Václav Havel

Construction of certain systems with two compositions

Commentationes Mathematicae Universitatis Carolinae, Vol. 6 (1965), No. 4, 413--428

Persistent URL: http://dml.cz/dmlcz/105033

Commentationes Mathematicae Universitatis Carolinae
6, 4 (1965)

CONSTRUCTION OF CERTAIN SYSTEMS WITH TWO COMPOSITIONS

Václav HAVEL, Brno

A <u>double quasigroup</u> is defined here as a triple $(S, +\, \square)$ where S, $\operatorname{card} S \geq 2$, is a set and $+, \square$ two binary compositions on S such that $(S, +)$ is a loop with a neutral element 0 satisfying $x \square 0 = 0 \square x = 0$ for all $x \in S$ and $(S \setminus \{0\}, \square)$ is a quasigroup. If $(S \setminus \{0\}, \square)$ has a neutral element, then $(S, +, \square)$ is called <u>double-loop</u> [4, p. 61].

Each double-quasigroup $(S, +, \square)$ with a prescribed additive loop $(S, +)$ may be constructed as follows,[7a]: Let $(B, \circ)$ be the group of all bijective mappings of S onto S reproducing the element 0, with a natural composition $\circ$; also, set $\underline{0} : S \longrightarrow \{0\}$. Choose any mapping $\eta : S \longrightarrow B \cup \{\underline{0}\}$ satisfying $\eta(0) = \underline{0}$, $\eta(x) \neq \underline{0}$ for $x \in S \setminus \{0\}$ such that $\eta(S \setminus \{0\})$ operates on $S \setminus \{0\}$ simply transitively, and define the composition $\square$ on S by $x \square y = \eta(x) y$ for all $x, y \in S$. Then $(S, +, \square)$ is the double-quasigroup associated with η, and every double-quasigroup $(S, +, \square)$ with a prescribed additive loop may be obtained in this way.

Now we exhibit the familiar algebraical properties of a given $(S, +, \square)$ in the following way:

A^+ $(x+y)+z = x+(y+z)$ for all $x, y, z \in S$. (<u>Associativity.</u>)

C^+ $x + y = y + x$ for all $x, y \in S$. (<u>Commutativity.</u>)

$RD^{+\Box}$ $x \Box (y+z) = x \Box y + x \Box z$ for all $x, y, z \in S$. (Right distributivity.)

$LD^{+\Box}$ $(x+y) \Box z = x \Box z + y \Box z$ for all $x, y, z \in S$. (Left distributivity.)

$RP^{+\Box}$ Any equation $-a \Box x + b \Box x = c$ has a unique solution $x \in S$ for any given a, b, c of S with $a \neq b$. (Right planarity.)

$LP^{+\Box}$ Any equation $x \Box a - x \Box b = c$ has a unique solution for any given a, b, c of S with $a \neq b$. (Left planarity.)

In the theory of incidence structures (partial planes) in the sense of [4 , p.2] and in the theory of systems with generalized parallelity [7b] there are important the double-quasigroups satisfying the axioms A^+, $LP^{+\Box}$ or A^+, $RP^{+\Box}$.

In the sequel we shall use modifications of the Moulton construction from the classical paper [1] (also see, e.g., [4], [5], [6]), and we wish to obtain some double-quasigroups $(S, +, \Box)$ satisfying A^+, C^+, $RP^{+\Box}$ or A^+, C^+, $LP^{+\Box}$ respectively. It remains an open question whether double-quasigroups in which exactly one of the laws $RP^{+\Box}$, $LP^{+\Box}$ holds are obtainable by this process.

We note that in a double-quasigroup $(S, +, \Box)$ from A^+, $RD^{+\Box}$ or A^+, $LD^{+\Box}$ there follows $LP^{+\Box}$ or $RP^{+\Box}$ respectively.

A double-quasigroup $(S, +, \Box)$ satisfying A^+, $RP^{+\Box}$, $LP^{+\Box}$ and either $RD^{+\Box}$ or $LD^{+\Box}$ is usually called a right or left quasifield, respectively, [4, p. 92].

We shall begin with the additive loop $(S, +)$ of a double-quasigroup $(S,+,\cdot)$ and also a mapping $\eta : S \to B$, and construct the associated double-quasigroup.

1. Let $F = (S,+,\cdot)$ be a left quasifield and $\Phi : S \to S$ a bijection with $\Phi(0) = 0$. For arbitrary $a \in S$ let $\eta(a)$ be the mapping $x \to \Phi(a)x$, $x \in S$; then the associated double-quasigroup $(S,+,\square)$ is also a left quasifield. $RD^{+\square}$ holds if and only if Φ is an additive mapping.

Proof. The validity of A^+, C^+ in $(S,+,\square)$ is, of course, trivial. - There is $x \square (y+z) = \Phi(x)(y+z) = \Phi(x)y + \Phi(x)z = x\square y + x \square z$, so that $LD^{+\square}$ holds. Any equation $-a\square x + b \square x = c$, for given $a, b, c \in S\setminus\{0\}$, $a \neq b$, may be rewritten as $-\Phi(a)x + \Phi(b)x = c$, and the unique solvability follows from RP^+: If Φ is not additive, there exist $a_0, b_0 \in S$ such that $\Phi(a_0+b_0) \neq \Phi(a_0) + \Phi(b_0)$, and this implies $(a_0+b_0)\square z = \Phi(a_0+b_0).z \neq \Phi(a_0)z + \Phi(b_0)z = a_0 \square z + b_0 \square z$ for all $z \in S\setminus\{0\}$; thus $RD^{+\square}$ is violated. If Φ is additive, then $RD^{+\square}$ follows directly.

One simple special case can be stated as follows: Let F be an ordered left quasifield [4 , p. 237]; the set of all negative elements of F will be denoted by N. We choose $\Phi(a) = a$ for all $a \geq 0$ and $\Phi(N) = N$ (so that Φ must map N bijectively onto N), and suppose $\Phi(n) \neq n$ for some $n \in N$. Then Φ is not additive and the assumption of theorem 1 is fulfilled.

If $\eta(a)$ is taken as $x \to \theta(\Phi(a)\cdot\Psi(x))$, $x \in S$, where Φ, Ψ, θ are the bijections of S onto S with $\Phi(0) = \Psi(0) = \theta(0) = 0$, then the associated double-quasigroup $(S, +, \Box)$ fulfils $LP^{+\Box}$ and $RP^{+\Box}$ whereas $LD^{+\Box}$ or $RD^{+\Box}$ is satisfied precisely if Φ, θ or Ψ, θ respectively are additive. This is the special case of the known notion of weak-isotopy, introduced in [3 , p. 460]. In theorem 1 only a special case of this weak-isotopy was used. The connection between weak isotopic double-quasigroups and their associated systems with generalized parallelity [7b] can be investigated when the corresponding ternary composition T is introduced by $T(a, b, c) = a \Box b + c$.

If $(S, +, \cdot)$ is a double-quasigroup, then we may choose Φ as the identity mapping on S, Ψ as the mapping $x \to \alpha \backslash^{x\beta}$, $x \in S$, and θ as the mapping $x \to x/\beta$, $x \in S$; here $\alpha, \beta \in S \setminus \{0\}$ are fixed elements, and $\backslash$ and $/$ denote, respectively the left and right division in $(S \setminus \{0\}, \cdot)$. The associated double-quasigroup $(S, +, \Box)$ satisfies $\alpha \Box x = x \Box \alpha = x$ for all $x \in S \setminus \{0\}$, and was introduced in [3 , p. 461] (but only under the assumption of $LP^{+\cdot}$ and $RP^{+\cdot}$) according to the description given by Hall.

For our aims, the most important special case of theorem 1 is that in which $F = (S, +, \cdot)$ is a skew-field. If Φ is not additive, then $(S, +, \Box)$ is a proper left quasifield without the identity element. According to [3, p. 463], in this manner the left quasifields, which are a form of "generalized natural field" [3, p. 451] of desarguesian planes, may be obtained.

2. Let $F = (S,+,\cdot)$ be a double-quasigroup. Take a mapping $\eta : S \to B \cup \{0\}$ with $\eta(0) = 0$ such that each $\eta(a)$, $a \in S \setminus \{0\}$ has the form $x \to a\Phi_a(x)$, $x \in S$, where $\Phi_a : S \to S$ is an additive bijection with $\Phi_a(0) = 0$ and $\eta(S \setminus \{0\})$ acts simply transitively on $S \setminus \{0\}$. Then the associated double-quasigroup $(S,+,\square)$ satisfies $RD^{+\square}$; the axiom $RP^{+\square}$ is fulfilled precisely if the mappings $x \to -a\Phi_a(x)+b\Phi_b(x)$, $x \in S$ are bijective for all distinct a, b of $S \setminus \{0\}$.

Proof. We have $x\square(y+z) = x\Phi_x(y)+x\Phi_x(z) = x\square y + x\square z$ for all $x, y, z \in S$, so that $RD^{+\square}$ holds. The rest of the theorem is obvious.

The André quasifield [4, p. 206] is constructed as described in theorem 2 on taking a field for F, and Φ_a, $a \in S \setminus \{0\}$, as suitable automorphisms of F leaving fixed each element of some proper subfield of F.

3a. Let $F = (S,+,\cdot)$ be an ordered non-commutative field and Φ and additive bijective mapping of S onto S satisfying $x > 0 \Rightarrow \Phi(x) > 0$ and $x < 0 \Rightarrow \Phi(x) < 0$. Define $\eta(a)$ as the mapping $x \to a\cdot x$, $x \in S$ if $a \geq 0$, and as the mapping $x \to \Phi(x)\cdot a$, $x \in S$ if $a < 0$.

Then the associated double-quasigroup $(S, +, \square)$ satisfies A^{+}, C^{+}, $RD^{+\square}$ (and thus also $LP^{+\square}$) and does not satisfy $LD^{+\square}$. Moreover, $RP^{+\square}$ holds if and only if the mappings $x \to -a \cdot x + \Phi(x) \cdot b$, $x \in S$, for $a > 0 > b$ and $x \to -\Phi(x) \cdot a + b \cdot x$, $x \in S$ for $a < 0 < b$ are bijections of S onto S.

Proof. For $x \geq 0$ we have $x \square (y+z) = x \square y + x \square z$, and for $x < 0$ we have $x \square (y+z) = \Phi(y+z) \cdot x = (\Phi(y) + \Phi(z)) \cdot x = x \square y + x \square z$, so that $RD^{+\square}$ is valid. If we choose $x_0, y_0, z_0 \in S$ such that $x_0 > 0 > y_0$, $x_0 + y_0 > 0$, $z_0 = 1$, then $(x_0 + y_0) \square z_0 = x_0 \cdot z_0 + y_0 \cdot z_0 \neq x_0 \cdot z_0 + \Phi(z_0) \cdot y_0 = x_0 \square z_0 + y_0 \square z_0$; thus $LD^{+\square}$ is violated. It remains to examine a mapping $x \to -a \square x + b \square x$, $x \in S$ for given distinct elements a, b of $S \setminus \{0\}$. We distinguish the following four alternatives:

$-a_0 \square x + b \square x = -a \cdot x + \Phi(x) \cdot b$ for $a > 0$, $b < 0$,

$-a_0 \square x + b \square x = (-a + b) \cdot x$ for $a > 0$, $b > 0$,

$-a_0 \square x + b \square x = \Phi(x) \cdot (-a + b)$ for $a < 0$, $b < 0$,

$-a_0 \square x + b \square x = -\Phi(x) \cdot a + b \cdot x$ for $a < 0$, $b > 0$.

In the second and third cases the required bijectivity is easily obtained; the first and fourth alternative figure explicitely in the last condition of the theorem.

3b. Let $F = (S, +, \cdot)$ be an ordered non-commutative s-field, and Φ an order preserving mapping of S onto S with $\Phi(0) = 0$. Define $\eta(a)$ as the mapping $x \to a \cdot x$, $x \in S$, if $a \geq 0$, and $x \to \Phi(x) \cdot a$, $x \in S$, if $a < 0$.

Then the associated double-quasigroup $(S, +, \square)$ satisfies $A^{+}, C^{+}, LP^{+\square}$ and does not satisfy $LD^{+\square}$. Moreover $RD^{+\square}$ holds if and only if the mapping $x \to -a \cdot x + \Phi(x) \cdot b$, $x \in S$, for $a > 0 > b$ and $x \to -\Phi(x) \cdot a + b \cdot x$, $x \in S$ for $a < 0 < b$ are surjections of S onto S.

The proof is analogous to that of theorem 3a with the exception of the axiom $LP^{+\square}$. But any mapping $x \to x \square a - x \square b$, $x \in S$ has the form $x \to x \cdot a - x \cdot b = x \cdot (a - b)$ for $x \geq 0$ and $x \to \Phi(a) \cdot x - \Phi(b) \cdot x = (\Phi(a) - \Phi(b)) \cdot x$ for $x < 0$. From the order-preservation of Φ there follows bijectivity of the mapping considered. At the end of the theorem, we have utilized surjectivity, this being possible because Φ is order-preserving.

For the construction of a non-bijective mapping $x \to \alpha \cdot x + \Phi(x) \cdot \alpha$, $x \in S$, for some positive α (if such situation occurs at all), the known ordered non-commutative sfield of Hilbert does not seem to be sufficiently general.

4. Let $F = (S, +, \cdot)$ be an ordered sfield and let N be the set of all negative elements of F. We denote by $\Phi : N \to N$ an order-preserving bijection and $\psi : S \to S$ an order-preserving bijection with $\psi(0) = 0$. Let $\eta(a)$ be the mapping $x \to \psi(a) \cdot x$, $x \in S$, if $a \geq 0$ and $x \to \psi(a) \cdot x$, $x \geq 0$ and $x \to \Phi(a) x$, $x < 0$ if $a < 0$. The associated double-quasigroup $(S, +, \square)$ satisfies $A^{+}, C^{+}, RP^{+\square}$. Moreover, $LP^{+\square}$ holds if and only if the mappings $x \to \psi(x) \cdot a - \Phi(x) \cdot b$, $x \in N$ for $a > 0, b < 0$, and $x \to \Phi(x) \cdot a - \psi(x) \cdot b$, $x \in N$ for

$a < 0, b > 0$ are surjections of N onto N .

Proof. Consider the mapping $x \to -a \square x + b \square x$, $x \in S$, for given distinct a, b of $S \setminus \{0\}$. Without loss of generality we may restrict ourselves to the case $a < 0, a < b$. Then we distinguish three cases

$$-a \square x + b \square x = (-\Phi(a) + \psi(b)) \cdot x \qquad \text{for } x < 0, b \geq 0,$$
$$-a \square x + b \square x = (-\Phi(a) + \Phi(b)) \cdot x \qquad \text{for } x < 0, b < 0,$$
$$-a \square x + b \square x = (-\Phi(a) + \psi(b)) \cdot x \qquad \text{for } x \geq 0 .$$

Since Φ and Ψ are order-preserving and $\psi(0) = 0$; the considered mapping is bijective. Analogously, consider the mapping $x \to x \square a - x \square b$, $x \in S$, where one may suppose without loss of generality, that $b < 0$, $a > b$. Then we distinguish three alternatives:

$$x \square a - x \square b = \psi(x) \cdot a - \Phi(x) \cdot b \qquad \text{for } x < 0, \ a > 0,$$
$$x \square a - x \square b = \Phi(x) \cdot (a - b) \qquad \text{for } x < 0, \ a < 0,$$
$$x \square a - x \square b = \psi(x)(a - b) \qquad \text{for } x \geq 0 .$$

In the second and the third case the required bijectivity follows directly, and in the first case it is stated in the last condition of the theorem. As Φ and Ψ are order-preserving, bijectivity can be replaced by surjectivity. From this the rest of the proof follows. The bijection Φ and Ψ can be chosen in such a way that $RD^{+\square}$ and $LD^{+\square}$ are both violated [6 , pp. 93-94].

If $\psi(x)$ for $x \in S$ and $\Phi(x) = p x$ for $x \in N$ for fixed $p > 0$, we obtain the classical case of the construction, especially the initial case of [1].

5. Let $F = (S, +, \cdot)$ be a pseudoordered field [6 , p. 427], and denote by N the set of all negative elements of F .

Let $\Phi : S \to S$, $\Psi : S \to S$ be pseudoorder-preserving bijections [6, p. 428] with $\Phi(0) = 0$ and $\psi(0) = 0$. Suppose that $\eta(a)$ is a mapping $x \to \psi(a)\cdot x$, $x \geq 0$ or $x \to \Phi(a)\cdot x$, $x < 0$ for every $a \in S$. Then the associated double-quasigroup $(S, +, \square)$ satisfies A^+, C^+, $RP^{+\square}$. Moreover, $LP^{+\square}$ holds if and only if any mapping $x \to \Phi(x)\cdot a - \psi(x)\cdot b$, $x \in S$ for given a, b with opposite signs (in the sense of [6, p. 427]) is a surjection of S onto S.

Proof. The validity of $RP^{+\square}$ must be obtained in a manner different from that of the proof of theorem 4. Following [6, p.90], we replace the requirement of the unique solvability in $R^{\cdot}P^{+\square}$ by requiring only the existence of solutions

$$U \quad a\,\square\, c - a\,\square\, d = b\,\square\, c - b\,\square\, d \Rightarrow c = d \quad \text{for } a, b, c, d \in S;$$

if $c \geq 0$, $d \geq 0$, then $a\,\square\, c - a\,\square\, d = b\,\square\, c - b\,\square\, d \Rightarrow$
$\Rightarrow \psi(a)\cdot(c-d) = \psi(b)\cdot(c-d)$;

if $c < 0$, $d < 0$, then $a\,\square\, c - a\,\square\, d = b\,\square\, c - b\,\square\, d \Rightarrow$
$\Rightarrow \Phi(a)\cdot(c-d) = \Phi(b)\cdot(c-d)$;

if $c < 0$, $d \geq 0$, then $a\,\square\, c - a\,\square\, d = b\,\square\, c - b\,\square\, d \Rightarrow$
$\Rightarrow (\Phi(a) - \Phi(b))\cdot c = (\psi(a) - \psi(b))\cdot d$;

and if $c \geq 0$, $d < 0$, then $a\,\square\, c - a\,\square\, d = b\,\square\, c - b\,\square\, d \Rightarrow$
$\Rightarrow (\psi(a) - \psi(b))\cdot c = (\Phi(a) - \psi(b))\cdot d$.

In the first and the second case $c = d$ follows, whereas in the third case $\frac{c}{d} < 0 \Rightarrow \frac{\Phi(a)-\Phi(b)}{\psi(a)-\psi(b)} =$

$$= \frac{\Phi(a)-\Phi(b)}{a-b} : \frac{\psi(a)-\psi(b)}{a-b} < 0 \Rightarrow \operatorname{sg} \frac{\Phi(a)-\Phi(b)}{a-b} \neq \operatorname{sg} \frac{\psi(a)-\psi(b)}{a-b}$$

and one of the mappings Φ, Ψ cannot be pseudoorder-preserving, contradicting the hypothesis. The fourth case may be studied analogously. Thus the condition U holds in $(S, +, \square)$. We verify that any equation $a \square x - b \square x = c$ has at least one solution $x \in S$ for given $a, b, c \in S$, $a \neq b$. Indeed, for $x \geq 0$ this equation can be rewritten as $(\psi(a) - \psi(b)) \cdot x = c$, thus if $\frac{c}{\psi(a) - \psi(b)} > 0$, we may use the solution $x = \frac{c}{\psi(a) - \psi(b)}$. For $x < 0$ one may rewrite as $(\Phi(a) - \Phi(b)) \cdot x = c$, so that for $\frac{c}{\Phi(a) - \Phi(b)} < 0$ we may put $x = \frac{c}{\Phi(a) - \Phi(b)}$. It is clear that $\frac{c}{\psi(a) - \psi(b)} = \frac{c}{a-b} : \frac{\psi(a) - \psi(b)}{a-b} > 0 \Longleftrightarrow \frac{c}{a-b} : \frac{\Phi(a) - \Phi(b)}{a-b} > 0$, while in the contrary case one of the mappings Φ, Ψ is not pseudoorder-preserving.

Finally, we investigate any equation $x \square a - x \square b = c$ for given $a, b, c \in S$, $a \neq b$. For $a \geq 0$, $b \geq 0$ or for $a < 0$, $b < 0$ we have $\psi(x) \cdot (a - b) = c$ or $\Phi(x) \cdot (a - b) = c$ respectively, and the unique solvability follows from the definition of ψ and Φ. The remaining cases $a \geq 0$, $b < 0$ and $a < 0$, $b \geq 0$ yield the equations $\psi(x) \cdot a - \Phi(x) \cdot b = c$ and $\Phi(x) \cdot a - \Phi(x) \cdot b = c$ respectively, stated in the last condition of our theorem.

If we neglect the postulate of unique solvability for $x \in S \setminus \{0\}$ or for $y \in S \setminus \{0\}$ of the equation $x \square y = z$ for given $y, z \in S \setminus \{0\}$ or $x, z \in S \setminus \{0\}$ respectively, then we may construct, by the method of theorem 5, a system $(S, +, \square)$ such that $(S, +)$ is an Abelian group with neutral element 0, $x \square 0 = 0 \square x = 0$ for all $x \in S$ and $(S \setminus \{0\}, \square)$ is a groupoid satisfying condition U. In the assumptions of theorem 5 it is sufficient to replace the requirement that Φ be a pseudo-order-bijection by that Φ is to be a pseudoorder-injection. Then the resulting $(S, +, \square)$ satisfies A^+, C^+, U, $RP^{+\square}$ and does not satisfy $LP^{+\square}$. To obtain a concrete case choose $F = (S, +, \cdot)$ to be the field $F_0(\xi)$ of rational expressions over the basic field $F_0 = (S_0, +, \cdot)$ and define the pseudoorder on F as follows [6, p. 428]: if

$$a = \frac{f(\xi)}{g(\xi)} \in S$$

has the lowest form with non-zero polynomials $f(\xi)$, $g(\xi)$, then set $x > 0$ or $x < 0$ according as $\deg f(\xi) - \deg g(\xi)$ is even or odd. Next, choose $\Phi(a) = a^3$, $a \in S$, and $\Psi(a)$, $a \in S$; it may be shown that Φ is pseudoorder-preserving injection which is not a surjection and the same conclusion holds for the mapping $x \to 1 \cdot \psi(x) + 1 \cdot \Phi(x) = x + x^3$, $x \in S$ (e.g. for ξ there is no x such that $x^3 = \xi$ or $x + x^3 = \xi$). Another example is obtained if $F = (S, +, \cdot)$ is the rational field with the following pseudoorder [6, p. 427]: choose some prime p and express every rational in the form $p^n \frac{a}{b}$ where a, b are the lowest integers prime to p, and then say that this rational is positive or negative according as n is even or odd. Now set $\psi(a) = a$, $a \in S$ and $\Phi(a) = a^3$, $a \in S$. It may be proved that Φ is a pseudoorder-preserving injection which is not a surjection and that also the mapping $x \to 1 \cdot \psi(x) + 1 \cdot \Phi(x) = x + x^3$ is of the same type. - The so-obtained systems $(S, +, \square)$ may be interpreted as near-planar ternary rings, which are not planar (see the following definition) if the corresponding ternary composition T on S is introduced by $T(x, u, v) = x \square u + v$ for all $x, u, v \in S$.

Now we use theorem 5 for rational field $F = (S, +, \cdot)$ with the pseudoorder described above and put $\psi(a) = a$, $a \in S$, and $\Phi(p^m \frac{a_1}{a_2}) = p^m \frac{a_2}{a_1}$ for $p^m \frac{a_1}{a_2}$ in canonical form in $S \setminus \{0\}$, whereas $\Phi(0) = 0$.*) Then Φ is a pseudoorder-preserving bijection because for $\alpha = p^m \frac{a_1}{a_2}$, $\beta = p^n \frac{b_1}{b_2} \in S \setminus \{0\}$ with $m - n \geq 0$ there is $\frac{\Phi(\alpha) - \Phi(\beta)}{\alpha - \beta} = p^0 \frac{a_2 b_2}{a_1 b_1} \cdot \frac{p^{m-n} a_1 b_2 - a_2 b_1}{p^{m-n} a_2 b_1 - b_2 a_1} > 0$.
Then, for $p = 2$, the mapping $x \to x + \Phi(x)$, $x \in S$, is not surjective since the equation $2^m \frac{x_1}{x_2} + 2^m \frac{x_2}{x_1} = 2^1(-1) \iff (\frac{x_1}{x_2})^2 + 2^{1-m}(\frac{x_1}{x_2}) + 1 = 0$ has only a non-rational solution $\frac{x_1}{x_2} = 2^{-m} \pm \sqrt{2^{-2m} - 1}$, $m = 0, \pm 1, \pm 2, \dots$; the element $2^1 \cdot (-1) \in S$ does not have an inverse image with regard to Φ. The obtained system $(S, +, \square)$ can be interpreted as a near-planar ternary ring which is not planar

---

*) The existence of such Φ was orally communicated to me by O. Kowalski.

(see the following definition) if the corresponding ternary composition T on S is introduced by $T(x, u, v) = x \square u + v$ for all $x, u, v \in S$.

This ternary ring satisfies the condition of "symmetry": $T(x, u, v) = y$ is uniquely solvable in $x \in S$ for given $u, v, y \in S$, $u \neq 0$. The existence of such ternary rings is important because it shows that the notion of symmetric near-planar ternary rings ([7b]) is in fact more general than that of planar ternary rings.

By a <u>ternary ring</u> (S, T) is meant here a non-empty set S with a ternary composition on T satisfying $T(S, S, S) = S$. The ternary ring (S, T) is called <u>near-planar</u> if

1^o there exists an element $0 \in S$ such that $T(x, 0, v) = v$, $T(0, u, v) = v$ for all $x, u, v \in S$,

2^o any equation $T(a, b, v) = d$ is, for given $a, b, d \in S$, uniquely solvable in $v \in S$,

3^o for given $x_1, y_1, x_2, y_2 \in S$ with $x_1 \neq x_2$, the equations $T(x_i, u, v) = y_i$, $i = 1, 2$ have a unique solution $x \in S$.

The near-planar ternary ring (S, T) is said to be <u>planar</u> if

4^o for given $u_1, v_1, u_2, v_2 \in S$ with $u_1 \neq u_2$ the equation $T(x, u_1, v_1) = T(x, u_2, v_2)$ has a unique solution $x \in S$.

6. Let $F = (S, +, \cdot)$ be a pseudoordered field and $\Phi : S \to S$ a bijection with fixed element 0. We define a ternary composition T on S as follows:

$T(x,u,v) = x \cdot u + v$ for $u \geq 0$ and $T(x,u,v) = \Phi^{-1}(\Phi(x) \cdot u + v)$ for $u < 0$. Then (S, T) is the ternary ring satisfying 1° and 2°; moreover 3° holds precisely if Φ is a pseudoorder-monotonic (in the sense of [6, p. 428]).

Proof. According to the definition of Φ and T, 0 satisfies condition 1°. Condition 2° is obvious for $u \geq 0$ and follows from the bijectivity of Φ if $u < 0$. Given the equations $y_i = T(x_i, u, v)$, $i = 1, 2$, with $x_1, y_1, x_2, y_2 \in S$, $x_1 \neq x_2$, $y_1 \neq y_2$, we distinguish two cases:

(1) $y_i = x_i \cdot u + v$, $i = 1, 2$ for $u \geq 0$,

(2) $\Phi(y_i) = \Phi(x_i) \cdot u + v$, $i = 1, 2$ for $u < 0$.

Thus from (1) there follows $(x_1 - x_2) \cdot u = y_1 - y_2$, $sg(x_1 - x_2) = sg(y_1 - y_2)$ and from (2) there follows $(\Phi(x_1) - \Phi(x_2)) \cdot u = \Phi(y_1) - \Phi(y_2)$, $sg(\Phi(x_1) - \Phi(x_2)) \neq sg(\Phi(y_1) - \Phi(y_2))$. We conclude that 3° is satisfied precisely if $\dfrac{y_1 - y_2}{x_1 - x_2} > 0 \iff$

$$\iff \frac{\Phi(y_1) - \Phi(y_2)}{\Phi(x_1) - \Phi(x_2)} > 0 \quad \text{or} \quad \frac{y_1 - y_2}{x_1 - x_2} \cdot \frac{\Phi(x_1) - \Phi(x_2)}{\Phi(y_1) - \Phi(y_2)} > 0 \quad \text{or}$$

$sg \dfrac{\Phi(x_1) - \Phi(x_2)}{x_1 - x_2} = sg \dfrac{\Phi(y_1) - \Phi(y_2)}{y_1 - y_2}$, all of which mean that Φ is a pseudoorder-monotone. Condition 4° holds in (S, T) precisely if for $u_1 < 0 < u_2$ each $\Phi(x) \cdot u_1 + v_1 = \Phi(x \cdot u_2 + v_2)$ is uniquely solvable in $x \in S$. For F the real field and $\Phi(x) = x^3$, $x \in S$, we obtain the situation investigated in [2].

7. Let $F = (S, +, \cdot)$ be a pseudoordered field and $\Phi : S \to S$ a bijection with $\Phi(0) = 0$; let T be

the ternary composition on S defined as follows: $T(x,u,v)=\Phi(x)\cdot u+v$ for $u\geq 0$ and $T(x,u,v)=\Phi^{-1}(x\cdot u+\Phi(v))$ for $u<0$. Then (S,T) is a ternary ring satisfying 1° and 2°; moreover 3° holds precisely if Φ is pseudoorder-monotone.

Proof. Condition 1° is obviously satisfied. Condition 2° is valid for $u\geq 0$ trivially, and for $u<0$ follows from bijectivity of Φ. Thus we need only consider condition 3°: assume given $x_1, y_1, x_2, y_2 \in S$, $x_1 \neq x_2$, $y_1 \neq y_2$, and distinguish two alternatives:

(3) $y_i = \Phi(x_i)\cdot u + v,\quad i=1,2$ for $u\geq 0$.

(4) $\Phi(y_i) = x_i\cdot u+\Phi(v),\quad i=1,2$ for $u<0$.

From (3) there follows $(\Phi(x_1)-\Phi(x_2))\cdot u = y_1-y_2$, so that $\dfrac{\Phi(x_1)-\Phi(x_2)}{y_1-y_2} > 0$; from (4) there follows

$(x_1-x_2)\cdot u=\Phi(y_1)-\Phi(y_2)$, so that $\dfrac{\Phi(y_1)-\Phi(y_2)}{x_1-x_2}<0$.

We conclude that $\dfrac{\Phi(x_1)-\Phi(x_2)}{y_1-y_2}$ and $\dfrac{\Phi(y_1)-\Phi(y_2)}{x_1-x_2}$ simultaneously have the same sign, which implies that Φ is pseudoorder-monotone (and conversely). Condition 4° holds in (S,T) if and only if, for $u_1<0<u_2$, each $x u_1 + \Phi(v_1) = \Phi(\Phi(x)u_2+v_2)$ is uniquely solvable in $x\in S$. If F is taken to be a rational field and $\Phi = 1$ chosen according to André's procedure [5, p. 204-205], one obtains the planar ternary ring investigated in [5].

L i t e r a t u r e :

[1] F.R. MOULTON, A simple non-Desarguesian plane, Trans. Amer.Math.Soc.3(1902),192-195.

[2] N.F. TSCHETWERUCHIN, Eine Bemerkung zu den Nicht-Desarguesschen Liniensystemen, Jahresbericht der Deutschen Mathematiker-Vereinigung 36 (1927), 154-156.

[3] L.A. SKORNJAKOV, Natural fields of Veblen-Wedderburn planes (in Russian),Izv.Ak.nauk SSSR 13 (1949),447-472.

[4] G. PICKERT, Projektive Ebenen, Berlin-Göttingen-Heidelberg 1955.

[5] J. ANDRÉ, Affine Ebenen mit genügend vielen Translationen, Math.Nachrichten 19(1958),203-210.

[6] V. HAVEL, Verallgemeinerte Gewebe I, (to appear in Publ.Fac.Sci.Brno)

[7] V. HAVEL, Near-planar ternary rings, (to appear in Journal of Algebra)

(Received August 23, 1965)