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CONTRIBUTION TO THE SOLUTION OF NON-LINEAR EQUATIONS

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Let us denote by X a Banach space, by H a Hilbert (separable and complete) space.

Let an equation

$$(1) \qquad F(x) = f$$

be given, where $F(x)$ is an arbitrary operator which maps X into X.

The generalization of Wiarda's method of solution of non-linear functional equations is based on the following theorems:

<u>Theorem 1.</u> [1] Let $F(x)$ be an arbitrary operator which maps X into X and let P be a linear bounded operator in X such that P^{-1} exists. Let the following conditions be fulfilled:

1) There exists the set $E \subset B$ and a real number α $(0 < \alpha < 1)$ such that for every $u, v \in E$

$$\| R(u) - R(v) \| \leqq \alpha \| u - v \| , \text{ where } R = I - PF .$$

2) The closed sphere $\Omega(x_1, r)$, where

$$x_1 = x_0 - PF(x_0) + Pf , \quad r = \frac{\alpha}{1-\alpha} \| x_1 - x_0 \|$$

and x_0 is an arbitrary element from E, lies in E. Then the equation (1) has a unique solution x^* in the sphere $\Omega(x_1, r)$. The sequence $\{x_n\}$ defined by

$$x_{n+1} = x_n - PF(x_n) + Pf$$

is convergent in the norm of X to the one of (1) and the error of the approximation x_n satisfies the inequality

$$\| x_n - x^* \| = \frac{\alpha^n}{1-\alpha} \| x_1 - x_0 \| \ .$$

Theorem 2. [1] . Let P be a linear bounded operator in H such that P^{-1} exists. Let $F(x)$ be an operator which maps H into H and has continuous Gateaux derivative $F'(x)$ on the set $E \subset H$. For every $x \in H$ let $PF'(x)$ be a symmetric operator in H such that the inequalities

$$\| P \| \sup_{x \in E} \| F'(x) \| < 1 \ , \quad (PF'(x)\, h,\, h) \geqq m \| h \|^2 \ , \quad m > 0$$

hold for every $x \in E$, $h \in H$.

Let us put

(2) $x_{n+1} = x_n - PF(x_n) + Pf$, $n = 0,1,2,\ldots$

$$\alpha = \sup_{x \in E} \| I - PF'(x) \| \ , \quad r = \frac{\alpha}{1-\alpha} \| x_1 - x_0 \| \ ,$$

where x_0 is an arbitrary element from E .
Let $\Omega(x_1, r)$ be a closed sphere contained in E . Then the equation (1) has a unique solution x^* in the sphere $\Omega(x_1, r)$. The sequence $\{x_n\}$ defined by (2) converges in the norm of H to the solution x^* of (1) and the error satisfies

$$\| x^* - x_n \| \leq \frac{\alpha^n}{1-\alpha} \| x_1 - x_0 \| \ .$$

Remark 1. From the theorem it also follows that α may be replaced by α' , where $\alpha < \alpha' = 1 - m < 1$.

If $F(x) = x - \lambda\phi(x)$, where λ is a real parameter, $\phi(x)$ is a non-linear operator which maps X into X , we get the following theorem:

Theorem 3. [1]. Let $\phi(x)$ be an operator which maps H

into H and has continuous Gateaux derivative $\phi'(x)$ on the set $E \subset H$ for every $x \in H$; assume it is a symmetrical operator in H and such that the inequality $(\lambda \phi (x) h, h) \geq 0$ holds for every $x \in E$ and $h \in H$. Let ϑ satisfy the inequality

$$0 < \vartheta < \frac{1}{1 + \sup\limits_{x \in E} \| \lambda \phi'(x) \|} .$$

Let us put

(3) $x_{n+1} = (1 - \vartheta) x_n + \vartheta \lambda \phi (x_n) + \vartheta f$

$$\alpha = \sup_{x \in E} \| I - \vartheta (I - \lambda \phi'(x)) \| ,$$

$r = \frac{\alpha}{1 - \alpha} \| x_1 - x_0 \|$. Let $\Omega (x_1, r)$ be a sphere which lies in E. Then the equation $x - \lambda \phi (x) = f$ has a unique solution x^* in the sphere $\Omega (x_1, r)$. The sequence $\{x_n\}$ defined by (3) is convergent in the norm of H to the solution x^* and the error $\| x^* - x_n \|$ of the approximation x_n satisfies

$$\| x^* - x_n \| \leqq \frac{\alpha^n}{1 - \alpha} \| x_1 - x_0 \| .$$

Remark 2. The number α may be replaced by α', where $\alpha < \alpha' = 1 - \vartheta < 1$. If we put $P = [F'(x_0)]^{-1}$ in (2) we obtain the Newton-Kantorowitch iterative process. The assumption of symmetricity of the operator $F'(x)$ in the theorems 2,3 may be replaced by an assumption of potentiality of the operator $F(x)$ (cf. [2] § 5, theorem 5.1).

We shall generalize the theorem 1 in two directions. At the first we replace in the inequality $\| R(u) - R(v) \| \leqq \alpha \| u - v \|$

the number α by $\alpha = 1$. Further we set $R(x) = PF(x) - A$, where A is a linear bounded operator in Banach space X.

(4) Consider the functional equation $F(x) = 0$.

If $G(x) = f$, then we set $F(x) = G(x) - f$.

Theorem 4. Let E be a convex closed bounded set in Hilbert space H. Let $F(x)$ be a potential operator on E which maps E in E. Let the inequality $(PF'(x)h, h) \geqq 0$ hold for every $x \in E$ and $h \in H$, where P is a linear operator in H such that P^{-1} exists and $0 < \| P \| < 1/ \sup_{x \in E} \| F'(x) \|$. Then the equation (4) has at least one solution x^* in E. The sequence $\{x_n\}$ defined by

$$x_{n+1} = x_n - QF(x_n), \quad Q = \frac{1}{2} P$$

is convergent weakly to the solution x^*.

This theorem generalizes the following result of M.M. Vajnberg [2, § 10] : If a potential operator $G(x)$ maps Hilbert space H in H and for every $x \in H$, $h \in H$ the inequality $(G'(x)h, h) \geq m \| h \|^2$; $m > 0$ holds, then the equality $G(x) = f$ has at least one solution x^* in H.

Theorem 5. Let $F(x)$ be a non-linear operator which maps X into X and let A be a linear bounded operator which maps X onto X. Let the following conditions be fulfilled:

1) There exist the linear bounded operator P such that P^{-1} is bounded and closed set $E \subset X$ such that for every u, $v \in E$

$$\| PF(u) - PF(v) - A(u - v) \| \leqq \alpha \| u - v \| .$$

2) The closed sphere $\Omega(x_1, r) \subset E$, where x_1 is

defined by equality $y_0 = A(x_1 - x_0)$, x_0 is an arbitrary element from E , $y_0 = - PF(x_0)$, $r = \frac{\beta}{1-\beta} \| x_1 - x_0 \|$, $\beta = \alpha \delta < 1$, where δ is the number from the inequality $\| x \| \leq \delta \| Ax \|$.

Then the equation (4) has a unique solution x^* in the sphere $\Omega(x_1, r)$. The sequence $\{x_n\}$ defined by equalities

$$y_{n-1} = A(x_n - x_{n-1}) ,$$

$$y_n = - PF(x_n)$$

converges in the norm of X to the solution x^* of (4) and the error satisfies

$$(5) \qquad \| x_n - x^* \| \leq \frac{\beta^n}{1-\beta} \| x_1 - x_0 \| .$$

If $F(x)$ maps X into X , A has a bounded inverse A^{-1} , the condition 1) from theorem 5 is fulfilled and the sphere $\Omega(x_1, r) \subset E$, where $x_1 = x_0 - A^{-1} PF(x_0)$, $x_0 \in E$, $r = \frac{\beta}{1-\beta} \| x_1 - x_0 \|$, $\beta = \alpha \| A^{-1} \| < 1$, then the equation (4) has a unique solution x^* in the sphere $\Omega(x_1, r)$.
The sequence $\{x_n\}$ defined by the inequality $x_{n+1} = x_n - A^{-1} PF(x_n)$ converges in the norm of X and the error of the approximation x_n satisfies the inequality (5).

Theorem 6. Let $F(x)$ be an arbitrary operator which maps X into X and let A, P be linear operators such that A^{-1}, P^{-1} are bounded. Let there exist $x_0 \in X$ such that

$\| PF(u) - PF(v) - A(u-v) \| \leq \alpha \| u-v \|$ holds for every u, v from the closed sphere $\Omega(x_0, r)$, where $r = \frac{1}{1-\alpha k} k \| y_0 \|$, $y_0 = PF(x_0)$, $k = \| A^{-1} \|$, $\alpha k < 1$.

Then the equation (4) has a unique solution x^* in the sphere $\Omega(x_0, r)$. The sequence $\{x_n\}$ defined by

$$x_{n+1} = x_n - A^{-1} PF(x_n) \tag{6}$$

converges in the norm of X to the solution x^* of (4) and

$$\| x_n - x^* \| \leqq \frac{(\alpha k)^n}{1 - \alpha k} \| x_1 - x_0 \| . \tag{7}$$

From this theorem we get immediately the following theorems:

Theorem 7. Let $F(x)$ be a non-linear operator which maps X into X, has the continuous Gateaux's derivative $F'(x)$, on the closed sphere $\Omega(x_0, r)$ and $[F'(x_0)]^{-1}$ is bounded. Then there exists a constant $k > 0$ such that $\|F(x_0)\| \leqq k$ the equation (4) has a unique solution x^* in the sphere $\Omega(x_0, r)$. The sequence $\{x_n\}$ defined by (6), where $A = F'(x_0)$ converges in the norm of X to the solution x^*.

Theorem 8. Let $F(x)$ map X into X and have continuous Gateaux's derivative $F'(x)$ in the closed sphere $\Omega(x_0, r)$. Let the following conditions be fulfilled:

1) There exists $[F'(x_0)]^{-1}$ and $\| [F'(x_0)]^{-1} \| \leq k$.

2) $\| F'(x) - F'(x_0) \| \leq \alpha$ for $x \in \Omega(x_0, r)$,

$\alpha k < 1$ and $\| F(x_0) \| \leqq \frac{1}{k} r(1 - \alpha k)$.

Then the equation (4) has a unique solution x^* in $\Omega(x_0, r)$. The sequence $\{x_n\}$ defined by (6), where $A = F'(x_0)$ converges in the norm of X to x^* and the inequality (7) holds.

Let $F(x)$ be differentiable in the sense of Gateaux's in the closed sphere $\Omega(x_0, r)$. We introduce monotone non negative increasing functions $\psi(\rho)$ defined on the set $0 \leqq \rho < r$ by

$$\psi(\rho) = \sup_{x \in \Omega(x_0, r)} \| F'(x) - F'(x_0) \| .$$

We have that $\varphi(0) = 0$. Let there exist bounded operator $[F'(x_0)]^{-1}$. We set $\mu = \dfrac{1}{\| [F'(x_0)]^{-1} \|}$.

Let ρ_0 be the supremum of all numbers ρ such that $\psi(\rho) < \mu$. This condition is fulfilled when $F'(x)$ is continuous at the point $x = x_0$.

<u>Theorem 9</u>. Let $F(x)$ be Gateaux differentiable in the sphere $\Omega(x_0, \rho_0)$. Let there exist a bounded operator $[F'(x_0)]^{-1}$. Then the equation (4) has a unique solution x^* in $\Omega(x_0, \rho_0)$. The sequence $\{x_n\}$ defined by (6), where $A = F'(x_0)$ converges in the norm of X to the solution x^* and the error satisfies (7).

R e f e r e n c e s

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