the sublemma, there exists a unique extension of the homomorphism $\Phi / H_\ell(a,a)$, namely the homomorphism $\overline{\Psi}: H_h(a,a) \to H_k((a)\Phi, (a)\Phi)$, such that $(\mu)\overline{\Psi} = \mu'$. Now define $\Psi: h \to K$ so that $(\nu)\Psi = (\nu)\Phi$ for all $\nu \in \ell^m$, $(\sigma)\Psi = (\sigma)\overline{\Psi}$ for $\sigma \in H_h(a,a)$, $(\rho)\Psi = (\nu)\Phi \cdot (\alpha_2 \cdot \ldots \cdot \alpha_n)\overline{\Psi}$ for $\rho = \langle \nu, \alpha_2, \ldots, \alpha_n \rangle \in H_h(c,a)$, $c \neq a$. To prove $(\alpha)\Psi \cdot (\beta)\Psi = (\alpha \cdot \beta)\Psi$ it is sufficient to consider the following two cases only:

α) $\alpha \in H_h(c,a), c \neq a$, $\beta \in H_h(a,a)$: let $\alpha = \langle \nu, \alpha_2, \ldots, \alpha_n \rangle$ and let $\rho_1 \cdot \ldots \cdot \rho_s$ be the standard decomposition of $\alpha_2 \cdot \ldots \cdot \alpha_n \cdot \beta$. Then

$(\alpha)\Psi \cdot (\beta)\Psi = (\nu)\Phi \cdot (\alpha_2 \cdot \ldots \cdot \alpha_n)\overline{\Psi} \cdot (\beta)\overline{\Psi} = (\nu)\Phi \cdot (\rho_1)\overline{\Psi} \cdot \ldots \cdot (\rho_s)\overline{\Psi} = (\nu \cdot \rho_1)\Phi \cdot (\rho_2)\overline{\Psi} \cdot \ldots \cdot (\rho_s)\overline{\Psi} = (\langle \nu \cdot \rho_1, \rho_2, \ldots, \rho_s \rangle)\Psi = (\alpha \cdot \beta)\Psi$.

β) $\alpha \in H_h(a,a)$, $\beta \in H_h(a,c)$, $c \neq a$. It is easy to see that then $(\alpha)\Psi \cdot (\beta)\Psi = (\alpha \cdot \beta)\Psi$ if either $\alpha \in H_\ell(a,a)$ or $\alpha = \mu$; and by induction according to n this is easily verified for $\alpha = \alpha_1 \cdot \ldots \cdot \alpha_n \in H_h(a,a)$ where $\alpha_1 \cdot \ldots \cdot \alpha_n$ is the standard decomposition of α.

If κ is an infinite regular cardinal such that $\operatorname{card} \ell^m \leq \kappa$, then evidently $\operatorname{card} h^m \leq \kappa$. Moreover, if κ is uncountable and $\operatorname{card} H_\ell(c,d) < \kappa$ for all $c, d \in \ell^\sigma$, then $\operatorname{card} H_h(c,d) < \kappa$ for all $c, d \in h^\sigma$; this follows from the definition of h.

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