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THE CONTINUUM PROBLEM AND POWERS OF ALEPHS

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Every cardinal is an aleph in a set theory with the axiom of choice. In particular, $\aleph_\alpha^{\aleph_\beta}$ is an aleph $\aleph_\gamma$. The reccurence formulas are well known for the calculation of $\aleph_\gamma$ (The Hausdorff formula for γ isolated, two formulas by Tarski for γ a limit ordinal, see (2,1)-(2,3)).The formula (2,3) is based on a calculation of an infinite cardinal product, and therefore it is not possible to use it generally for a calculation of $\aleph_\gamma$.

The present paper contains an exact definition of the notion of calculability. We introduce a continuum function $\varkappa$ and describe its properties. A new property of the continuum function is proved in theorem 3.2. We examine the calculability of $\aleph_\gamma$ (i.e. $\mu(\alpha;\beta)$) relative to $\varkappa$ and other functions.

Throughout this paper, we use the notation and definitions introduced in [G]. We use two kinds of considerations: mathematical and metamathematical. Therefore we use the symbols: $f, g, k, \dots, \alpha, \beta, \gamma, \delta, \pi, \varkappa, \mu, \xi$ for mathematical objects, and $\mathfrak{M}, \nabla, \varphi, \psi, \dots, n, m, k$ for metamathematical ones.

In the case of mathematical considerations, we work in the set theory Σ^* of Gödel (i.e. we use the axioms of groups A-E). By $\vdash \varphi$ we denote that φ is provable in Σ^*.

§ 1. Calculable functions

Metadefinition 1.1 A normal formula φ is called an of-formula iff there is a number (metamathematical) n such that

$$\vdash (\exists ! X)\varphi(X)$$

$$\vdash (X)(\varphi(X) \rightarrow . X\, Fn\, O_n^n \,\&\, W(X) \subseteq O_n)$$

We say that φ defines an ordinal n-ary function.

An ordinal function is a constant of the theory Σ^* defined by an of-formula. To simplify the considerations we always speak about an ordinal function instead of the formula which defines the function. Thus, the expression " let f be an n-ary ordinal function" is an abbreviation for the expression "let φ be an of-formula which defines an n-ary function f ". The formula $\psi(f)$ is an abbreviation for $(X)(\varphi(X) \rightarrow \psi(X))$.

Example: The of-formula $\varphi_0(X) \equiv X = \{\emptyset\} \times On$ defines an 1-ary ordinal function Z. $\varphi_1(X) \equiv . (x)(y)(\langle x\, y\rangle \in \in X \equiv . y \in On \,\&\, x = y \dot{+} 1) \,\&\, X \subseteq On \times On$ defines a function S. P is the function defined by 7.9 in [G], C_1, C_2 are functions: $P(C_1(\alpha), C_2(\alpha)) = \alpha$.

It is easy to find the formulas which define the following functions:

$$U_i^n(\alpha_1, \dots \alpha_n) = \alpha_i; \quad 1 \leq i \leq n, \quad n = 1, 2, \dots$$

$$sg(\alpha; \beta) = \begin{cases} 0 \text{ for } \alpha \leq \beta \\ 1 \text{ for } \alpha > \beta \end{cases} \qquad eq(\alpha; \beta) = \begin{cases} 0 \text{ for } \alpha \neq \beta \\ 1 \text{ for } \alpha = \beta \end{cases}$$

$$cf(\alpha) = \begin{matrix} \alpha \\ \gamma \end{matrix} \text{ for } \alpha \in K_I$$

γ is the least β for which α is confinal with ω_β, for $\alpha \in K_{II}$.

__Metadefinition 1.2__ The operation of composition associates with the ordinal functions $f_0(\alpha_1, \dots \alpha_m)$, $f_1(\alpha_1, \dots \alpha_m)$, $\dots f_n(\alpha_1, \dots \alpha_m)$ the function

$$f(\alpha_1, \dots \alpha_m) = f_0(f_1(\alpha_1, \dots \alpha_m), \dots f_n(\alpha_1, \dots \alpha_m)).$$

The operation of induction associates with the ordinal functions $f_0(\alpha_1, \dots \alpha_n)$, $f_1(\alpha_1, \dots \alpha_{n+1})$, $f_2(\alpha_1, \dots \alpha_{n+2})$, $f_3(\alpha_1, \dots \alpha_{n+2})$ the function defined in the following way

$$f(0, \alpha_1, \dots \alpha_n) = f_0(\alpha_1, \dots \alpha_n)$$

$$f(\alpha+1, \alpha_1, \dots \alpha_n) = f_2(\alpha, f(\alpha, \alpha_1, \dots \alpha_n), \alpha_1, \dots \alpha_n)$$

for $\alpha \in K_{II}$ $f(\alpha, \alpha_1, \dots \alpha_n) = f_1(\lim\limits_{\xi \in \alpha} f_3(\xi, f(\xi, \alpha_1, \dots \alpha_n), \alpha_1, \dots \alpha_n), \alpha_1, \dots \alpha_n)$.

__Remark__: These operations are metamathematical ones on of-formulas, e.g. for composition: if $\varphi_0, \varphi_1, \dots \varphi_n$ are of-formulas (which define $f_0, f_1, \dots f_n$) then the formula

$$\varphi(X) \equiv . X \subseteq On^{m+1} \,\&\, (\alpha_0)(\alpha_1)\dots(\alpha_m)(\langle \alpha_0, \alpha_1, \dots \alpha_m \rangle \in X \equiv$$

$$\equiv (X_0)(X_1)\dots(X_n)(\beta_1)\dots(\beta_n)(\varphi_0(X_0) \,\&\, \dots \,\&\, \varphi_n(X_n) . \rightarrow$$

$$\rightarrow . \langle \alpha_0, \beta_1, \dots \beta_n \rangle \in X_0 \,\&\, \langle \beta_1, \alpha_1, \dots \alpha_m \rangle \in X_1 \,\&\, \dots \&$$

$$\& \langle \beta_n, \alpha_1, \dots \alpha_m \rangle \in X_n))$$

defines the ordinal function f (composition of $f_0, f_1, \dots f_n$).

__Example__: $\overline{sg}(\alpha) = sg(f(\alpha), U_1^1(\alpha))$ is a composition of sg, f, U_1^1 where $f(\alpha) = S(Z(\alpha))$.

__Metadefinition 1.3__ *) An ordinal function is called calculable relative to the ordinal functions $k_1, \dots k_m$ iff it can be obtained by a finite number of applications of composition and induction beginning with the functions of the following list:

a) $k_1, \dots k_n$

b) $S, Z, sg, P, C_1, C_2, U_i^m, \ i = 1, \dots m, \ m = 1, 2, \dots$

Example: $eq, \overline{sg}, \alpha \dot{+} \beta, \alpha \times \beta$ are calculable *) ordinal functions ($\alpha \dot{+} \beta$, $\alpha \times \beta$ are ordinal summ and product).

Definition 1.3 enables to demonstrate the calculability of an ordinal function. But we cannot prove the uncalculability of a function directly. The notion of an invariant function and theorem 1.5 will be useful for this purpose.

A model of set theory is defined in [V1]. It is a meta-concept (a pair of formulas). If $\mathfrak{M}$ is a model of set theory then the corresponding concepts of set theory in the model $\mathfrak{M}$ are denoted by " m " . In particular, if f is an ordinal function (i.e. φ is an of-formula which defines f, then $f^{\mathfrak{M}}$ is the corresponding function in model $\mathfrak{M}$).

If there is no danger of misunderstanding, we shall simplify the notation.

Metadefinition 1.4 Let $k_1, \dots k_n$ be ordinal functions, φ an of-formula. We say that φ defines a function f invariant with respect to $k_1, \dots k_n$ iff the following implication holds:

If $\mathfrak{M}$ is a weakly regular standart (see [V1]) model of set theory for which there is a class F with the properties

a) $\vdash F \ \mathrm{Isom}_{E, \varepsilon^{m}} \ On^{m}$

b) $(\alpha_1) \dots (\alpha_{m_i})(\alpha_1, \dots \alpha_{m_i} \in On \rightarrow$
$\rightarrow F(k_i(\alpha_1, \dots \alpha_{m_i})) = k_i^{m}(F(\alpha_1), \dots F(\alpha_{m_i}))), \quad i = 1, \dots n$

then $\vdash (\alpha_1) \dots (\alpha_e)(\alpha_1, \dots \alpha_e \in On \rightarrow F(f(\alpha_1, \dots \alpha_e)) = f^{m}(F(\alpha_1), \dots F(\alpha_e)))$.

A function is called invariant iff it is invariant with respect to the empty sequence of functions.

Example: It is easy to see that functions sg, U_i^n, Z, S, P, C_1, C_2 are invariant.

Metatheorem 1.5 If f is calculable relative to $k_1, \dots k_m$, then f is invariant with respect to $k_1, \dots k_m$.

Proof: Let $\mathcal{M}$ be a model with properties of definition 1.4. Then $F(0) = 0^{\mathcal{M}}$, $F(\alpha \dot{+} 1) = F(\alpha) \dot{+} 1^{\mathcal{M}}$. It suffices to prove that a composition of functions invariant with respect to $k_1, \dots k_m$ has also this property and the same for induction (as functions sg, U_i^n, Z, S, P, C_1, C_2 are invariant).

For composition:

$$F(f_0(f_1(\alpha_1, \dots \alpha_m), \dots f_n(\alpha_1, \dots \alpha_m))) =$$
$$= f_0^{\mathcal{M}}(F(f_1(\alpha_1, \dots \alpha_m)), \dots F(f_n(\alpha_1, \dots \alpha_m))) = f_0^{\mathcal{M}}(f_1^{\mathcal{M}}(F(\alpha_1), \dots$$
$$\dots F(\alpha_m)), \dots f_n^{\mathcal{M}}(F(\alpha_1), \dots F(\alpha_m))).$$

Now, we prove $F(\lim_{\xi \in \alpha} f(\xi)) = \lim^{\mathcal{M}}_{\epsilon^{\mathcal{M}} F(\alpha)} f^{\mathcal{M}}(\eta)$, where f is a function invariant with respect to $k_1, \dots k_m$.

Let us denote $\beta = \lim_{\xi \in \alpha} f(\xi)$. Thus, $\xi \in \alpha \to f(\xi) \subseteq \beta$. Let $\eta \in^{\mathcal{M}} F(\alpha)$. Then $F^{-1}(\eta) \in \alpha$ and $f(F^{-1}(\eta)) \leq \beta$ i.e. $F(f(F^{-1}(\eta))) = f^{\mathcal{M}}(\eta) \leq F(\beta)$.

Let $\gamma \in^{\mathcal{M}} On^{\mathcal{M}}$ and $\eta \in^{\mathcal{M}} F(\alpha) \to f^{\mathcal{M}}(\eta) \overset{\mathcal{M}}{\leq} \gamma$. For every $\xi \in \alpha$, we have $F(\xi) \in^{\mathcal{M}} F(\alpha)$ therefore $f^{\mathcal{M}}(F(\xi)) \leq^{\mathcal{M}} \gamma$ and $f(\xi) \leq F^{-1}(\gamma)$. It follows that $\lim^{\mathcal{M}}_{\epsilon^{\mathcal{M}} F(\alpha)} f^{\mathcal{M}}(\eta) = F(\beta)$. The theorem follows immediately.

Example: The function $\aleph$ defined by 8.57 (see [G]) is not invariant. By [V2], there is a model ∇ such that the cardinals of model Δ_∇ (i.e. Δ-model constructed in ∇) are not absolute. It follows: $\aleph$ is not calculable.

The function cf is not invariant: if $F(\alpha) = \omega_1^{\Delta_\nabla}$, $\omega_1^{\Delta_\nabla} \neq \omega_1^{\nabla}$ (there is such a model ∇), then $cf^{\Delta_\nabla}(F(\alpha)) = 1^{\Delta_\nabla}$, $cf^{\nabla}(F(\alpha)) = 0^{\nabla}$

§ 2. Almost constant functions

The following assertions *) are well known:

(2.1) $\aleph_{\alpha+1}^{\aleph_\beta} = \aleph_\alpha^{\aleph_\beta} \cdot \aleph_{\alpha \dot{+} 1}$

(2.2) $\alpha \in K_{II} \ \& \ \beta < cf(\alpha) . \rightarrow \aleph_\alpha^{\aleph_\beta} = \sum_{\xi \in \alpha} \aleph_\xi^{\aleph_\beta}$

(2.3) If $\alpha = \lim_{\xi \in \omega_\beta} \tau_\xi$, τ_ξ is an increasing sequence, $\alpha \in K_{II}$, then $\aleph_\alpha^{\aleph_\beta} = \prod_{\xi \in \omega_\beta} \aleph_{\tau_\xi}$.

The proof of the following lemma is trivial:

Lemma 2.1 Let τ_ξ be an increasing sequence, $\alpha = \lim_{\xi \in \omega_\beta} \tau_\xi$. If ω_β is a regular cardinal, then $cf(\alpha) = \beta$.

In this paragraph, a function is always a class f for which

$$f \, Fn \, On \ \& \ W(f) \subseteq On .$$

Definition 2.2 Let f be a non-decreasing function, $X \subseteq On$. We say that f has a gap on X iff there is $\alpha \in X$ such that $(f(\alpha) \dot{+} 1 - \alpha \dot{+} 1) \cap K_{II} \neq 0$, i.e. there is $\beta \in K_{II}$, $\alpha < \beta \leq f(\alpha)$. $G(f; X) = \{\alpha; (f(\alpha) \dot{+} 1 - \alpha \dot{+} 1) \cap K_{II} \neq 0 \ \& \ \alpha \in X\}$ is called the class of gaps of function f on X. If $G(f; On) = 0$, we say that f has no gap.

We say that f is almost constant on X iff the ordinal type of $W(f \upharpoonright X)$ is not confinal with the type of X.

Lemma 2.3 Let f be a non-decreasing function for which: $\xi \in On \rightarrow f(\xi) \geq \xi$. Let f be almost constant on $\alpha \in K_{II}$. Then

a) the typ of $G(f; \alpha)$ is confinal with α (and f has a gap on α);

b) there is $\xi_0 \in \alpha$ such that $f(\xi_0) = f(\eta)$ for every $\eta \in \alpha$, $\eta \geq \xi_0$.

Proof: There is $\beta \in \alpha$ and a function $g : g \ \mathrm{Isom}_{E,E}\ \beta, W(f \upharpoonright \alpha)$. By assumptions, α is not confinal with β.

a) Let us suppose: $\xi \in \alpha \rightarrow f(\xi) \in \alpha$ i.e. $W(f \upharpoonright \alpha) \subseteq \subseteq \alpha$. A contradiction ($\alpha = \lim_{\xi \in \beta} g(\xi)$) follows from $f(\xi) \geq \xi$. Hence, there is $\xi_0 \in \alpha$ and $f(\xi_0) \geq \alpha$. $\alpha - \xi_0$ is confinal with α and $\alpha - \xi_0 \subseteq G(f; \alpha)$.

b) Let us suppose: $(\xi)(\xi \in \alpha \rightarrow (\exists \eta)(\eta \in \alpha \ \& \ \xi \in \eta \ \& \ f(\xi) \in \in f(\eta)))$. We denote $h(\xi)$ the least η for which $g(\xi) = = f(\eta)$. h is a non-decreasing function and $D(h) = \beta \ \& \ \& \ W(h) \subseteq \alpha$. For every $\xi \in \alpha$ there is the least $\eta > \xi$ such that $f(\xi) \in f(\eta)$. Thus, $h(g^{-1}(f(\eta))) = \eta$ and therefore $\lim_{\xi \in \beta} h(\xi) = \alpha$ which is a contradiction.

q.e.d.

§ 3. The continuum function

Let φ_c denote the following of-formula

$$X \subseteq On \times On \ \& \ (\alpha)(\beta)(\langle \alpha \, \beta \rangle \in X \equiv 2^{\aleph_\beta} = \aleph_\alpha)$$

The function $\varkappa$ defined by φ_c is called a continuum function.

The generalized continuum hypothesis is equivalent to

$$(\alpha)(\varkappa(\alpha) = \alpha \dotplus 1)$$

Lemma 3.1. a) $(\alpha)(\varkappa(\alpha) > \alpha)$

b) $(\alpha)(\beta)(\alpha \leq \beta \rightarrow \varkappa(\alpha) \leq \varkappa(\beta))$

c) $(\alpha)(\alpha < cf(\varkappa(\alpha)))$.

Proof is trivial: a) is Cantor theorem, b) follows from definition and c) from König inequality.

Theorem 3.2 Let α be a limit ordinal, $cf(\alpha) \neq \alpha$. If $\varkappa$ is almost constant on α then there is an $\xi_0 \in \alpha$ such that $\varkappa(\alpha) = \varkappa(\xi_0)$.

Proof: Let ξ_0 be the least ordinal for which $\varkappa(\xi_0) \geq \alpha$ and $\xi_0 \in \xi \in \alpha \to \varkappa(\xi_0) = \varkappa(\xi)$. Its existence follows from lemma 2.3 b).

a) Let $\alpha < \omega_\alpha$. We denote ξ_1 an ordinal: $\xi_1 > \xi_0, \xi_1 \in \alpha, \aleph_{\xi_1} \geq card\ \alpha$. Then

$$\aleph_{\varkappa(\xi_0)} \leq 2^{\aleph_\alpha} = 2^{\sum_{\xi\in\alpha}\aleph_\xi} = \prod_{\xi\in\alpha} 2^{\aleph_\xi} \leq \prod_{\xi\in\alpha} 2^{\aleph_{\xi_1}} \leq (2^{\aleph_{\xi_1}})^{card\ \alpha} =$$
$$= 2^{\aleph_{\xi_1}\cdot card\ \alpha} = 2^{\aleph_{\xi_1}} = 2^{\aleph_{\xi_0}}.$$

b) Let $\alpha = \omega_\alpha$, ω_α is singular. There is $\beta < \alpha$ and an increasing sequence τ_ξ for which $\omega_\alpha = \lim_{\xi\in\omega_\beta} \omega_{\tau_\xi}$. Let ξ_1 be an ordinal: $\xi_1 \in \alpha$, $\xi_0 \in \xi_1$, $\beta \in \xi_1$. We may suppose $\tau_\xi > \xi_1$ for $\xi \in \omega_\beta$.

Then

$$\aleph_{\varkappa(\xi_0)} = 2^{\aleph_{\xi_0}} \leq 2^{\aleph_\alpha} = 2^{\sum_{\xi\in\omega_\beta}\aleph_{\tau_\xi}} = \prod_{\xi\in\omega_\beta} 2^{\aleph_{\tau_\xi}} =$$
$$= (\aleph_{\varkappa(\xi_1)})^{\aleph_\beta} = 2^{\aleph_{\xi_1}\cdot\aleph_\beta} = 2^{\aleph_{\xi_1}} = 2^{\aleph_{\xi_0}}.$$

q.e.d.

Theorem 3.3 Let α be a limit ordinal. If $\varkappa$ is not almost constant on α then $\varkappa(\alpha) > \lim_{\xi\in\alpha} \varkappa(\xi)$.

Proof: We define f, $f\,Fn\,\alpha$ in the following way:

$f(\xi)$ denotes the least $\eta \in \alpha$ such that $\varkappa(\xi) < \varkappa(\eta)$.

The existence of such a function follows from the fact that the typ of $W(\varkappa \upharpoonright \alpha)$ is confinal with α. It is easy to see that $\xi < f(\xi)$. The equality $\sum_{\xi\in\alpha} \aleph_\xi = \sum_{\xi\in\alpha} \aleph_{f(\xi)}$ and the König inequality imply:

$$\aleph_{\lim_{\xi\in\alpha} \varkappa(\xi)} = \sum_{\xi\in\alpha} \aleph_{\varkappa(\xi)} < \prod_{\xi\in\alpha} \aleph_{\varkappa(f(\xi))} = 2^{\sum\aleph_{f(\xi)}} = 2^{\sum\aleph_\xi} =$$
$$= 2^{\aleph_\alpha} = \aleph_{\varkappa(\alpha)}.$$

q.e.d

Theorems 3.1 - 3.3 give necessary conditions for function $\varkappa$. The function $\varkappa$ is defined by a cardinal operation. We are interested in its calculability and relation to ordinal operations. This question is solved by

Metatheorem 3.4 The continuum function is not calculable relative to $\aleph$, cf.

Proof: By [V2], there is a model ∇ with the following properties:

$2^{\aleph_0} = \aleph_2$, $2^{\aleph_\alpha} = \aleph_{\alpha+1}$ for $\alpha > 0$ holds in ∇,

the cardinals of Δ-model constructed in ∇ are absolute, cf is absolute.

Let F be the identity function defined on class On^∇ of model ∇.

Then: $(\alpha)(F(\omega_\alpha) = \omega^\Delta_{F(\alpha)})$

$(\alpha)(cf(\alpha) = cf^\Delta(F(\alpha)))$

but $F(\varkappa(0)) \neq \varkappa^\Delta(F(0))$

$\varkappa(0) = 2$, $\varkappa^\Delta(F(0)) = 1$, everything in ∇.

Thus, $\varkappa$ is not invariant with respect to $\aleph$, cf and therefore, is not calculable relative to $\aleph$, cf. q.e.d.

§ 4. The function μ

Let $\varphi_m(X)$ denote the formula

$$(\alpha)(\beta)(\gamma)[\langle\gamma\alpha\beta\rangle \in X \equiv \aleph_\alpha^{\aleph_\beta} = \aleph_\gamma] \,\&\, X \subseteq On^3 .$$

φ_m is an of-formula. Let μ denote the corresponding ordinal function.

The following properties of μ are almost trivial:

(4.1) $\alpha \leq \beta \rightarrow \mu(\alpha;\beta) = \varkappa(\beta)$

(4.2) $\alpha > \beta \rightarrow \alpha \leq \mu(\alpha;\beta) \leq \varkappa(\alpha)$

(4.3) $\mu(\alpha \dotplus 1;\beta) = Max\{\mu(\alpha;\beta);\alpha \dotplus 1\}$

We shall use the following notations:

$l(\alpha) = \beta \equiv . (\beta \in K_{II} \vee \beta = 0) \;\&\; \beta \leq \alpha \;\&\; (\gamma)(\gamma > \beta \;\&\; \gamma \in K_{II} . \rightarrow \alpha \in \gamma)$,

$r(\alpha)$ is the least $\beta \in \omega_0$ for which $l(\alpha) \dot{+} \beta = \alpha$; if f is an ordinal function, $\overleftarrow{f}(\alpha)$ is the least β such that $f(\beta) \geq \alpha$.

The following lemma is an immediate consequence of these notations:

<u>Lemma 4.1</u> a) $\xi \in G(f; \alpha) \equiv (\exists \beta)(\beta \in K_{II} \;\&\; \beta \in \alpha \;\&\; \overleftarrow{f}(\beta) \leq \xi < \beta)$,

b) $\overleftarrow{\varkappa}(\alpha) \leq \alpha$,

c) $\varkappa$ has no gap on $On \equiv (\beta)(\beta \in K_{II} \rightarrow \overleftarrow{\varkappa}(\beta) = \beta)$.

By induction, (4.3) implies

<u>Lemma 4.2</u> a) $\mu(\alpha; \beta) = Max\{\mu(l(\alpha); \beta); \alpha\}$

b) $\beta \geq l(\alpha) \rightarrow \mu(\alpha; \beta) = Max\{\varkappa(\beta); \alpha\}$.

Using this lemma, the calculation of $\mu(\alpha; \beta)$ reduces to the calculation of $\mu(\alpha; \beta)$ for α limit. For α limit, the calculation of $\mu(\alpha; \beta)$ is more complicated.

<u>Theorem 4.3</u> If $\alpha \in K_{II} \cup \{0\}$, $\overleftarrow{\varkappa}(\alpha) = \alpha$, then

a) $cf(\alpha) \leq \beta \leq \alpha \rightarrow \mu(\alpha; \beta) = \varkappa(\alpha)$,

b) $\beta < cf(\alpha) \rightarrow \mu(\alpha; \beta) = \alpha$.

<u>Proof</u>: a) Let α_ξ be an increasing sequence, $\lim\limits_{\xi \in \omega_{cf(\alpha)}} \alpha_\xi = \alpha$.

We define a function f from $\omega_{cf(\alpha)}$ into $\omega_{cf(\alpha)}$:

let $f(0)$ be the least ξ for which $\varkappa(\alpha_0) \leq \alpha_\xi$.

Let us suppose that f is defined for $\delta \in \eta \; (\in \omega_{cf(\alpha)})$.

If $\varkappa(\alpha_\eta) > \alpha_\xi \eta$ for every $\xi \in \omega_{cf(\alpha)}$, then $\varkappa(\alpha_\eta) \geq \alpha$ – contradicts with $\overleftarrow{\varkappa}(\alpha) = \alpha$. If for every $\xi \in \alpha$, $\varkappa(\alpha_\eta) \leq \alpha_\xi$, there is an $\delta \in \eta$, $\xi \leq f(\delta)$, then $\lim\limits_{\xi \in} \alpha_{f(\xi)} = \alpha$ – a contradiction.

Thus, there is ξ such that $\varkappa(\alpha_\eta) \leq \alpha_\xi$ and $\xi > f(\delta)$ for every $\delta \in \eta$. $f(\eta)$ is the least ξ with these properties.

f is an one-to-one function from $\omega_{cf(\alpha)}$ into itself and $\alpha_{f(\xi)} \geq \varkappa(\alpha_\xi)$. Now, if $x \in \prod_{\xi \in \omega_{cf(\alpha)}} \omega_{\varkappa(\alpha_\xi)}$, then we denote by $g(x)$ the function Y defined by

$$Y(\xi) = \begin{array}{ll} X(\eta) & \text{for } \xi = f(\eta) \\ 0 & \text{for } \xi \notin W(f) . \end{array}$$

g is an one-to-one function into $\prod_{\xi \in \omega_{cf(\alpha)}} \omega_{\alpha_\xi}$, thus $\prod_{\xi \in \omega_{cf(\alpha)}} 2^{\aleph_{\alpha_\xi}} \leq \prod_{\xi \in \omega_{cf(\alpha)}} \aleph_{\alpha_\xi}$.

Using (2.3), we have

$$\aleph_\alpha^{\aleph_{cf(\alpha)}} = \prod \aleph_{\alpha_\xi} \geq \prod 2^{\aleph_{\alpha_\xi}} = 2^{\sum \aleph_{\alpha_\xi}} = 2^{\aleph_\alpha}$$

and the theorem follows immediately.

b) $\aleph_\alpha^{\aleph_\beta} = \sum_{\xi \in \alpha} \aleph_\xi^{\aleph_\beta} \leq \sum_{\xi \in \alpha} (2^{\aleph_\xi})^{\aleph_\beta} = \sum_{\beta \leq \xi < \alpha} (2^{\aleph_\xi})^{\aleph_\beta} =$

$= \sum_{\beta \leq \xi < \alpha} 2^{\aleph_\xi} \leq \aleph_\alpha$. q.e.d.

__Theorem 4.4__ If $\alpha \in K_{II}$, $\overleftarrow{\varkappa}(\alpha) < \alpha$, then

a) $\overleftarrow{\varkappa}(\alpha) \leq \beta < \alpha \rightarrow \mu(\alpha; \beta) = \varkappa(\beta)$,

b) $cf(\alpha) \leq \beta < \overleftarrow{\varkappa}(\alpha) \rightarrow \alpha < \mu(\alpha; \beta) \leq \varkappa(\overleftarrow{\varkappa}(\alpha))$.

__Proof__: a) $2^{\aleph_\beta} \leq \aleph_\alpha^{\aleph_\beta} \leq (2^{\aleph_{\overleftarrow{\varkappa}(\alpha)}})^{\aleph_\beta} = 2^{\aleph_\beta}$.

b) König inequality implies $\aleph_\alpha < \aleph_\alpha^{\aleph_{cf(\alpha)}}$. But $\aleph_\alpha^{\aleph_{cf(\alpha)}} \leq \aleph_\alpha^{\aleph_\beta} \leq (2^{\aleph_{\overleftarrow{\varkappa}(\alpha)}})^{\aleph_\beta} = 2^{\aleph_{\overleftarrow{\varkappa}(\alpha)}} = \aleph_{\varkappa(\overleftarrow{\varkappa}(\alpha))}$. q.e.d.

The author does not know how to prove a stronger theorem than 4.4 b) and therefore, he cannot prove the calculability of the function μ relative to cf, $\varkappa$, but only a weaker result

__Theorem 4.5__ In the set theory Σ^* with the axiom $(\alpha)(\alpha \in K_{II} \rightarrow \overleftarrow{\varkappa}(\alpha) = \alpha)$ the function μ is calculable relative to $\varkappa$, cf.

__Proof__: We define

$$m(0;\beta) = \varkappa(\beta)$$

$$m(\alpha \dotplus 1;\beta) = \mathrm{Max}\{m(\alpha;\beta);\ \alpha \dotplus 1\}$$

$$\alpha \in K_{II}: m(\alpha;\beta) = sg(\beta \dotplus 1;\alpha) \times \varkappa(\beta) \dotplus sg(\alpha;\beta) \times$$

$$\times [sg(\beta \dotplus 1;\ cf(\alpha)) \times \varkappa(\alpha) \dotplus sg(cf(\alpha);\beta) \times \alpha].$$

It is easy to see that m is calculable relative to $\varkappa$, cf. Using the axiom $(\alpha)(\alpha \in K_{II} \rightarrow \overleftarrow{\varkappa}(\alpha) = \alpha)$ and theorem 4.3, we can prove the equality

$$(\alpha)(\beta)(m(\alpha;\beta) = \mu(\alpha;\beta)). \qquad \text{q.e.d.}$$

Remark: The assumption $(\alpha)(\alpha \in K_{II} \rightarrow \overleftarrow{\varkappa}(\alpha) = \alpha)$ (i.e. $\varkappa$ has no gap) is consistent with Σ^* it holds e.g. in the Gödel's Δ- model. It follows from [V2], that the assumption is independant.

There is a model ∇ where

$$2^{\aleph_0} = \aleph_1,\ 2^{\aleph_1} = 2^{\aleph_2} \dots = 2^{\aleph_{\omega_0}} = \aleph_{\omega_0 + 2},\ 2^{\aleph_\alpha} = \aleph_{\alpha+1} \text{ for } \alpha \geq \omega_0 \dotplus 1,$$

$$\aleph_{\omega_0}^{\aleph_0} = \aleph_{\omega_0 + 1}.$$

Generally, can prove neither $\aleph_{\omega_0}^{\aleph_0} < \aleph_{\omega_0 + 2}$ nor $\aleph_{\omega_0}^{\aleph_0} = \aleph_{\omega_0 + 2}$. The positive solution of the following problem implies the non-calculability of μ relative to $\varkappa$, cf.

Problem:

There are two models ∇_1, ∇_2 and a mapping F between On^{∇_1}, On^{∇_2} satisfying the conditions of definition 1.4 *) (with $K_1 = \varkappa$, $K_2 = cf$) and the following ones:

(i) $\varkappa(0) = 1$, $\varkappa(\alpha) = \omega_0 + 2$ for $0 \leq \alpha \leq \omega_0 \dotplus 1$, $\varkappa(\alpha) = \alpha \dotplus 1$ for $\alpha > \omega_0$, everything in ∇_1 and ∇_2,

(ii) $\aleph_{\omega_0}^{\aleph_0} = \aleph_{\omega_0 + 2}$ in ∇_1,

$\aleph_{\omega_0}^{\aleph_0} = \aleph_{\omega_0 \dotplus 1}$ in ∇_2.

§ 5. The function π

We define a function π in following way:

$$(\alpha)(\beta)[\pi(\alpha) = \beta \equiv \aleph_\alpha^{\aleph_{cf(\alpha)}} = \aleph_\beta] .$$

It has been conjectured by P.Vopěnka that the calculation of $\aleph_\alpha^{\aleph_\beta}$, $2^{\aleph_\beta}$ can be reduced to π .

In this paragraph, we prove this assumption, namely, we prove that μ and $\varkappa$ are calculable relative to π, cf .

We define $\pi^*(\alpha) = \pi(\alpha \dot{+} 1)$.

Theorem 5.1 a) $\pi(\alpha) = \varkappa(\alpha)$ for α regular (i.e. $\alpha = cf(\alpha)$).

b) $\alpha < \pi(\alpha)$.

c) $cf(\alpha) < cf(\pi(\alpha))$.

d) π^* is non-decreasing .

Proof: a) If α is regular, then $cf(\alpha) = cf(\omega_\alpha) = \alpha$ and $\pi(\alpha) = \mu(\alpha; \alpha) = \varkappa(\alpha)$.

b) For α regular, the Cantor theorem implies b), for α singular, it follows from König inequality.

c) $$\aleph_{\pi(\alpha)}^{\aleph_{cf(\alpha)}} = (\aleph_\alpha^{\aleph_{cf(\alpha)}})^{\aleph_{cf(\alpha)}} = \aleph_{\pi(\alpha)} < \aleph_{\pi(\alpha)}^{\aleph_{cf(\pi(\alpha))}}$$

hence

$$\aleph_{cf(\alpha)} < \aleph_{cf(\pi(\alpha))} .$$

d) We have $\pi^*(\alpha) = \varkappa(\alpha \dot{+} 1) \leqslant \varkappa(\beta \dot{+} 1) = \pi^*(\beta)$ for $\beta \geqslant \alpha$. q.e.d.

Theorem 5.2 Let α be a limit ordinal.

a) If π^* is almost constant on α, $cf(\alpha) \neq \alpha$, then there is $\xi_0 \in \alpha$, $\xi_0 \in K_I$ such that $\varkappa(\alpha) = \pi(\xi_0) = \lim_{\xi \in \alpha} \pi(\xi)$.

b) If π^* is not almost constant on α, then $\varkappa(\alpha) = \pi(\lim_{\xi \in \alpha} \pi(\xi))$.

Proof: a) If π^* is almost constant on α, then, by lemma

2.3, there is $\xi_0 \in \alpha : \pi^*(\xi_0) = \pi^*(\xi)$ for $\xi_0 \in \xi \in \alpha$. The theorem follows from theorem 3.2 .

b) Let β be $cf(\alpha)$. As π^* is not almost constant on α, there is a sequence $\tau_\xi \in K_I$ with the properties:

$\pi(\tau_\xi)$ is increasing,

$\lim\limits_{\xi \in \omega_\beta} \tau_\xi = \alpha$.

Then $2^{\aleph_\alpha} = 2^{\Sigma \aleph_{\tau_\xi}} = \prod\limits_{\xi \in \omega_\beta} 2^{\aleph_{\tau_\xi}} = \prod\limits_{\xi \in \omega_\beta} \aleph_{\pi(\tau_\xi)}$.

Using the facts: $\pi(\tau_\xi)$ is increasing, ω_β is regular, lemma 2.1 implies $cf(\lim\limits_{\xi \in \omega_\beta} \pi(\tau_\xi)) = \beta$. Then, by (2.3), it holds

$$\prod_{\xi \in \omega_\beta} \aleph_{\pi(\tau_\xi)} = \aleph^{\aleph_\beta}_{\lim\limits_{\xi \in \omega_\beta} \pi(\tau_\xi)} = \aleph_{\pi(\lim\limits_{\xi \in \omega_\beta} \pi(\tau_\xi))} \quad \text{q.e.d.}$$

<u>Corollary 5.3</u> If ϑ is a strongly inaccessible cardinal then $\pi(\vartheta) = \pi(\lim\limits_{\xi \in \vartheta} \pi(\xi))$, i.e. $\vartheta = \lim\limits_{\xi \in \vartheta} \pi(\xi)$.

If ϑ is a weakly inaccessible cardinal, π^* is not almost constant on ϑ, then $\pi(\vartheta) = \pi(\lim\limits_{\xi \in \vartheta} \pi(\xi))$.

<u>Theorem 5.4</u> If $\beta < cf(\alpha)$, $\alpha \in K_{II}$, then $\mu(\alpha;\beta) = \lim\limits_{\xi \in \alpha} \mu(\xi;\beta)$.

<u>Proof</u>: By (2.2), $\aleph_\alpha^{\aleph_\beta} = \sum\limits_{\xi \in \alpha} \aleph_\xi^{\aleph_\beta}$.

We define: $s(0) = 0$, $s(\eta) = \gamma \equiv \aleph_\gamma = \sum\limits_{\xi \in \eta} \aleph_\xi^{\aleph_\beta}$ for $\eta \leq \alpha$.

It holds $s(1) = \mu(0;\beta)$,

$s(\eta + 1) = Max\{s(\eta); \mu(\eta;\beta)\} = \mu(\eta;\beta)$

because of $\aleph_\xi^{\aleph_\beta} \leq \aleph_\eta^{\aleph_\beta}$ and $\bar{\bar{\eta}} \leq \aleph_\eta^{\aleph_\beta}$. It suffices to prove $s(\eta) = \lim\limits_{\xi \in \eta} s(\xi)$ for η limit.

Let $\lim\limits_{\xi \in} s(\xi) = \sigma$. But $s(\eta) \geq s(\xi)$ for $\xi \in \eta$ i.e. $s(\eta) \geq \sigma$. If $\xi \in \eta$, then $s(\xi + 1) \geq \xi$, $s(\xi + 1) \leq \sigma$.

Therefore: $\eta \leq \delta$. Thus, we have

$$\aleph_{s(\eta)} = \sum_{\xi \in \eta} \aleph_\xi^{\aleph_\beta} \leq \aleph_{\delta \cdot \overline{\overline{\eta}}} = \aleph_\delta \quad . \qquad \text{q.e.d.}$$

Theorem 5.5 Let $\alpha \in K_{II}$, $cf(\alpha) \leq \beta < \alpha$.

a) If $\mu(\xi; \beta)$ (as a function of ξ) is almost constant on α, then there is $\xi_0 \in \alpha$ such that $\mu(\alpha; \beta) = \mu(\xi_0, \beta)$, i.e. $\mu(\alpha; \beta) = \lim_{\xi \in \alpha} \mu(\xi; \beta)$.

b) If $\mu(\xi; \beta)$ is not almost constant on α, then

$$\mu(\alpha; \beta) = \pi(\lim_{\xi \in \alpha} \mu(\xi; \beta)).$$

Proof: a) Let $\alpha = \lim_{\xi \in \omega_{cf(\alpha)}} \tau_\xi$. Let ξ_0 be an ordinal choosen by lemma 2.3 . We may suppose $\tau_\xi > \xi_0$ for every $\xi \in \omega_{cf(\alpha)}$. Then

$$\aleph_\alpha^{\aleph_\beta} = \prod_{\xi \in \omega_{cf(\alpha)}} \aleph_{\tau_\xi}^{\aleph_\beta} = (\aleph_{\tau_{\xi_0}})^{\aleph_\beta \cdot \aleph_{cf(\alpha)}} = \aleph_{\tau_{\xi_0}}^{\aleph_\beta} \quad .$$

b) There is a sequence τ_ξ with the following properties:

$\alpha = \lim_{\xi \in \omega_{cf(\alpha)}} \tau_\xi$, $\mu(\tau_\xi; \beta)$ is increasing, $\tau_\xi \in K_I$. Then

$$\aleph_\alpha^{\aleph_\beta} = \prod_{\xi \in \omega_{cf(\alpha)}} \aleph_{\tau_\xi}^{\aleph_\beta} = \prod_{\xi \in \omega_{cf(\alpha)}} \aleph_{\mu(\tau_\xi; \beta)} \; .$$

It follows from (2.3) that $\prod_{\xi \in \omega_{cf(\alpha)}} \aleph_{\mu(\tau_\xi; \beta)} = \aleph_{\lim \mu(\tau_\xi; \beta)}^{\aleph_{cf(\alpha)}}$.

By lemma 2.1, $cf(\alpha) = cf(\lim_{\xi \in \omega_{cf(\alpha)}} \mu(\tau_\xi; \beta))$. Using

$\lim_{\xi \in \alpha} \mu(\xi; \beta) = \lim_{\xi \in \omega_{cf(\alpha)}} \mu(\tau_\xi; \beta)$, we have $\mu(\alpha; \beta) = \pi(\lim \mu(\xi; \beta))$.

q.e.d.

Theorem 5.6 The functions $\varkappa$, μ are calculable relative to π, cf.

Proof: We define two functions:

$$h(0) = 0$$

$$h(\alpha + 1) = h(\alpha) \dot{+} sg(\pi^*(\alpha \dot{+} 1); \pi^*(\alpha))$$

$$\alpha \in K_{II} : \; h(\alpha) = \lim_{\xi \in \alpha} h(\xi)$$

and

$k(0) = \pi(0)$

$k(\alpha \dotplus 1) = \pi(\alpha \dotplus 1)$

$\alpha \in K_{II} : k(\alpha) = \pi(\alpha) \times eq(cf(\alpha); \alpha) \dotplus sg(\alpha; cf(\alpha)) \times$
$\times [eq(cf(\alpha); cf(h(\alpha))) \times \lim_{\xi \in \alpha} \pi(\xi) \dotplus \overline{sg}(eq(cf(\alpha);$
$cf(h(\alpha))) \times \pi(\lim_{\xi \in \alpha} \pi(\xi))]$.

Using theorem 5.2, we can prove $(\alpha)(\varkappa(\alpha) = k(\alpha))$. Thus, $\varkappa$ is calculable relative to π, cf.

Now, we define a function t in such a way that, for β fixed, $C_1(t(\alpha; \beta))$ will be the typ of $W(\mu \restriction \alpha)$, $C_2(t(\alpha; \beta))$ will be $\mu(\alpha; \beta)$.

Let t be the function defined as follows:

$t(0; \beta) = P(0; \varkappa(\beta))$

$t(\alpha \dotplus 1; \beta) = P(C_1(t(\alpha; \beta) \dotplus sg(\alpha \dotplus 1; C_2(t(\alpha; \beta)));$

$Max\{C_2(t(\alpha; \beta)); \alpha \dotplus 1\})$

$\alpha \in K_{II} : t(\alpha; \beta) = P(\delta; \varepsilon)$

where

$\delta = \lim_{\xi \in \alpha} C_1(t(\xi; \beta))$, $\gamma = \lim_{\xi \in \alpha} C_2(t(\xi; \beta))$, $\varepsilon =$
$= sg(\beta \dotplus 1; \alpha) \times \varkappa(\beta) \dotplus sg(\alpha; \beta) \times [sg(cf(\alpha); \beta) \times \gamma \dotplus$
$\dotplus sg(\beta \dotplus 1; cf(\alpha)) \times (eq(cf(\delta); cf(\alpha)) \times \pi(\gamma) \dotplus \overline{sg}(eq(cf(\delta); cf(\alpha))) \times \gamma)]$.

t is calculable relative to $\varkappa$, π, cf. Using theorems 5.4, 5.5, we can prove $(\alpha)(\beta)(t(\alpha; \beta) = \mu(\alpha; \beta))$, thus μ is calculable relative to $\varkappa$, π, cf.

Theorem follows immediately.

q.e.d.

Remark:

The manuscript of this paper had been written before the author knew the Easton's paper [E], where on the pages 2 and 3, there is a conjecture that the conditions a) - c) of lemma 3.1 are sufficient for the continuum function. The conjecture is false as there is a function satisfying these conditions, which

does not fulfil the assertion of theorem 3.2.

B i b l i o g r a p h y :

[G] K. GÖDEL, The consistency of the axiom of choice and of the generalized continuum hypothesis with the axioms of set theory, Annals of Mathematics Studies, No. 3, Princeton 1940.

[V1] P. VOPĚNKA, Модели теории множеств, Zeitschr. für math. Logik und Grundlagen d. Math., Bd. 8, 1962.

[V2] P. VOPĚNKA, A forsheaf of relations on a topological space and models of set theory (Czech), mimeographed lecture.

[E] W.B. EASTON, Powers of regular cardinals, mimeographed.

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p. 183: *) The metadefinition determinates a system of ~~of~~-formulas.

p. 184: *) A function is called calculable iff it is calculable relative to the empty sequence of functions.

p. 186: *) (2.1) is Hausdorff reccurence formula, (2.2) and (2.3) are the reccurence formulas by Tarski.

p. 192: *) i.e. ∇_2 is a weakly regular standart model in ∇_1 and the conditions a) b) are relativized to ∇_1 .