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A PRINCIPLE OF DEHOMOGENIZATION FOR EIGENVALUE PROBLEMS

(Preliminary communication x))

Ivo MAREK, Praha

It is shown that some eigenvalue problems can be reduced to sequences of unhomogeneous equations by using iterative methods of Kellogg's type. By means of this procedure the problem of the accuracy-order of approximations of eigenelements in Banach spaces is investigated.

1. Notation and definitions.

Let h be a real parameter $0 < h < h_0$ and let X, X_h be Banach spaces. Symbols denoting norms in all these spaces will not be distinguished. Let $\tilde{X}_h \subset X$ be some subspaces such that $\tilde{X}_h$, X_h are isomorph. Let us denote this isomorphism by S_h. By X', X'_h the spaces of continuous linear forms on X and X_h with the usual norm (see [3]) will be denoted. If Y is a Banach space, then by $[Y]$ we shall denote the space of linear bounded mappings of Y into itself with the uniform topology (see [3]). Let P_h be a projection of X onto X_h such that

$S_h^{-1} P_h x \to x$ if $h \to 0$, $h \in (0, h_0)$, $0 < h_0 \leq +\infty$.

If T is a linear, generally unbounded operator, then we denote the definition domain of T by the symbol $\mathcal{D}(T)$ and the range of T by $\mathcal{R}(T)$ respectively.

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x) The complete text will be published in the Czech.Math.Journ.

In the following text we shall denote the constants which do not depend of h by only one symbol c without any further distinguishing.

Definition 1. Suppose that $\mathfrak{M} \subset X$ and $T \in [X]$, $T_h \in [X_h]$. If the relation

(1.1) $$\| P_h Tx - T_h P_h x \| \leq c(x) h^p$$

holds for each vector $x \in \mathfrak{M}$, where p is a positive integer or zero, then we say that the "approximative" operator T_h has the approximation-order p according to the operator T on the set $\mathfrak{M}$.

Let $y \in X$ be an arbitrary vector and let $y_h = P_h y$. By u, u^h we shall denote the solutions of the equations

(1.2) $$Tx = y, \quad T_h x_h = y_h .$$

Definition 2. Let the equations (1.2) have unique solutions u, u^h for y, y_h given. If the inequality

(1.3) $$\| P_h u - u^h \| \leq c h^r$$

holds for u, u^h, where r is a positive integer, then we say that T_h has the accuracy-order r with respect to T.

The equations involved in the preceding definitions have been assumed uniquely solvable. Therefore these definitions are not suitable for eigenvalue problems.

Definition 3. Let μ_0 be an eigenvalue of an operator $T \in [X]$ and $x_1, \dots, x_t$, $(t < +\infty)$, the corresponding eigenvectors. We say that the operator T_h has the accuracy-order p with respect to the eigenvalue μ_0 of the operator T, if for each eigenvector x_j corresponding to μ_0 there exists an eigenvalue μ_j^h of the operator T_h and an eigenvector x_j^h corresponding to μ_j^h such that the inequalities

(1.4) $$\| P_h x_j - x_j^h \| \leq c h^p ,$$
$$| \mu_0 - \mu_j^h | \leq c h^p$$

hold for $j = 1, \dots, t$.

Let M , C be linear operators mapping the domains $\mathfrak{D}(M), \mathfrak{D}(C)$ into X and let M_h , C_h be corresponding operators mapping the domains $\mathfrak{D}(M_h)$, $\mathfrak{D}(C_h)$ into X_h . The couple $\{ M_h , C_h \}$ will be called a scheme.

Definition 4. A scheme $\{ M_h , C_h \}$ has the accuracy-order p with respect to the equation $Mu = Cv$, where $v \in \mathfrak{D}(C)$ is some given vector, if the inequality
$$\| P_h u - u^h \| \leq c h^p$$
holds for all solutions u, u^h of the equations $Mu = Cv$, $M_h u^h = C_h P_h v$ respectively.

Definition 5. A scheme $\{ M_h , C_h \}$ has the accuracy-order p with respect to the characteristic value λ_0 if one of the operators $T_h = M_h^{-1} C_h$, $T_h = C_h M_h^{-1}$ has the accuracy-order p according to the eigenvalue $\mu_0 = 1/\lambda_0$ of the operator $T = M^{-1} C$ or $T = C M^{-1}$ respectively. If we use the scheme $\{ M_h , C_h \}$ for counting λ_0 we shall denote this fact by $\{ M_h , C_h ; \lambda_0 \}$.

2. Eigenvalue problems.

This paragraph is concerned with the investigation of eigenvalue problems of the form

(2.1) $$Mu = \lambda C u ,$$

where M, C are linear, generally unbounded, operators mapping the domains $\mathfrak{D}(M)$, $\mathfrak{D}(C)$ into X . Moreover it is assumed that $\mathfrak{D}(M)$ is dense in X and $\mathfrak{D}(M) \subset \mathfrak{D}(C)$.

Simultaneously with the problem (2.1) we shall consider the "approximative" eigenvalue problem

(2.2) $$M_h u^h = \lambda^h C_h u^h$$

assuming that the scheme $\{M_h, C_h\}$ has the accuracy-order p with respect to the equation $Mu = Cv$ according to the definition 4.

If we use this assumption we must reduce the equations (2.1) and (2.2) to sequences of unhomogeneous equations of the type

(2.3) $$Mx = y, \quad M_h x^h = y^h .$$

To do this we use Kellogg's iterative procedure. This procedure is applicable if the operators $M^{-1}C$, CM^{-1}, $M_h^{-1} C_h, C_h M_h^{-1}$ have suitable properties.

Let T, T_h be one of the couples mentioned above. It will be supposed that T, T_h be closed have dominant eigenvalues μ_0, μ_0^h i.e. in spectra $\sigma(T)$, $\sigma(T_h)$ there are points μ_0, μ_0^h such that the inequalities

(2.4) $$|\lambda| < |\mu_0|, \quad |\lambda^h| < |\mu_0^h|$$

hold for each $\lambda \in \sigma(T)$, $\lambda \neq \mu_0$ and each $\lambda^h \in \sigma(T_h)$, $\lambda^h \neq \mu_0^h$. Moreover, we shall assume that the points μ_0, μ_0^h are poles of the resolvents $R(\lambda, T) = (\lambda I - T)^{-1}$, $R(\lambda, T_h) = (\lambda I_h - T_h)^{-1}$, where I, I_h denote the identity-operators in X and X_h respectively. We remark that the last assumption is not necessary.

Let B_1, B_1^h be the operators defined by the following integrals

$$B_1 = \frac{1}{2\pi i} \int_{C_0} R(\lambda, T)\, d\lambda ,$$

(2.5)

$$B_1^h = \frac{1}{2\pi i} \int_{C_0^h} R(\lambda, T_h)\, d\lambda ,$$

where C_0, C_0^h denote the circles having the centres μ_0, μ_0^h and radii ρ_0, ρ_0^h such that for the sets $K = \{\lambda \mid |\lambda - \mu_0| \leq \rho_0\}$, $K_h = \{\lambda \mid |\lambda - \mu_0^h| \leq \rho_0^h\}$ the relations $K \cap \sigma(T) = \{\mu_0\}$, $K_h \cap \sigma(T_h) = \{\mu_0^h\}$ hold.

If $x'_h \in X'_h$, then we put

$$x'(x) = x'_h(P_h x), \quad x \in X,$$

so that $x' \in X'$.

Suppose that there exists a vector $x^{(0)} \in X$ for which

(2.6) $0 < c = |x'(B_1 x^{(0)})|, \quad 0 < c \leq |x'_h(B_1^h P_h x^{(0)})|$.

Now, we shall investigate the case of the operator $T = M^{-1}C$ and then we must assume that C is bounded. The corresponding Kellogg's iterations leading to unhomogeneous equations are defined as follows

(2.7 a) $M u^{(n+1)} = C u_{(n)}, \; u_{(n+1)} = \lambda_{(n)} u^{(n+1)}, \; u_{(0)} = x^{(0)}$,

(2.7 b) $$\lambda_{(n)} = \frac{x'(u_{(n)})}{x'(u^{(n+1)})} \quad ;$$

(2.8 a) $M_h u^{(n+1)} = C_h u_{(n)}^h, \quad u_{(n+1)}^h = \lambda_{(n)}^h u_h^{(n+1)}$,

$$u_{(0)}^h = P_h x^{(0)} \quad ,$$

(2.8 b) $$\lambda_{(n)}^h = \frac{x'_h(u_{(n)}^h)}{x'_h(u_h^{(n+1)})} \quad .$$

Let us investigate the case of the operator $T = CM^{-1}$. It is easy to see that the restrictive assumption $C \in [X]$ can be replaced by a weaker one. Let instead of $C \in [X]$ the inclusion

(2.9) $\mathcal{R}(M^{-1}) \subset \mathcal{D}(C)$

hold, where $\mathcal{R}(M^{-1})$ denotes the range of the operator M^{-1}. Similarly, let

(2.10) $$\mathcal{R}(M_h^{-1}) \subset \mathcal{D}(C_h) .$$

The Kellogg's iterations corresponding to the operators $T = CM^{-1}$, $T_h = C_h M_h^{-1}$ are defined as follows

(2.11a) $Mv^{(n)} = v_{(n)}, v_{(n+1)} = \lambda_{(n)} Cv^{(n)}, v_{(0)} = x^{(0)} = Cy^{(0)}, y^{(0)} \in X$,

(2.11b) $$\lambda_{(n)} = \frac{x'(v_{(n)})}{x'(Cv^{(n+1)})} ;$$

(2.12a) $M_h v_h^{(n)} = v_{(n)}^h, v_{(n+1)}^h = \lambda_{(n)}^h C_h v_h^{(n)}, v_{(0)}^h = C_h P_h y^{(0)}, y^{(0)} \in X$,

(2.12b) $$\lambda_{(n)}^h = \frac{x_h'(v_{(n)}^h)}{x_h'(C_h v_h^{(n+1)})} .$$

The convergence of the processes defined by (2.7),(2.8), (2.11),(2.12) is described in the following theorems.

Let $A \in [X]$ be an operator having a dominant eigenvalue μ_0. Let μ be a number with the following properties: (a) $\mu < |\mu_0|$; (b) $\lambda \in \sigma(A)$, $\lambda \neq \mu_0$ implies $\lambda \in H(A)$, where $H(A) = \{\lambda / |\lambda| < \mu\}$.

<u>Theorem A</u> [2] Suppose that

1. The values μ_0, μ_0^h are dominant eigenvalues of the operators $T = M^{-1}C$, $T_h = M_h^{-1} C_h$.
2. μ_0, μ_0^h are poles of the resolvents $R(\lambda, T)$, $R(\lambda, T_h)$.
3. The conditions (2.6) are fulfilled for a vector $x^{(0)} \in X$.
4. $\alpha = |\frac{\mu}{\mu_0}|$, $\alpha_h = |\frac{\mu^h}{\mu_0^h}|$, where μ, μ^h are radii of the sets $H(T)$, $H(T_h)$ defined above.

Then the relations

$$\| u_{(n)} - u_0 \| \leq c\alpha^n, \ |\lambda_{(n)} - \lambda_0| \leq c\alpha^n ,$$

$$\| u_{(n)}^h - u_0^h \| \leq c\alpha_h^n, \ |\lambda_{(n)}^h - \lambda_0^h| \leq c\alpha_h^n$$

(n sufficiently large)

hold for the sequences defined by (2.7),(2.8), where

$$\lambda_0 = \frac{1}{\mu_0}, \quad \lambda_0^h = \frac{1}{\mu_0^h}$$

and

$$M u_0 = \lambda_0 C u_0, \quad M_h u_0^h = \lambda_0^h C_h u_0^h,$$
$$u_0 \neq 0, \quad u_0^h \neq 0 .$$

Theorem B [2] Let the assumptions 1,2,4 of the theorem A be fulfilled for the operators $T = CM^{-1}$, $T_h = C_h M_h^{-1}$. Instead of the assumption 3 let the following one be fulfilled: The condition (2.6) holds for a vector $x^{(0)} = v_{(0)}$, where $v_{(0)} = Cy^{(0)}$, $y^{(0)} \in X$.

Then the relations

$$\| v_{(n)} - v_0 \| \leqq c\alpha^n, \quad |\lambda_{(n)} - \lambda_0| \leqq c\alpha^n,$$

$$\| v_{(n)}^h - v_0^h \| \leqq c\alpha_h^n, \quad |\lambda_{(n)}^h - \lambda_0^h| \leqq c\alpha_h^n$$

(n sufficiently large)

hold for the sequences defined by (2.11),(2.12), where

$$\lambda_0 = \frac{1}{\mu_0}, \quad \lambda_0^h = \frac{1}{\mu_0^h}$$

and

$$M v_0 = \lambda_0 C v_0, \quad M_h v_0^h = \lambda_0^h C_h v_0^h,$$

$$v_0 \neq 0, \quad v_0^h \neq 0 .$$

By means of the iterations given by (2.7),(2.8) and (2.11),(2.12) the initial eigenvalue problems (2.1),(2.2) are reduced to sequences of unhomogeneous equations of the type (2.3). These procedures form a base of the dehomogenization.

3. Theory of accuracy-order of eigenvalue problems.

In this paragraph the accuracy-order of eigenvalue problems of the type

$$M u = \lambda C u, \quad M_h u^h = \lambda^h C_h u^h$$

will be investigated. The basic assumption is the knowledge of the accuracy-order of the scheme $\{M_h, C_h\}$ with respect to the unhomogeneous equation $Mu = Cv$ with given vectors $v \in X$.

Theorem 1. Suppose that

1. The operators $T = M^{-1}C$, $T_h = M_h^{-1} C_h$, where the operatos $M, C; M_h, C_h$ map $\mathcal{D}(M), \mathcal{D}(C) = X$ into X and $\mathcal{D}(M_h), \mathcal{D}(C_h) = X_h$ into X_h, be bounded and have dominant eigenvalues μ_o, μ_o^h and these values are simple poles of the resolvents $R(\lambda, T)$, $R(\lambda, T_h)$.

2. The scheme $\{M_h, C_h\}$ has the accuracy-order p with respect to the problem $Mu = Cv$ with a given vector $v \in X$.

3. The relations (2.6) for a vector $x^{(o)} \in X$ and simultaneously the inequalities

$$\|x_h'\| \leq c \qquad \text{if} \quad 0 < h < h_o$$

hold for the system $\{x_h'\}$ of linear forms $x_h' \in X_h'$.

4. The inequalities

$$\| M_h^{-1} C_h \| \leq c$$

hold for all h, $0 < h < h_o$.

Then the scheme $\{M_h, C_h; \lambda_o\}$ has the accuracy-order p with respect to characteristic value $\lambda_o = 1/\mu_o$.

The case of an unbounded operator C is described in the following theorem.

Theorem 2. Suppose that

1. The values μ_o, μ_o^h are dominant eigenvalues of the closed operators $T = CM^{-1}$, $T_h = C_h M_h^{-1}$, where M, C, M_h, C_h map the domains $\mathcal{D}(M), \mathcal{D}(C)$ into X and $\mathcal{D}(M_h), \mathcal{D}(C_h)$ into X_h and the inclusions $\mathcal{R}(M^{-1}) \subset \mathcal{D}(C)$, $\mathcal{R}(M_h^{-1}) \subset \mathcal{D}(C_h)$ hold. Moreover, let μ_o, μ_o^h be simple poles of the resolvents

$R(\lambda, T)$, $R(\lambda, T_h)$.

2. The approximative operator M_h has the accuracy-order p with respect to the equation $Mu = v$ with a given $v \in X$.

3. For each vector $u \in \mathcal{R}(M^{-1})$ there exists a vector $y^h \in X_h$ such that the identity

$$C_h P_h u = P_h C u + y^h \quad ,$$

holds, where

$$\| y^h \| \leqq c h^p \; .$$

In other words - the approximation-order of the operator C_h is equal to p with respect to the operator C on the set $\mathcal{R}(M^{-1})$.

4. The relations (2.6) for a vector $x^{(0)} = Cy^{(0)}$, $y^{(0)} \in X$ and simultaneously the inequalities

$$\| x'_h \| \leqq c \quad \text{if} \quad 0 < h < h_0$$

hold for the system $\{ x'_h \}$ of linear forms $x'_h \in X'_h$.

5. The inclusions

$$\mathcal{R}(P_h M^{-1} X) \subset \mathcal{R}(M_h^{-1} X_h)$$

hold for $0 < h < h_0$.

6. The inequalities

$$\| C_h M_h^{-1} \| \leqq c \quad , \quad 0 < h < h_0 \quad ,$$

hold for the operators $T_h = C_h M_h^{-1}$ of the system $\{ T_h \}$.

Then the scheme $\{ M_h, C_h ; \lambda_0 \}$ has the accuracy-order p with respect to the characteristic value $\lambda_0 = 1/\mu_0$ of the eigenvalue problem

$$Mu = \lambda C u \; .$$

The assumption that the eigenvalue μ_0 is a dominant point of the spectrum $\sigma(T)$ can be weakened. Let us suppose that there is a finite number of eigenvalues $\mu_1, \dots, \mu_s$ on the circle $|\lambda| = \kappa(T)$, where $\kappa(T) = \lim_{n \to \infty} \sqrt[n]{\| T^n \|}$

is the spectral radius of the operator $T \in [X]$. In such a case it always is possible to find suitable complex numbers $\nu_1, \dots, \nu_s$ such that the operators

(3.1) $$S_j = T + \nu_j I \,, \quad j = 1, \dots, s \,,$$

have dominant eigenvalues $\tau_j = \mu_j + \nu_j$.

Now suppose that complex numbers $\nu_1, \dots, \nu_s$ are chosen so that the values $\tau_j = \mu_j + \nu_j$ are dominant points of $\sigma(S_j)$ $j = 1, \dots, s$, where $T = CM^{-1}$, $|\mu_j| = \kappa(T)$, $|\mu_j + \nu_j| = \kappa(S_j)$ and S_j , $j = 1, \dots, s$, are defined in (3.1.)

Put

$$L = M \,, \quad D = \nu_j M + C \,.$$

It is easy to see that the construction of the eigenvalue μ_j and the corresponding eigenvector x_j is equivalent to the construction of the dominant characteristic value $\rho_j = 1/\tau_j$ and the corresponding eigenvector of the equation

(3.2) $$(\nu_j M + C)u = \rho M u \,.$$

The solution of this problem can be obtained using the theorems 1 and 2 .

Let $L_{h,j}$, $D_{h,j}$, $S_{h,j}$ be operators corresponding to the operators L , D , S_j . This means that there exist complex numbers $\nu_1^h, \dots, \nu_s^h$ such that the values $\nu_1^h + \mu_1^h, \dots$ $\dots, \nu_s^h + \mu_s^h$ are dominant eigenvalues of the operators

$$S_{h,j} = T_h + \nu_j^h I_h \,, \quad j = 1, \dots, s \,.$$

Theorem 3. Suppose that

1. The assumptions of theorem 2 are fulfilled fr the operators M, C and forms $x' \in X'$, $x_h' \in X_h'$.
2. The operator $T = CM^{-1}$ has a finite number of eigenvalues $\mu_1, \dots, \mu_s$, $s \geq 1$, on the circle $|\lambda| = \kappa(T)$.

3. There exist complex numbers $\nu_1, \dots, \nu_s$ such that $\tau_j = \mu_j \pm \nu_j$ are dominant eigenvalues of the operators (3.1) and are simple poles of the resolvents $R(\lambda, S_j)$, $j = 1, \dots s$.

4. The inequalities

(3.3) $\quad 0 < c = |x'(B_{1j}\, x^{(0)}|, \quad 0 < c \leq |x'_h (B^h_{1j} P_h x^{(0)})|$

hold for a vector $x^{(0)} \in X$, where $x' \in X'$, $x'_h \in X'_h$ and B_{1j}, B^h_{1j} are elements of Laurent developments of the resolvents $R(\lambda, S_j) = (\lambda I - S_j)^{-1}$, $R(\lambda S_{h,j}) = (\lambda I_h - S_{h,j})^{-1}$ in neighbourhoods of the points τ_j, τ^h_j

$$R(\lambda, S_j) = \sum_{k=0}^{\infty} A_{kj} (\lambda - \tau_j)^k + \sum_{k=1}^{\infty} B_{kj} (\lambda - \tau_j)^{-k} ,$$

$$R(\lambda, S_{h,j}) = \sum_{k=0}^{\infty} A^h_{kj} (\lambda - \tau^h_j)^k + \sum_{k=1}^{\infty} B^h_{kj} (\lambda - \tau^h_j)^{-k} .$$

Then the schemes $\{ M_h , C_h ; \lambda_j \}$ have the accuracy-order p with respect to the characteristic values $\lambda_j = 1/\mu_j$, $j = 1, \dots, s$.

4. Applications.

The preceding theory can be applied to various problems of numerical analysis. If the spaces X_h are finite dimensional, the operators T_h corresponding to the operator T are given as finite matrices. Particularly, that is the case of net methods of numerical solution of differential equations (see [4]). Applications of the idea of dehomogenization of eigenvalue problems of this type were described by the author (see [1]) at the conference on basic problems of numerical analysis, Liblice (Czechoslovakia) 1964 . Other applications will be given in the complete text which will be submitted to the Czechoslovak Mathematical Journal.

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