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Petr Vopěnka; Lev Bukovský

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THE EXISTENCE OF A PCA - SET OF CARDINAL $\aleph_1$

Petr VOPĚNKA and Leo BUKOVSKÝ , Praha

The aim of this note is to prove that the axioms of set theory and the following theorem are consistent: There exists a projective (moreover PCA) subset of the Baire space the cardinal of which is $\aleph_1$ and $2^{\aleph_0} \geq \aleph_2$ holds.

We assume familiarity with [2], [3] and [5]. Throughout this note we use the notation introduced in the papers [2] and [5]. All our considerations are in system Σ^* of [2] (with the axioms of groups A - E). We denote by P_n the set of all subsets of the Baire space N^N ($N = \omega_0 - \{0\}$) of the projective class L_n (see [3], p. 361). Thus, the elements of P_0 are Borel sets, the elements of P_1 are analytical sets, the elements of P_3 are PCA-sets. We denote by P the sum of all P_n .

Definition 1. Let $\mathcal{A}$ be a set, $\alpha_1, \dots \alpha_k$ are uncountable cardinals. We define

$$\mathcal{X}_k(\mathcal{A}; \alpha_1, \dots \alpha_k) \equiv (\forall x)(x \in \mathcal{A} \rightarrow . \overline{\overline{x}} \leq \aleph_0 \vee \overline{\overline{x}} = \alpha_1 \vee \dots \vee \overline{\overline{x}} = \alpha_k).$$

Proofs of the following statements are given in the paper [3]:

$$\mathcal{X}_1(P_0; 2^{\aleph_0}), \mathcal{X}_1(P_1; 2^{\aleph_0}), \mathcal{X}_2(P_2; \aleph_1, 2^{\aleph_0}), \mathcal{X}_2(P_3; \aleph_1, 2^{\aleph_0}).$$

Kuratowski ([3], p.392) mentioned as an unsolved problem a question, whether $\mathcal{X}_1(P_3; 2^{\aleph_0})$ holds. We shall prove that the assertion $\neg \mathcal{X}_1(P_3;\ 2^{\aleph_0})$ is valid in the model ∇ with a suitable choise ω_μ, ν (see [5]).

Let L be the class of all constructible sets. It follows immediately from [1]:

Lemma 1. $L \cap N^N \in P_3$

It remains to prove that the power of $L \cap N^N$ is $\aleph_1$ (in the model ∇). If $\mathcal{A}$ is a concept in the theory Σ, then we denote by $\mathcal{A}_e$ the corresponding concept in Δ (see [2], Ch.V.).

Lemma 2. $\omega_0 = \omega_{0e}$

Proof: The following concepts are absolute: O_n, $\dot{+}\ 1$, $\cup$ (see [2], 11.42, 11.45, 11.20). Therefore K_I is absolute (2, 7.42). The rest follows from 8.4. q.e.d.

Lemma 3. $\omega_0 \in \omega_{1e} \subseteq \omega_1$

Proof: By lemma 2 $\omega_0 \in \omega_{1e}$. If $\alpha \in \omega_{1e}$, then there exists f such that

$$Un_{2e}(f)\ \&\ Rel_e(f)\ \&\ D_e(f) = \alpha\ \&\ W_e(f) = \omega_{0e}$$

By 10.21, 11.15, 11.12, 11.17 we have

$$Un_2(f)\ \&\ Rel\ (f)\ \&\ D(f) = \alpha\ \&\ W(f) = \omega_0$$

therefore, $\omega_1 \in \omega_{1e}$ cannot hold. q.e.d.

Lemma 4. $L \cap P(\omega_0) \subseteq F''\omega_{1e}$

Proof: In the model Δ , $P_e\ (F_e''\ \omega_{0e}\ ) \subseteq F_e''\omega_{1e}$ holds. It is easily to prove that $L \cap P(\omega_0) \subseteq P_e\ (F_e''\ \omega_{0e}\ )$. Because F is absolute, the lemma follows. q.e.d.

Lemma 5. $(\forall\alpha)\ (\exists\beta)\ [\alpha \in \omega_{1e} \rightarrow .\ \alpha \in \beta \in \omega_{1e}\ \&\ F'_\beta \subseteq \omega_0\ \&$
$\&\ (\forall\gamma)\ (\gamma \in \alpha \dot{+} 1 \longrightarrow F'\gamma \neq F'\beta\ )]$

Proof is more or less a modification of the proof $a < 2^a$ by Cantor. Let $\alpha \in \omega_{1e}$ be such that

(1) $(\forall\beta)\ [\alpha \in \beta \in \omega_{1e}\ \&\ F'\beta \subseteq \omega_0\ . \longrightarrow (\exists\gamma)\ (\gamma \in \alpha\ \dot{+}$
$\dot{+}\ 1\ \&\ F'\gamma = F'\beta\ )]$

From (1) and lemma 4, we have

(2) $\quad x \in L \,\&\, x \subseteq \omega_0 \,.\longrightarrow (\exists \gamma)\,(\gamma \in \alpha \dotplus 1 \,\&\, x = F'\gamma)$

From $\alpha \in \omega_{1\ell}$ and the definition of F, we derive $\overline{\overline{F''\alpha}}^{\ell} \leq_{\ell} \overline{\overline{\alpha}}^{\ell} = \omega_0$. Therefore, there is $f \in L$ such that

(3) $\quad f\ Fn_{\ell}\ \omega_0 \ \&\ W_{\ell}(f) = F''\alpha \dotplus 1$

We define a set d : $(\forall x)\,(x \in d \equiv .\ x \in \omega_0 \,\&\, x \notin f'x)$.
We can deduce $d = \omega_0 - D(f \cap Cnv(E))$. Thus $d \in L$ and $d \subseteq \omega_0$. By (2), there is $\gamma \in \alpha \dotplus 1$ such that $d = F'\gamma$. For every $x \in \omega_0$ the following implications hold:

$$x \in d \longrightarrow f'x \neq d \quad (\text{as } x \notin f'x)$$
$$x \notin d \longrightarrow f'x \neq d \quad (\text{as } x \in f'x)$$

A contradiction with (3) can be deduced from these implications and the fact that $d = F'\gamma$. q.e.d.

Lemma 6. $\overline{\overline{L \cap N^N}} = \overline{\overline{\omega_{1\ell}}}$

Proof: In the model Δ, we can prove $\overline{\overline{L \cap N^N}}^{\ell} = \overline{\overline{P_{\ell}(\omega_0)}}^{\ell}$. The result follows from lemma 5 and the following fact: $\overline{\overline{a}}^{\ell} = \overline{\overline{b}}^{\ell} \longrightarrow \overline{\overline{a}}^{\ell} = \overline{\overline{b}}^{\ell}$. q.e.d.

From ([5], p.42) it follows

Lemma 7. Let ν be a regular cardinal $\geq \omega_2$, $\omega_{\mu} = \omega_0$, both in ∇. Then $\omega_{1\ell} = \omega_1$. We can deduce from lemmas 1, 6, 7 the following theorems:

Theorem 1. Let $\omega_{\mu} = \omega_0$ and $\nu = \omega_2$ (or $\omega_3, \ldots \omega_{\omega_0+1}, \ldots$). Then there is a PCA-subset $\mathcal{A}$ of the Baire space, such that $\overline{\overline{\mathcal{A}}} = \aleph_1$ and $2^{\aleph_0} \geq \aleph_2$ (or $\geq \aleph_3, \ldots \aleph_{\omega_0+1}, \ldots$) hold, everything in model ∇.

Theorem 2. If the set theory Σ (with the axioms of groups A - D) is consistent, then the sentences $\chi_1(P_3;\, 2^{\aleph_0})$, $\chi_1(P;\, 2^{\aleph_0})$ are undecidable in Σ.

Lusin mentioned in ([4] p. 323): " ... le domaine des ensembles projectifs est un domaine où le tiers exclu ne s'applique plus..." . The theorem 2 fully confirms the assumptions by Lusin.

The authors do not know, whether the following equivalences

$$2^{\aleph_0} = \aleph_1 \equiv \mathcal{X}_1(P_3; 2^{\aleph_0}), \quad 2^{\aleph_0} = \aleph_1 \equiv \mathcal{X}_1(P; 2^{\aleph_0}$$

hold in the set theory.

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